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EXHAUSTION OF THE CURVE GRAPH VIA RIGID EXPANSIONS

JESÚS HERNÁNDEZ HERNÁNDEZ
Centro de Ciencias Matemáticas, UNAM, Campus Morelia,
Morelia, Mich. 58190, Mexico
e-mail: jhdezhdez@gmail.com, https://sites.google.com/site/jhdezhdez/

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Abstract. For an orientable surface S of finite topological type with genus $g \geq 3$, we construct a finite set of curves whose union of iterated rigid expansions is the curve graph $\mathcal{C}(S)$. The set constructed, and the method of rigid expansion, are closely related to Aramayona and Leiniger's finite rigid set in Aramayona and Leininger, *J. Topology Anal.* **5**(2) (2013), 183–203 and Aramayona and Leininger, *Pac. J. Math.* **282**(2) (2016), 257–283, and in fact a consequence of our proof is that Aramayona and Leininger's set also exhausts the curve graph via rigid expansions.

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1. Introduction. In this paper, we consider an orientable surface $S_{g,n}$ of finite topological type with genus $g \geq 3$ and $n \geq 0$ punctures. The mapping class group of $S_{g,n}$, denoted by $\text{Mod}(S_{g,n})$ is the group of orientation preserving self-homeomorphisms of $S_{g,n}$. The extended mapping class group of $S_{g,n}$, denoted by $\text{Mod}^*(S_{g,n})$ is the group of isotopy classes of self-homeomorphisms of $S_{g,n}$.

In order to study these groups, Harvey in 1979 (see [**7**]) introduced the curve complex of a surface as the simplicial complex whose vertices are isotopy classes of essential curves, and simplices are defined by disjointness (see Section 2 for details). We call the 1-skeleton of the curve complex the *curve graph*, which we denote by $\mathcal{C}(S_{g,n})$.

There is a natural link between the curve complex and $\text{Mod}(S_{g,n})$ and $\text{Mod}^*(S_{g,n})$. Ivanov (in [**14**]) linked the curve complex to $\text{Mod}^*(S_{g,n})$ via simplicial automorphisms, while Harer (in [**6**]) linked the curve complex with $\text{Mod}(S_{g,n})$ by their (co-)homology.

On one hand, in [**14**, **15**] and [**17**], it was proved that for most surfaces every automorphism of the curve graph is induced by a homeomorphism of $S_{g,n}$, with the well-known exception of $S_{1,2}$. Later on, there were generalisations of this result for larger classes of simplicial maps (see [**3**, **11–13**]), until Shackleton (see [**18**]) proved that any locally injective self-map of the curve graph is induced by a homeomorphism (for surfaces of high-enough complexity).

Thereafter, Aramayona and Leininger introduced in [**1**] the concept of a rigid set of the curve graph, which is a full subgraph Y such that any locally injective map from Y to $\mathcal{C}(S_{g,n})$ is the restriction to Y of an automorphism, unique up to the pointwise stabiliser of Y in $\text{Aut}(\mathcal{C}(S_{g,n}))$. By Shackleton's result, the curve graph itself is a rigid set. In [**1**], they also construct a finite rigid set for any orientable surface of finite topological type. See Section 5 below.

On the other hand, it is a well-known result by Harer [**6**] that the curve complex is homotopically equivalent to a bouquet of spheres, which is used to determine the virtual cohomological dimension of the mapping class group.

Later on, Birman, Broaddus and Menasco in [**4**] proved that Aramayona and Leininger's finite rigid set either is (for $g = 0$ and $n \geq 5$) or contains (for $g \geq 1$ and $n \leq 1$) a $\mathrm{Mod}(S_{g,n})$-module generator of the reduced homology of the curve complex. Thus, they link the (co-)homological and simplicial sides of the study of the mapping class group and curve complexes.

Afterwards, Aramayona and Leininger proved in [**2**] that for almost all surfaces of finite topological type, there exists an increasing sequence of finite rigid sets that exhaust the curve graph, each of which has trivial pointwise stabiliser in $\mathrm{Mod}^*(S_{g,n})$. Note that this is **not trivial**, given that there exist examples of supersets of a rigid set that are not rigid themselves.

While their proof is *effective* for the result, it does not lend itself to improving other results concerning simplicial maps. In this work, we prove a similar result to theirs via a different method: we use a method developed in [**2**] for expanding subgraphs that produces different supergraphs than those obtained in the main result of [**2**]. As a quick consequence, we can also recover their result (see Theorem B below). This method can be used to obtain new results concerning edge-preserving maps; the details of these results are given in [**8**] and will appear in a second paper [**9**]. We call this method *rigid expansion*.

We define the first rigid expansion of a subgraph Y, denoted as Y^1, as the union of Y with all the curves uniquely determined by subsets of Y, where a curve β is uniquely determined by a subset B of $\mathcal{C}(S_{g,n})$ if it is the unique curve disjoint from every element in B. We also define $Y^0 = Y$ and, inductively, $Y^k = (Y^{k-1})^1$. If the sequence $Y \subset Y^1 \subset \ldots \subset Y^i \subset \ldots \mathcal{C}(S_{g,n})$ exhausts $\mathcal{C}(S_{g,n})$, we say that Y is a seed subgraph of $\mathcal{C}(S_{g,n})$.

