COMPLETELY RIGHT INJECTIVE SEMIGROUPS THAT ARE UNIONS OF GROUPS†

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1. Introduction. A semigroup S with 0 and 1 is termed *completely right injective* provided every right unitary S-system is injective. A necessary condition for a semigroup to be completely right injective is given in **[2]**; namely, every right ideal is generated by an idempotent. An example in section **3** of this paper shows the existence of semigroups with 0 and 1 satisfying this condition which are not completely right injective. In **[3]**, it is shown that the condition that every right and left ideal is generated by an idempotent is necessary and sufficient in the case that S is both completely right and left injective (called *completely injective*). Such a semigroup is an inverse semigroup with 0 whose idempotents are dually well-ordered.

The purpose of this paper is to give a characterization for semigroups which are completely right injective and a union of groups and to determine a decomposition for such semigroups. We first develop several properties concerning the two-sided ideals of a semigroup which satisfies the condition that every right ideal is generated by an idempotent. We give equivalent conditions for semigroups of this type to be a union of groups. Using these properties, we are able to prove the characterization. The main theorem states that *a semigroup S is completely right injective and is a union of groups if and only if every right ideal I of S is generated by an idempotent which commutes with all the elements of S not in I.* It is shown that a semigroup of this type is a chain of right groups. In addition, all completely right injective semigroups which have a finite number of right ideals are unions of groups.

We follow the definitions and notations introduced in **[2]** and **[3]** and use freely the results proved there; otherwise the notation and terminology is that of Clifford and Preston **[1]**. Throughout this paper all semigroups will have 0 and 1 and all S-systems will be right unitary S-systems.

2. Completely right injective semigroups. *In this section, with the exceptions of Theorems* 2.10, 2.11, *and* 2.12, *S will always denote a semigroup with* 0 *and* 1 *such that every right ideal is generated by an idempotent. In the aforementioned theorems, S will denote a completely right injective semigroup.* As in **[3]**, the lattice of right ideals of S under set inclusion is dually well-ordered. In addition, S is a regular semigroup **[1,** p. 27]. An inverse of an element s in S will usually be denoted by s', i.e., $s = ss's$ and $s' = s'ss'$, although s' need not be unique. Consequently, if $s \in S$ and $sS = eS$ for some $e \in E(S)$, where $E(S)$ denotes the subsemigroup of all idempotents in S, there exists an inverse s' of s such that $ss' = e$. Moreover, $sS = ss'S$ and $Ss = Ss's$.

Since the right ideals of S are linearly ordered we have

2.1. Proposition. *If $Se = Sf$, for $e, f \in E(S)$, then $e = f$.*

2.2. Proposition. *If $e \in E(S)$, $s \in S$, then $Ses = Ss'es$.*

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Proof. We need only show that $Ss'es$ contains es. If $sS \supseteq eS$, then $s(s'es) = (ss')es = es$. If $sS \subseteq eS$, then $es = s$ and $es(s'es) = es$.

For each $e \in E(S)$, we have $s'es \in E(S)$. Consequently, 2.1 and 2.2 imply

2.3. PROPOSITION. *If s' and s'' are inverses of an element s in S, then $s'es = s''es$.*

As defined in [**1**, pp. 47–48], $\mathscr{H}$, $\mathscr{R}$ and $\mathscr{L}$, $\mathscr{J}$ will denote Green's equivalence relations on the semigroup S. $L_a[R_a, H_a]$ denotes the $\mathscr{L}$-[$\mathscr{R}$-, $\mathscr{H}$-] class of S containing the element a.

2.4. PROPOSITION. *Each $\mathscr{L}$-class of S contains exactly one idempotent.*

Proof. Since S is regular, every $\mathscr{L}$-class contains an idempotent. By 2.1, it is unique.

The following proposition is true for any regular semigroup.

2.5. PROPOSITION. *If $xsS = sxS$, where x is an inverse of s, then there exists an inverse s' of s such that $s's = ss'$.*

Proof. Now $xsS = sxS$ implies that $(sx)(xs) = xs$ and $xss = s$. Set $s' = x^2s$. Then

$$\begin{aligned} ss's &= (sx)(xss) = sxs = s, \\ s'ss' &= x(xss)(x^2s) = (xsx)(xs) = x^2s = s', \\ ss' &= (sx)(xs) = xs = x(xss) = (x^2s)s = s's. \end{aligned}$$

2.6. PROPOSITION. *$a\mathscr{L}b$ implies $a'\mathscr{R}b'$ for all $a, b \in S$.*

Proof. Now $a\mathscr{L}b$ implies $Sa'a = Sb'b$. By 2.1 we have $a'a = b'b$. Thus $a' = a'aa' = b'ba$ and $a'S \subseteq b'S$. Similarly, $b'S \subseteq a'S$.