Note in particular that if β is uniquely determined by B, then for every $h \in \mathrm{Mod}^*(S_{g,n})$, we have that $h(\beta)$ is uniquely determined by $h(B)$.

Now, we can state the main result of this work.

THEOREM A. *Let $S_{g,n}$ be an orientable surface of finite topological type with genus $g \geq 3$, $n \geq 0$ punctures, and empty boundary. There exists a finite seed subgraph of $\mathcal{C}(S_{g,n})$, i.e., there exists a finite subgraph of $\mathcal{C}(S_{g,n})$ whose union of iterated rigid expansions is equal to $\mathcal{C}(S_{g,n})$.*

The proof of Theorem A is divided into two cases: the closed surface case (see Theorem 3.1 in Section 3) and the punctured surface case (see Theorem 4.3 in Section 4). We begin by defining a particular set of curves (based on the rigid set introduced in [**1**]) that we call the *principal set*, and use a Humphries–Lickorish generating set of $\mathrm{Mod}(S_{g,n})$ to see that the positive and negative translations of the principal set are contained in some rigid expansion of it (the principal set); afterwards, the iterated use of this result allows us to see that most topological types of curves are in some rigid expansion, while the rest of the topological types are uniquely determined by finite sets of curves from the previous cases.

Afterwards, in Section 5, we reintroduce the rigid set of [**1**], denoted by $\mathfrak{X}(S_{g,n})$. Note that while Birman, Broaddus and Menasco's homological spheres in [**4**] (which is a subset of $\mathfrak{X}(S_{g,n})$ for $g \geq 1$ and $n \leq 1$) are not contained in the principal set of a closed surface, they *are* contained in their first rigid expansion. Then, noting that a supergraph of a seed subgraph is a seed subgraph itself, we use Theorem A to obtain an analogous result for Aramayona and Leininger's finite rigid set.

THEOREM B. *Let $S_{g,n}$ be an orientable surface of genus $g \geq 3$, $n \geq 0$ punctures and empty boundary. Then, $\mathfrak{X}(S_{g,n})$ is a rigid seed subgraph, i.e., $\bigcup_{i\in\mathbb{N}} \mathfrak{X}(S_{g,n})^i = \mathcal{C}(S_{g,n})$.*

There are many other questions that are derived from the results of this work. For example, since there exist both finite seed subgraphs and finite rigid seed subgraphs of $\mathcal{C}(S_{g,n})$, what is the size of the smallest seed subgraph of $\mathcal{C}(S_{g,n})$? What is the size of the smallest rigid seed subgraph of $\mathcal{C}(S_{g,n})$? The first one is obviously smaller, but how much smaller is it? Are there (rigid) seed graphs of other simplicial graphs related to $S_{g,n}$?

We must remark that this work is the published version of the first two chapters of the author's Ph.D. thesis, and as was mentioned before these results are used to obtain new results on simplicial maps of different graphs. In particular, we use these results in **[9]** to prove that under certain conditions on the surfaces, all edge-preserving maps between a priori different curve graphs are actually induced by homeomorphisms between the underlying surfaces.

2. Preliminaries. We suppose $S_{g,n}$ is an orientable surface of finite topological type with empty boundary, genus $g \geq 3$ and n punctures. The *mapping class group of* $S_{g,n}$, denoted by $\mathrm{Mod}(S_{g,n})$, is the group of isotopy classes of orientation preserving self-homeomorphisms of $S_{g,n}$; the *extended mapping class of* $S_{g,n}$, denoted by $\mathrm{Mod}^*(S_{g,n})$, is the group of isotopy classes of *all* self-homeomorphisms of $S_{g,n}$. Note that $\mathrm{Mod}(S_{g,n})$ is an index 2 subgroup of $\mathrm{Mod}^*(S_{g,n})$.

A *curve* α is a topological embedding of the unit circle into the surface. We often abuse notation and call "curve" the embedding, its image on $S_{g,n}$ or its isotopy class. The context makes clear which use we mean.

A curve is *essential* if it is neither null-homotopic nor homotopic to the boundary curve of a neighbourhood of a puncture.

Also, recall that an essential curve α on S is *separating* if $S\backslash\{\alpha\}$ is disconnected, and it is called *nonseparating* otherwise.

The (geometric) *intersection number* of two (isotopy classes of) curves α and β is defined as follows:

$$i(\alpha, \beta) := \min\{|a \cap b| : a \in \alpha, b \in \beta\}.$$

Let α and β be two curves on $S_{g,n}$. As a convention for this work, we say α and β are *disjoint* if $i(\alpha, \beta) = 0$ **and** $\alpha \neq \beta$.