The following results give some special properties concerning (two-sided) ideals of S.

2.7. PROPOSITION. *Let I be an ideal of S and $a, b, c \in S$.*

(i) *If $a \in I$, then every inverse a' of a is in I.*
(ii) *If $a \notin I$ and $c \in I$, then $Sac = Sc$.*
(iii) *I is a prime ideal of S.* [**2**, p. 40].
(iv) *The relation ρ, defined by $a\rho b$ if and only if either $a, b \in I$ and $a\mathscr{L}b$ or $a, b \notin I$, is a right congruence on S.*

Proof. The first part follows from the fact that $a' = a'aa'$ and I is an ideal of S. Now $a \notin I$ implies $a'a \notin I$. If $c \in I$, then we have $cS \subseteq a'aS$ so that $a'ac = c$. This proves (ii). Moreover, either $a'aS \subseteq cc'S$ or $cc'S \subseteq a'aS$. The former implies $a = a(cc')(a'a)$ and the latter $c = (a'a)(cc')c$. Consequently, $ac \in I$ implies either $a \in I$ or $c \in I$. This completes the proof of (iii).

The relation ρ defined in (iv) is clearly an equivalence relation on S. Suppose $a\rho b$ and $c \in S$. Since $\mathscr{L}$ is a right congruence on S we may assume $a, b \notin I$. If $c \in I$, then, by (ii), $Sac = Sc = Sbc$. If $c \notin I$, then (iii) implies that ac and bc are not elements of I. In either case we have $ac\rho bc$.

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Let $D(S)$ denote the subset of $E(S)$ consisting of all elements which generate the (two-sided) ideals of S. Since the collection $\mathfrak{I}(S)$ of all ideals of S is a dually well-ordered set with respect to set inclusion, then we can write the chain of all ideals in the following manner.

$$S = d_0 S \supset d_1 S \supset d_2 S \supset \ldots \supset d_\alpha S \supset \ldots, \tag{2.8}$$

where the subscripts belong to the set M_γ of all ordinals less than the ordinal γ of the dual of $\mathfrak{I}(S)$, and $d_\alpha \in D(S)$.

2.9. PROPOSITION. *For each ordinal α in M_γ, let us define $T_\alpha = d_\alpha S \backslash d_{\alpha+1} S$. Then T_α is a subsemigroup of S for which $a \in T_\alpha$ implies that $a' \in T_\alpha$, where a' is any inverse of a. Moreover $\{T_\alpha \mid \alpha \in M_\gamma\}$ is the set of all $\mathcal{J}$-classes of S.*

Proof. Applying 2.7 (iii), one can easily show that T_α is a subsemigroup of S. Let $a \in T_\alpha$. Since $a' = a'aa'$, $a \in d_\alpha S$, and $d_\alpha S$ is an ideal of S, it follows that $a' \in d_\alpha S$. On the other hand, since $a = aa'a$, $a \notin d_{\alpha+1} S$ and $d_{\alpha+1} S$ is an ideal of S, we must have that $a' \notin d_{\alpha+1} S$. Hence $a \in T_\alpha$ implies that $a' \in T_\alpha$.

Let $\alpha \in M_\gamma$. We show that T_α is precisely the $\mathcal{J}$-class of S containing the idempotent d_α. Let $a \in T_\alpha$. Then $SaS \subseteq Sd_\alpha S = d_\alpha S$. Since the ideals of S are linearly ordered and $a \notin d_{\alpha+1} S$, it follows that $d_{\alpha+1} S = Sd_{\alpha+1} S \subset SaS$. Therefore $d_{\alpha+1} S \subset SaS \subseteq d_\alpha S$, and because $d_{\alpha+1} S$ is the maximal ideal of S contained in $d_\alpha S$, this implies that $SaS = d_\alpha S$. Thus $a \mathcal{J} d_\alpha$. On the other hand, if b is an element of S for which $b \mathcal{J} d_\alpha$, then $SbS = d_\alpha S$ which, in turn, implies that $b \in T_\alpha$.