Under the conditions on $S_{g,n}$ imposed above, we define the *curve graph of* $S_{g,n}$, denoted by $\mathcal{C}(S_{g,n})$, as the simplicial graph whose vertices are the isotopy classes of essential curves on $S_{g,n}$, and two vertices span an edge if the corresponding curves are disjoint.

Let β be an essential curve on $S_{g,n}$ and B a set of curves on $S_{g,n}$. We say β is *uniquely determined by* B, denoted $\beta = \langle B\rangle$, if β is the unique essential curve on $S_{g,n}$ that is disjoint from every element in B, i.e.,

$$\{\beta\} = \bigcap_{\gamma\in B} \mathrm{lk}(\gamma),$$

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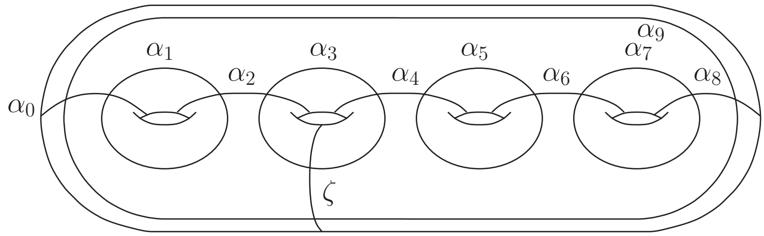

Figure 1. The set $\mathscr{C} = \{\alpha_0, \ldots, \alpha_{2g+1}\}$ and the curve ζ – one of the curves of the bounding pair associated to $\{\alpha_0, \alpha_1, \alpha_2\}$.

where $\mathrm{lk}(\gamma)$ denotes the link of γ in $\mathcal{C}(S_{g,n})$.

Let $Y \subset \mathcal{C}(S_{g,n})$; the first *rigid expansion* of Y is defined as

$$Y^1 := Y \cup \{\beta : \beta = \langle B \rangle, B \subset Y\};$$

we also define $Y^0 = Y$ and, inductively, $Y^k = (Y^{k-1})^1$.

3. Closed surface case. In this section, we suppose that S is a closed surface of genus $g \geq 3$. This section is divided as follows: Section 3.1 gives some definitions, fixes the principal set, states the main result of the section, and gives the proof of said result pending the proof of a technical lemma; Sections 3.2–3.4 give the proofs of the claims for the technical lemma.

3.1. Statement and proof of Theorem 3.1. Let $k \in \mathbb{Z}^+$ and $C = \{\gamma_0, \ldots, \gamma_k\}$ be an ordered set of $k + 1$ curves in S. We call C a *chain* of length $k + 1$ if $i(\gamma_i, \gamma_{i+1}) = 1$ for $0 \leq i \leq k - 1$, and γ_i is disjoint from γ_j for $|i - j| > 1$. On the other hand, C is called a *closed chain* of length $k + 1$ if $i(\gamma_i, \gamma_{i+1}) = 1$ for $0 \leq i \leq k$ modulo $k + 1$, and γ_i is disjoint from γ_j for $|i - j| > 1$ (modulo $k + 1$); a closed chain is maximal if it has length $2g + 2$. A *subchain* is an ordered subset of either a chain or a closed chain which is itself a chain, and its length is its cardinality.

Recalling that $k \geq 1$, note that if C is a chain (or a subchain), then every element of C is a nonseparating curve. Also, if C has odd length, a closed regular neighbourhood $N(C)$ has two boundary components; we call these curves the bounding pair associated to C.

Let $\mathscr{C} = \{\alpha_0, \ldots, \alpha_{2g+1}\}$ be the closed chain in S depicted in Figure 1. Observe it is a maximal closed chain, and given any other maximal closed chain C there exists an element of $\mathrm{Mod}(S)$ that maps C to $\mathscr{C}$ (see [**5**]).

We define the set $\mathscr{B}$ as the union of the bounding pairs associated to the subchains of odd length of $\mathscr{C}$.

Now, we are able to state the main result for the closed surface case.

THEOREM 3.1. *Let S be an orientable closed surface with genus $g \geq 3$, and let $\mathscr{C}$ and $\mathscr{B}$ be defined as above. Then, $\bigcup_{i \in \mathbb{N}} (\mathscr{C} \cup \mathscr{B})^i = \mathcal{C}(S)$.*

The idea of the proof is as follows. Let ζ be the curve depicted in Figure 1; we define the set $\mathscr{G} = \{\alpha_0, \ldots, \alpha_{2g-1}, \zeta\}$. Note that Humphries and Lickorish proved that the Dehn twists along the elements of $\mathscr{G}$ generate $\mathrm{Mod}(S)$ (see [**10**]).

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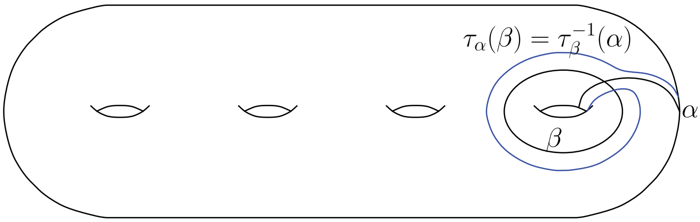

Figure 2. (Colour online) The curves α and β in black, with the curve $\tau_\alpha(\beta)$ in blue.

First, we prove that the image of $\mathscr{C} \cup \mathscr{B}$ under the Dehn twist along any element of $\mathscr{G}$ is contained in $(\mathscr{C} \cup \mathscr{B})^4$. Afterwards, we note that any nonseparating curve in $\mathcal{C}(S)$ is the image of an element in $\mathscr{G}$ under an orientation preserving mapping class, and thus is contained in $(\mathscr{C} \cup \mathscr{B})^k$ for some k. Finally, we show that every separating curve in $\mathcal{C}(S)$ is uniquely determined by some finite subset of nonseparating curves, and thus also lies in $(\mathscr{C} \cup \mathscr{B})^k$ for some k.

Before passing to the proof of Theorem 3.1, we give the necessary notation and state a technical lemma.

Let $\alpha, \beta \in \mathcal{C}(S)$ and $A, B \subset \mathcal{C}(S)$. We denote by $\tau_\alpha(\beta)$ the right Dehn twist of β along α, $\tau_\alpha(B) = \bigcup_{\gamma \in B}\{\tau_\alpha(\gamma)\}$, and $\tau_A(B) = \bigcup_{\gamma \in A} \tau_\gamma(B)$. Observe that if α and β are such that $i(\alpha, \beta) = 1$, we have

$$\tau_\alpha(\beta) = \tau_\beta^{-1}(\alpha) \qquad \tau_\alpha^{-1}(\beta) = \tau_\beta(\alpha); \tag{1}$$

See Proposition 3.9 in [**2**] or Figure 2 for a proof.

The key technical lemma for the proof of Theorem 3.1 is the following.

Lemma 3.2. $\tau_{\mathscr{G}}^{\pm 1}(\mathscr{C} \cup \mathscr{B}) \subset (\mathscr{C} \cup \mathscr{B})^4$.

Note that, as was mentioned in the Introduction, if $\beta = \langle B \rangle$, we have that for any $h \in \mathrm{Mod}^*(S)$, $h(\beta) = \langle h(B) \rangle$. This allows the iterated use of the lemma.

Assuming this lemma (which we prove in the following subsections), we embark on the proof of Theorem 3.1.

Proof of Theorem 3.1. Let γ be a nonseparating curve and $\alpha \in \mathscr{G}$. There exists an orientation preserving mapping class $h \in \mathrm{Mod}(S)$ such that $\gamma = h(\alpha)$. As was mentioned above, the Dehn twists along the elements of $\mathscr{G}$ generate $\mathrm{Mod}(S)$. Thus, for some $\gamma_1, \ldots, \gamma_m \in \mathscr{G}$ (not necessarily different), we have that $\gamma = \tau_{\gamma_1} \circ \cdots \circ \tau_{\gamma_m}(\alpha)$. By an inductive use of Lemma 3.2, we have that $\gamma \in (\mathscr{C} \cup \mathscr{B})^{4m}$. Hence, every nonseparating curve is an element of $\bigcup_{i \in \mathbb{N}}(\mathscr{C} \cup \mathscr{B})^i$.

Let γ be a separating curve. Note that, up to homeomorphism, there exist only a finite number of separating curves. Moreover, as can be seen in Figure 3, every such curve can be uniquely determined by a pair of chains of cardinalities $2g'$ and $2g''$, where g' and g'' are the genera of the connected components of $S \backslash \{\gamma\}$. Then, there exist chains C_1 and C_2 such that $\gamma = \langle C_1 \cup C_2 \rangle$. By the previous case, $C_1 \cup C_2 \subset (\mathscr{C} \cup \mathscr{B})^k$ for some $k \in \mathbb{N}$; thus, $\gamma \in (\mathscr{C} \cup \mathscr{B})^{k+1}$. Therefore, $\mathcal{C}(S) = \bigcup_{i \in \mathbb{N}}(\mathscr{C} \cup \mathscr{B})^i$. □

As stated before, the rest of this section is dedicated to the proof of Lemma 3.2, which (using that $\zeta \in \mathscr{B}$) is divided as follows:

Claim 1: $\tau_{\mathscr{C}}^{\pm 1}(\mathscr{C}) \subset (\mathscr{C} \cup \mathscr{B})^2$.

Claim 2: $\tau_{\mathscr{C}}^{\pm 1}(\mathscr{B}) \cup \tau_{\mathscr{B}}^{\pm 1}(\mathscr{C}) \subset (\mathscr{C} \cup \mathscr{B})^3$.