Since each element of S belongs to some T_α, then the above implies that each $\mathcal{J}$-class of S coincides with some T_α. Thus the set, $\{T_\alpha \mid \alpha \in M_\gamma\}$, is the set of all $\mathcal{J}$-classes of S.

2.10. THEOREM. *Let S be a completely right injective semigroup and let I be an ideal of S. There exists an idempotent $d \in S$ such that $I = dS$, and $ds = sd$ for all $s \notin I$.*

Proof. If $I = S$, the statement is trivially true. Thus we assume that I is a proper ideal of S. Let ρ be the right congruence on S defined in 2.7 (iv). We consider the right S-system S/ρ consisting of all the ρ-classes of S, where the system product is given by $(x\rho)s = (xs)\rho$, $x\rho \in S/\rho$ and $s \in S$. Let $N = \{x\rho \mid x \in I\}$. Since I is an ideal, N is an S-subsystem of S/ρ. Also we note that $x\rho \subseteq I$ if $x \in I$.

Since S is completely right injective, the identity mapping $1_N : N \to N$ can be extended to an S-homomorphism $\pi : S/\rho \to N$. By 2.4, if an equivalence class $x\rho$ is in N, then it contains one and only one idempotent; namely, the idempotent $x'x$. Consequently, we can write $\pi(1\rho) = d\rho$, where d is an idempotent in I. If $I = eS$, where $e \in E(S)$, then $dS \subseteq eS$. However,

$$e\rho = 1_N(e\rho) = \pi(e\rho) = \pi(1\rho)e = (d\rho)e = (de)\rho.$$

Thus $e = de$, and it follows that $dS = eS = I$.

Let $s \notin I$. Then $\pi(1\rho) = \pi(s\rho) = \pi(1\rho)s = (ds)\rho$. By 2.2, we have $(ds)\rho = (s'ds)\rho$. Therefore $d\rho = (s'ds)\rho$ which, in turn, implies $d = s'ds$. Since $s \notin I$, then $ss' \notin I$, and we have $sd = s(s'ds) = ds$.

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2.11. PROPOSITION. *Let S be a completely right injective semigroup and let I be an ideal of S. Then K is a left [right, two-sided] ideal of I if and only if K is a left [right, two-sided] ideal of S contained in I.*

Proof. Assume K is a left ideal of I. Let $s \in S$, $s \notin K$ and $k \in K$. If $s \in I$, then $sk \in K$, for K is a left ideal of I. If $s \notin I$, then $sk = s(dk) = (sd)k = (ds)k \in K$, where d is the idempotent, defined in 2.10, which generates I.

Suppose K is a right ideal of I. Let $s \in S$, $s \notin K$ and $k \in K$. Now $k \in K$ implies $k'k \in I$ which, in turn, gives $dk'k = k'k$. Hence $ks = k(dk'ks) \in KI$. Since $KI \subseteq K$, we have $ks \in K$.

2.12. PROPOSITION. *If S is completely right injective, then the semigroups $T_\alpha(\alpha < \gamma)$ of 2.9 are simple.*

Proof. Let $K \neq \emptyset$ be a (two-sided) ideal of T_α. Then $K \cup d_{\alpha+1}S$ is an ideal of $d_\alpha S$. By 2.11, $K \cup d_{\alpha+1}S$ is an ideal of S and $d_{\alpha+1}S \subset K \cup d_{\alpha+1}S \subseteq d_\alpha S$. It follows that $K \cup d_{\alpha+1}S = d_\alpha S$ which, in turn, implies $K = T_\alpha$.

3. Completely right injective semigroups that are unions of groups. We begin with a theorem which does not require the injective property.

3.1. THEOREM. *Let S be a semigroup with 0 and 1 such that every right ideal is generated by an idempotent. Then the following are equivalent.*

(i) *S is the union of groups.*
(ii) *Every $\mathscr{L}$-class of S is a group.*
(iii) *Every right ideal of S is two-sided.*

Proof. (i) *implies* (ii). Since S is a union of groups, each $\mathscr{H}$-class of S is a group [**1**, Theorem 4.3]. We will have (ii) provided we show that $\mathscr{H} = \mathscr{L}$. Suppose $a\mathscr{L}b$. Then $a, b \in L_e$, where, according to 2.4, e is the unique idempotent belonging to L_e. Since $H_a \subseteq L_e$, $H_b \subseteq L_e$, and since both H_a and H_b contain idempotents, we have $e \in H_a \cap H_b$. Hence $H_a = H_b$ so that $a\mathscr{H}b$. This proves (ii). Since S is a union of its $\mathscr{L}$-classes, (ii) implies (i).

(ii) *implies* (iii). Let eS, where $e \in E(S)$, be a right ideal of S. Let $a \in eS$ and $s \in S$. We want to show $sa \in eS$. Since eS is a subsemigroup, we may assume that $s \notin eS$. This implies that $aS \subset eS \subset sS$. Since S is a union of its $\mathscr{L}$-classes, $s \in L_f$ for some $f \in E(S)$. Because L_f is a group with identity f, there exists $t \in L_f$ such that $ts = f$. From $aS \subset sS = fS$ we conclude that $a = fa$. Therefore $a = fa = (ts)a = t(sa) \in Ssa$. This implies that $Sa = Ssa$ and hence $sa \in L_a$. Since L_a is a group, there exists $u \in L_a$ such that $sa = au$. Thus $sa \in eS$.

(iii) *implies* (ii). Let L_e be an $\mathscr{L}$-class of S, where e is the unique idempotent of S contained in L_e. We show $L_e = H_e$ which, together with Theorem 2.16 of [**1**], implies that L_e is a group. By 2.6, $a\mathscr{L}e$ implies that $a'\mathscr{R}e$, where a' is any inverse of a. However, $a'S$ is a two-sided ideal of S, so that $a = aa'a \in a'S = eS$. Hence $aS \subseteq eS$. On the other hand, from $a'a \in L_e$ and 2.4 we can conclude that $a'a = e$. Since aS is a two-sided ideal of S, $e = a'a \in aS$ so that $eS \subseteq aS$. Therefore $aS = eS$ and $a \in R_e$. Hence $L_e \subseteq R_e$ from which we conclude that $L_e = H_e$.

3.2. Main Theorem. *Let S be a semigroup with 0 and 1. Then S is completely right injective and a union of groups if and only if every right ideal I is generated by an idempotent d such that $ds = sd$ for all $s \notin I$.*

Proof. The necessity follows from 3.1 and 2.10.

Assume that the right ideals of S satisfy the condition in the statement of the theorem. We first prove that every right ideal of S is two-sided. It then follows, by 3.1, that S is a union of groups. Let I be a right ideal of S. It suffices to show that $sa \in I$ for all $a \in I$ and $s \in S \backslash I$. Since $s \notin I$, our assumption implies that $sa = s(da) = (sd)a = (ds)a \in I$.

To show that S is completely right injective we use the technique employed in the proof of 2.6 of [2]. Let M, P, and R be S-systems, where $P \subseteq R$, and let $f: P \to M$ be an S-homomorphism of P into M. As in [2, 2.6], we can use Zorn's Lemma to obtain a maximal pair (P_0, f_0) consisting of a subsystem P_0 of R, where $P_0 \supseteq P$, and an S-homomorphism $f_0: P_0 \to M$, where f_0 extends f. To show that M is injective it suffices to show $P_0 = R$.

Suppose that $P_0 \subset R$ and let $r \in R$ be such that $r \notin P_0$. Set $A = \{a \in S \mid ra \in P_0\}$. In the two cases, A non-empty or A empty, we will be able to define an S-homomorphism h of rS into M which agrees with f_0 on $P_0 \cap rS$.

If A is empty, define $h: rS \to M$ by $h(x) = m0$ for all $x \in rS$, where m is an arbitrary but fixed element of M. Then $P_0 \cap rS$ is empty and $h(x)s = (m0)s = m0 = h(xs)$ for all $x \in rS$ and $s \in S$. Thus h is an S-homomorphism of rS into M.