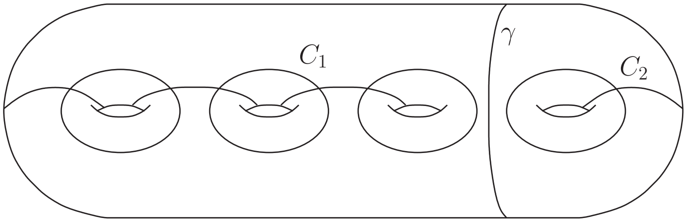

Figure 3. A separating curve γ and chains C_1 and C_2 that uniquely determine it.

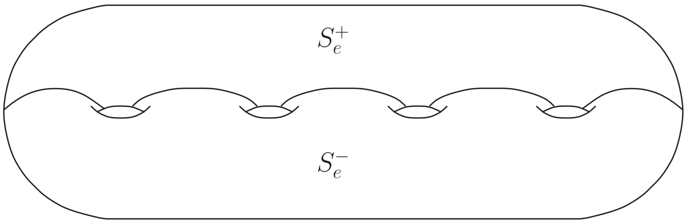

Figure 4. The set $\mathscr{C}'_e$ and the corresponding S_e^+ and S_e^-.

Claim 3: $\tau_\zeta^{\pm 1}(\mathscr{B}) \subset (\mathscr{C} \cup \mathscr{B})^4$.

Before going further, we introduce the notation used in the proofs of said claims.

Let $\mathscr{C}' = \{\gamma_0, \ldots, \gamma_{2g+1}\}$ be a maximal closed chain in S. The sets $\mathscr{C}'_o = \{\gamma_i \in \mathscr{C}' : i \text{ is odd}\}$ and $\mathscr{C}'_e = \{\gamma_i \in \mathscr{C}' : i \text{ is even}\}$, satisfy that $S \backslash \mathscr{C}'_e$ and $S \backslash \mathscr{C}'_o$ have two connected components, each homeomorphic to $S_{0,g+1}$. We denote by S_e^+ and S_e^- the connected components of $S \backslash \mathscr{C}'_e$, and by S_o^+ and S_o^- the connected components of $S \backslash \mathscr{C}'_o$. See Figure 4 for an example. Let $1 \leq k \leq g-1$, and $\{\gamma_i, \ldots, \gamma_{i+2k}\}$ (with the indices modulo $2g+2$) be a subchain of $\mathscr{C}'$. We denote by $[\gamma_i, \ldots, \gamma_{i+2k}]^+$ the curve in the associated bounding pair that is contained in either S_o^+ or S_e^+. Analogously, we denote by $[\gamma_i, \ldots, \gamma_{i+2k}]^-$ the curve in the associated bounding pair contained in either S_o^- or S_e^-.

Remark 3.3. Note that according to this notation, $\zeta = [\alpha_0, \alpha_1, \alpha_2]^-$.

We partition the set $\mathscr{B}$ into $\mathscr{B}_o^+$, $\mathscr{B}_o^-$, $\mathscr{B}_e^+$ and $\mathscr{B}_e^-$, depending on whether $\beta \in \mathscr{B}$ is contained in S_o^+, S_o^-, S_e^+, or S_e^-, respectively. We write $\mathscr{B}^+ = \mathscr{B}_o^+ \cup \mathscr{B}_e^+$ and $\mathscr{B}^- = \mathscr{B}_o^- \cup \mathscr{B}_e^-$.

3.2. Proof of Claim 1: $\tau_{\mathscr{C}}^{\pm 1}(\mathscr{C}) \subset (\mathscr{C} \cup \mathscr{B})^2$**.** To prove the claim, we start with a pair of particular curves and we show that is enough to prove the claim via the action of a particular subgroup of $\mathrm{Mod}(S)$.

The following lemma is heavily based on Lemma 5.3 in [**2**]. However, its proof has been modified to emphasise the arguments that are used to obtain a more general result, which is repeatedly used in the following subsections.

Lemma 3.4. $\tau_{\alpha_{2g}}^{\pm 1}(\alpha_{2g-1}) \in (\mathscr{C} \cup \mathscr{B})^2$.

Proof. Using the set

$$C_+ = \{\alpha_{2g+1}, \alpha_1, \alpha_2, \ldots, \alpha_{2g-4}, \alpha_{2g-2}, [\alpha_{2g-3}, \alpha_{2g-2}, \alpha_{2g-1}]^+, [\alpha_2, \ldots, \alpha_{2g-2}]^+\},$$

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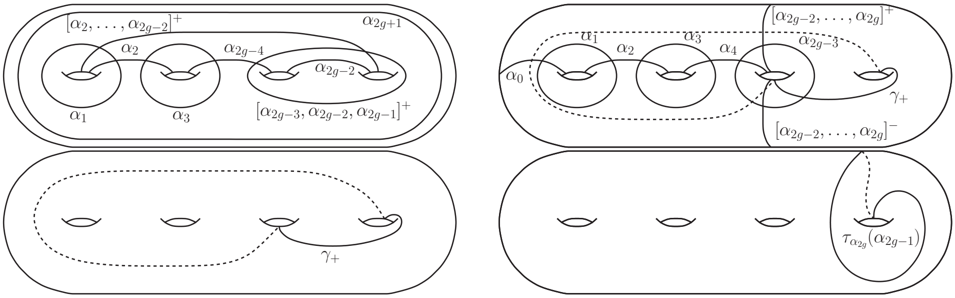