Suppose that A is non-empty. Then A is a right ideal of S and hence by hypothesis, $A = dS$, where d is an idempotent of S such that $sd = ds$ for all $s \notin A$. Define h by $h(rs) = f_0(rds)$ for all $s \in S$. From the definition of the set A we conclude that $h(rs) \in M$ for all $s \in S$. First of all, we have that $rs_1 = rs_2$, where $s_1, s_2 \in S$, implies that $rds_1 = rds_2$. Indeed, the definition of the set A yields that both s_1 and s_2 either are or are not members of A. In either situation we conclude that $rds_1 = rds_2$; the latter uses the fact that s_1 and s_2 commute with d. This together with the single-valued property of f_0 implies that

$$h(rs_1) = f_0(rds_1) = f_0(rds_2) = h(rs_2).$$

Hence $h: rS \to M$ is a map of rS into M. Since f_0 is an S-homomorphism, then h is an S-homomorphism. Also if $x \in P_0 \cap rS$, then $x = ra \in P_0$, where $a \in A$. Since $da = a$, then

$$h(x) = h(ra) = f_0(rda) = f_0(ra) = f_0(x).$$

Thus h is an S-homomorphism of rS into M which agrees with f_0 on $P_0 \cap rS$.

Set $P^* = P_0 \cup rS$ and let $f^*: P^* \to M$ be the map defined by $f^*(x) = f_0(x)$, if $x \in P_0$, and $f^*(x) = h(x)$, if $x \in rS$, where $h(x)$ is the map defined above, according to the appropriate case where A is empty or non-empty. It follows that f^* is an S-homomorphism of P into M which extends f_0. Hence $(P^*, f^*) > (P_0, f_0)$, which contradicts the maximality of the pair (P_0, f_0). Thus $P_0 = R$ and M is injective.

Let S be a completely right injective semigroup which is a union of groups. By applying

(2.8), the chain of all right (and hence two-sided) ideals of S can be exhibited in the following manner.

$$S = d_0 S \supset d_1 S \supset d_2 S \supset \ldots \supset d_\alpha S \supset \ldots, \tag{3.3}$$

where $\alpha \in M_\gamma$ and, by 3.1 (iii), d_α is an idempotent of S which commutes with all elements of S not in $d_\alpha S$.

3.4. THEOREM. *Let S be a completely right injective semigroup which is a union of groups. Then $T_\alpha = d_\alpha S \backslash d_{\alpha+1} S (\alpha < \gamma)$, is a right group. In addition, S is a chain M_γ of right groups $T_\alpha (\alpha \in M_\gamma)$.*

Proof. Let $a \in T_\alpha$. Since $d_{\alpha+1}S$ is the maximal right ideal of S contained in $d_\alpha S$, we must have $d_\alpha S = aS$. Hence there exists an inverse a' of a such that $aa' = d_\alpha$. Since $d_\alpha S$ and $d_{\alpha+1}S$ are two-sided ideals and since $a = aa'a$ and $a' = a'aa'$, it follows that $a' \in T_\alpha$. If $b \in T_\alpha$, then $b = d_\alpha b = aa'b$. By 2.9, T_α is a subsemigroup of S. Thus $a'b \in T_\alpha$ so that $b \in aT_\alpha$. This proves that $T_\alpha = aT_\alpha$ for all $a \in T_\alpha$. Therefore T_α is right simple and contains an idempotent. Applying Theorem 1.27 (ii) of [**1**, p. 38], we have that T_α is a right group.

Clearly S is the disjoint union of right groups $T_\alpha (\alpha \in M_\gamma)$. Following the terminology of [**1**, p. 25], we will have that S is a chain M_γ of right groups $T_\alpha (\alpha \in M_\gamma)$ if we can show that $T_\alpha T_\beta \subseteq T_\beta$ and $T_\beta T_\alpha \subseteq T_\beta$ for all $\alpha, \beta \in M_\gamma$, where $\alpha < \beta$. Let $\alpha, \beta \in M_\gamma$, where $\alpha < \beta$, $a \in T_\alpha$ and $b \in T_\beta$. We have that $d_{\beta+1}S \subset d_\beta S \subseteq d_{\alpha+1}S \subset d_\alpha S$. Since $d_\beta S$ is two-sided and $b \in d_\beta S$, it follows that ab and ba are elements in $d_\beta S$. By 2.9, we have that a, a', $a'a$ and aa' all belong to T_α. Consequently, $aS = a'S = aa'S = a'aS = d_\alpha S$. Likewise, $bS = b'S = bb'S = b'bS = d_\beta S$. Because $bS \subset a'aS$, it follows that $b = a'ab$. In addition, since $b'bS \subset aa'S$, we have that $b'b = aa'b'b$ which, in turn, implies that $b = baa'b'b$. The expression $b = a'ab = baa'b'b$ together with the fact that $d_{\beta+1}S$ is two-sided implies that neither ab nor ba belongs to $d_{\beta+1}S$; for otherwise, in both cases, we will have that $b \in d_{\beta+1}S$, which is not true. Thus ab and ba belong to T_β.

Using known properties of right groups, we can apply 3.4 to give additional properties of a semigroup S which is completely right injective and a union of groups. Because of Theorem 1.27 (iii) of [**1**, p. 38], each of the right groups $T_\alpha (\alpha < \gamma)$ is the direct product of a group G_α and a right zero semigroup E_α. In addition, Problem 3 of [**1**, p. 39] implies that T_α is the union of isomorphic disjoint groups; namely $T_\alpha = \bigcup L_g$, where the union ranges over all idempotents g in T_α. This reminds one of the decomposition of semi-simple rings.

3.5. THEOREM. *If S is completely right injective and has a finite number of right ideals, then S is a union of groups.*

Proof. Let $a \in S$ and let a' be an inverse of a. The mapping $h: a'aS \to aS (= aa'S)$ defined by $h(a'as) = as$, for all $s \in S$, is an S-isomorphism of the S-subsystem $a'aS$ onto the S-subsystem $aa'S$. This S-isomorphism requires that the number of right ideals in the chain of all right ideals of S contained in $a'aS$ equals the number in the chain of all right ideals of S contained in $aa'S$. Hence we cannot have either $aa'S \subset a'aS$ or $aa'S \supset a'aS$. That is, $aa'S = a'aS$ and from 2.5 we conclude that $aa'' = a''a$ for some inverse a'' of a. Since $a\mathcal{R}aa''$

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and $a''a\mathscr{L}a$, this implies that $a\mathscr{H}aa''$. Hence H_a contains an idempotent and, by Theorem 2.16 of [**1**, p. 59], H_a is a group. Since S is the union of its $\mathscr{H}$-classes we have our result.

In view of 3.5 and the obvious fact that an idempotent semigroup is a union of groups we can apply the main theorem to prove the following result.

3.6. THEOREM. *A semigroup with* 0 *and* 1 *which is either idempotent or contains a finite number of right ideals is completely right injective if and only if each right ideal I of S contains an idempotent generator which commutes with all elements not in I.*

An example of an idempotent semigroup which is completely right injective can be constructed as follows.

Let E and F be two disjoint right zero semigroups. Define $ef = fe = e$ for all $e \in E$ and $f \in F$. This product together with the product already defined in E and F make $E \cup F$ into a semigroup. If we adjoin 0 and 1 to $E \cup F$, then the resultant semigroup is completely right injective. Also $T = E \cup F \cup 0 \cup 1$ can be made into a completely right injective semigroup by defining $fe = e$ and $ef = e^*$ for all $e \in E$, $f \in F$, where e^* is a fixed element of E. For both semigroups we can show that every right ideal has the property stated in 3.6. All the right ideals in the latter semigroup T are listed according to the chain $T \supset fT \supset e^*T \supset 0$, where $f \in F$, $e^*T = E \cup 0$ and $fT = E \cup F \cup 0$. The idempotent generator of e^*T which commutes with all elements not in this ideal is the idempotent e^*.

We now give an example of an idempotent semigroup S in which every right ideal is generated by an idempotent, but such that S is not completely right injective. Let $S = \{0, 1, e_1, e_2, f_1, f_2\}$ where 0 and 1 are the zero and identity elements of S, respectively. Define

$$e_i e_j = e_j, \quad f_i f_j = f_j, \quad f_i e_j = e_j \qquad (i, j = 1, 2),$$

$$e_1 f_1 = e_1, \quad e_1 f_2 = e_2, \quad e_2 f_1 = e_1, \quad e_2 f_2 = e_2.$$

Every right ideal of S is generated by an idempotent; in fact, all the right ideals of S can be exhibited in the chain $S \supset f_i S \supset e_i S \supset 0$. The right ideal $e_i S$ contains no idempotent which commutes with every f_j. By 3.6, it follows that S is not completely right injective.

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