Figure 5. Above, the sets C_+ (left) and C'_+ (right). Below, the curves γ_+ (left) and $\tau_{\alpha_{2g}}(\alpha_{2g-1})$ (right), uniquely determined by the sets C_+ and C'_+, respectively.

we obtain the curve $\gamma_+ \in (\mathscr{C} \cup \mathscr{B})^1$ as the curve uniquely determined by C_+, see Figure 5. Then, letting

$$C'_+ = \{\alpha_0, \ldots, \alpha_{2g-3}, [\alpha_{2g-2}, \alpha_{2g-1}, \alpha_{2g}]^+, [\alpha_{2g-2}, \alpha_{2g-1}, \alpha_{2g}]^-, \gamma_+\},$$

we have that $\tau_{\alpha_{2g}}(\alpha_{2g-1}) = \langle C'_+ \rangle \in (\mathscr{C} \cup \mathscr{B})^2$.

Analogously, we have that $\tau^{-1}_{\alpha_{2g}}(\alpha_{2g-1}) = \langle C'_- \rangle \in (\mathscr{C} \cup \mathscr{B})^2$. See **[8]** for more details. Therefore, $\tau^{\pm 1}_{\alpha_{2g}}(\alpha_{2g-1}) \in (\mathscr{C} \cup \mathscr{B})^2$. □

Note that the proof of Lemma 3.4 does not work as it is for the case of $g = 2$.

Let $H_{\mathscr{C}} < \mathrm{Mod}(S)$ be the setwise stabiliser of $\mathscr{C}$.

REMARK 3.5. Observe that $H_{\mathscr{C}}(\mathscr{B}) = \mathscr{B}$ (thus, $H_{\mathscr{C}}$ is a subgroup of the setwise stabiliser of $\mathscr{B}$ in $\mathrm{Mod}^*(S)$), and for all $h \in H_{\mathscr{C}}$, we have that $h(\mathscr{B}^+), h(\mathscr{B}^-) \in \{\mathscr{B}^+, \mathscr{B}^-\}$. Moreover, $H_{\mathscr{C}}$ can be partitioned as $H_{\mathscr{C}} = H^+_{\mathscr{C}} \sqcup \iota H^+_{\mathscr{C}}$, where $H^+_{\mathscr{C}}$ is the subgroup such that $H^+_{\mathscr{C}}(\mathscr{B}^+) = \mathscr{B}^+$, and ι the hyperelliptic involution (which exchanges S^+_o (resp. S^+_e) and S^-_o (resp. S^-_e)). Also, note that $H^+_{\mathscr{C}}$ *acts transitively* on $\mathscr{C}$.

LEMMA 3.6. *Let $Y \subset \mathcal{C}(S)$ and Γ be the setwise stabiliser of Y. Then, Γ is a subgroup of the setwise stabiliser of Y^k for $k \in \mathbb{N}$. In particular, if $h \in H_{\mathscr{C}}$, $k \in \mathbb{N}$ and $\gamma \in (\mathscr{C} \cup \mathscr{B})^k$, then $h(\gamma) \in (\mathscr{C} \cup \mathscr{B})^k$.*

Proof. We proceed by induction; if $k = 0$, we obtain the result by construction. If $k \geq 1$ let $\gamma \in Y^k \backslash Y^{k-1}$, as such $\gamma = \langle C_0 \rangle$ with $C_0 \subset Y^{k-1}$; then if $h \in \Gamma$, $h(\gamma) = \langle h(C_0) \rangle$; but by induction $h(C_0) \subset Y^{k-1}$, thus $h(\gamma) \in Y^k$.

The last part of the lemma follows from Remark 3.5. □

Armed with Lemma 3.6, we are ready to prove Claim 1.

Proof of Claim 1. Let $\alpha_i, \alpha_j \in \mathscr{C}$ with $i \neq j$. We want to prove that $\tau^{\pm 1}_{\alpha_i}(\alpha_j) \in (\mathscr{C} \cup \mathscr{B})^2$. If $|i - j| > 1$ (modulo $2g + 2$), then the curves are disjoint and we have that $\tau^{\pm 1}_{\alpha_i}(\alpha_j) = \alpha_j \in \mathscr{C}$. Suppose then that $|i - j| = 1$. There exists an element $h \in H^+_{\mathscr{C}}$ such that either $h(\alpha_{2g}) = \alpha_i$ and $h(\alpha_{2g-1}) = \alpha_j$ if $i = j + 1$, or $h(\alpha_{2g}) = \alpha_j$ and $h(\alpha_{2g-1}) = \alpha_i$ if $j = i + 1$. Repeating the procedure of the proof of Lemma 3.4, precomposing by h and using Lemma 3.6, we obtain that $\tau^{\pm 1}_{\alpha_i}(\alpha_j) \in (\mathscr{C} \cup \mathscr{B})^2$. □

This finishes the proof of Claim 1. However, the proofs of Lemma 3.4 and Claim 1 give us a slightly more general result, which is often used in the rest of this section. Its objective is to reduce the problems posed in the following claims when finding

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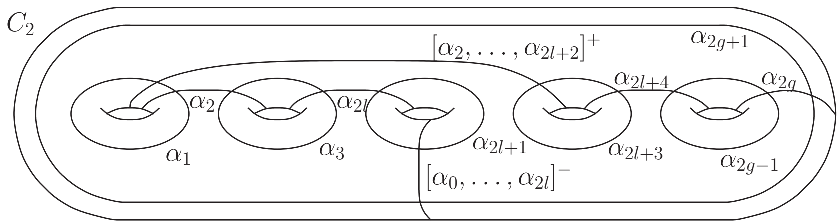

Figure 6. $\mathscr{C}_2$ for the genus 5 surface and the first reordering.

convenient maximal closed chains and showing that particular curves dependent on said chains are uniquely determined by elements in the expansions of $\mathscr{C} \cup \mathscr{B}$.

LEMMA 3.7. *Let* $\{\gamma_0, \ldots, \gamma_{2g+1}\}$ *be a maximal closed chain in* S *that is contained in* $(\mathscr{C} \cup \mathscr{B})^k$ *for some* $k \in \mathbb{N}$. *If* $[\gamma_{2g-3}, \gamma_{2g-2}, \gamma_{2g-1}]^+, [\gamma_2, \ldots, \gamma_{2g-2}]^\pm \in (\mathscr{C} \cup \mathscr{B})^{k+1}$ *and* $[\gamma_{2g-2}, \gamma_{2g-1}, \gamma_{2g}]^\pm \in (\mathscr{C} \cup \mathscr{B})^{k+2}$, *then* $\tau_{\gamma_{2g}}^{\pm 1}(\gamma_{2g-1}) \in (\mathscr{C} \cup \mathscr{B})^{k+3}$. *Moreover,* $\tau_{f(\gamma_{2g})}^{\pm 1}(f(\gamma_{2g-1})) \in (\mathscr{C} \cup \mathscr{B})^{k+3}$ *for any* $f \in H_{\mathscr{C}}^+$.

Proof. Given that *up to the action of* $\mathrm{Mod}(S)$, $\mathscr{C}$ is the only maximal closed chain, there exists $h \in \mathrm{Mod}(S)$ such that $h(\alpha_i) = \gamma_i$ for $i \in \{0, \ldots, 2g+1\}$. Then, we repeat the procedure of the proof of Lemma 3.4 precomposing by h and using Lemma 3.6, getting $\tau_{\gamma_{2g}}^{\pm 1}(\gamma_{2g-1}) = \tau_{h(\alpha_{2g})}^{\pm 1}(h(\alpha_{2g-1})) \in (\mathscr{C} \cup \mathscr{B})^{k+3}$.

Let $f \in H_{\mathscr{C}}^+$. Using Lemma 3.6, we can apply the result above to $fh(\mathscr{C})$ and we get that $\tau_{f(\gamma_{2g})}^{\pm 1}(f(\gamma_{2g-1})) \in (\mathscr{C} \cup \mathscr{B})^{k+3}$. □

3.3. Proof of Claim 2: $\tau_{\mathscr{C}}^{\pm 1}(\mathscr{B}) \cup \tau_{\mathscr{B}}^{\pm 1}(\mathscr{C}) \subset (\mathscr{C} \cup \mathscr{B})^3$. As in Section 3.2, we first note that letting $\alpha \in \mathscr{C}$ and $\beta \in \mathscr{B}$, if α and β are disjoint, there is nothing to prove and so we assume $i(\alpha, \beta) \neq 0$. By construction, we then have that $i(\alpha, \beta) = 1$. In part 1, we first establish the claim for a particular family of maximal closed chains that verify the conditions of Lemma 3.7, proving that $\tau_{[\alpha_0,\ldots,\alpha_{2l}]^-}^{\pm 1}(\alpha_{2l+1}) \in (\mathscr{C} \cup \mathscr{B})^3$ for all $1 \leq l \leq g-2$; then, via the action of $H_{\mathscr{C}}^+$, we prove that $\tau_{\mathscr{B}^-}^{\pm 1}(\mathscr{C}) \cup \tau_{\mathscr{C}}^{\pm 1}(\mathscr{B}^-) \subset (\mathscr{C} \cup \mathscr{B})^3$. In part 2, we finish the proof via the action of the hyperelliptic involution $\iota : S \to S$ mentioned in Remark 3.5.

Part 1: Let $1 \leq l \leq g-2$. We define the following maximal closed chain:

$$\mathscr{C}_l = \{\alpha_1, \alpha_2, \ldots, \alpha_{2l}, \alpha_{2l+1}, [\alpha_0, \ldots, \alpha_{2l}]^-, \alpha_{2g+1}, \alpha_{2g}, \alpha_{2g-1}, \ldots, \alpha_{2l+3}, [\alpha_2, \ldots, \alpha_{2l+2}]^+\}.$$

We refer the reader to Figure 6 for an example of such a maximal closed chain.

We now prove, using Lemma 3.7, that $\tau_{\mathscr{B}^-}^{\pm 1}(\mathscr{C}) \cup \tau_{\mathscr{C}}^{\pm 1}(\mathscr{B}^-) \subset (\mathscr{C} \cup \mathscr{B})^3$.

In order to facilitate the use of Lemma 3.7, we cyclically reorder the elements of $\mathscr{C}_l$ as follows:

$$\gamma_0 = \alpha_{2g},\ \gamma_1 = \alpha_{2g-1},\ \ldots,\ \gamma_{2g-1} = \alpha_{2l+1},\ \gamma_{2g} = [\alpha_0, \ldots, \alpha_{2l}]^-, \text{ and } \gamma_{2g+1} = \alpha_{2g+1}.$$

Again, see Figure 6 for an example.

By inspection, we can verify that $\mathscr{C}_l$ satisfies the conditions of Lemma 3.7, and thus $\tau_{\gamma_{2g}}^{\pm 1}(\gamma_{2g-1}) = \tau_{[\alpha_0,\ldots,\alpha_{2l}]^-}^{\pm 1}(\alpha_{2l+1}) \in (\mathscr{C} \cup \mathscr{B})^3$; however, for the sake of completeness, we give a detailed account of which set of curves uniquely determines the needed curves.

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For $1 \leq l \leq g-2$, , we have

$$\begin{aligned}[][\gamma_{2g-3}, \gamma_{2g-2}, \gamma_{2g-1}]^+ &= [\alpha_{2l-1}, \alpha_{2l}, \alpha_{2l+1}]^+ \\ [\gamma_2, \ldots, \gamma_{2g-2}]^+ &= \langle \alpha_{2g-2}, \ldots, \alpha_{2l+3}, [\alpha_2, \ldots, \alpha_{2l+2}]^+, \alpha_1, \alpha_2, \ldots, \alpha_{2l}, \\ &\quad [\alpha_0, \ldots, \alpha_{2l}]^-, \alpha_{2g+1}, \alpha_{2g}, [\alpha_{2l+2}, \ldots, \alpha_{2g-2}]^- \rangle \\ [\gamma_2, \ldots, \gamma_{2g-2}]^- &= [\alpha_{2l+2}, \ldots, \alpha_{2g-2}]^-.\end{aligned}$$

In the case of $l = 1$, we have

$$\begin{aligned}[][\gamma_{2g-2}, \gamma_{2g-1}, \gamma_{2g}]^+ &= \alpha_0 \\ [\gamma_{2g-2}, \gamma_{2g-1}, \gamma_{2g}]^- &= \langle \alpha_2, \alpha_3, [\alpha_0, \alpha_1, \alpha_2]^-, \alpha_0, [\alpha_2, \alpha_3, \alpha_4]^+, \alpha_5, \alpha_6, \ldots, \alpha_{2g} \rangle.\end{aligned}$$

In the cases with $l > 1$, we have

$$\begin{aligned}[][\gamma_{2g-2}, \gamma_{2g-1}, \gamma_{2g}]^+ &= [\alpha_0, \ldots, \alpha_{2l-2}]^- \\ [\gamma_{2g-2}, \gamma_{2g-1}, \gamma_{2g}]^- &= \langle \alpha_{2l}, \alpha_{2l+1}, [\alpha_0, \ldots, \alpha_{2l}]^-, [\alpha_0, \ldots, \alpha_{2l-2}]^-, \alpha_{2l-2}, \alpha_{2l-3}, \ldots, \alpha_1, \\ &\quad [\alpha_2, \ldots, \alpha_{2l+2}]^+, \alpha_{2l+3}, \alpha_{2l+4}, \ldots, \alpha_{2g} \rangle.\end{aligned}$$

Letting l vary from 1 to $g-2$, and applying Lemma 3.7, we have that $\tau_{\gamma_{2g}}^{\pm 1}(\gamma_{2g-1}) = \tau_{[\alpha_0,\ldots,\alpha_{2l}]^-}^{\pm 1}(\alpha_{2l+1}) \in (\mathscr{C} \cup \mathscr{B})^3$ for all $1 \leq l \leq g-2$.

Now, using the fact that $H_{\mathscr{C}}^+ < \mathrm{Mod}^*(S)$ acts transitively on $\mathscr{C}$, we have as a consequence that it also acts transitively on each of the sets $\{[\alpha_i, \ldots, \alpha_{i+2l}]^- : 0 \leq i \leq 2g+1\}$ for $1 \leq l \leq g-2$. This implies that given $[\alpha_i, \ldots, \alpha_{i+2l}]^- \in \mathscr{B}^-$, there exists $h \in H_{\mathscr{C}}^+$ such that $h([\alpha_i, \ldots, \alpha_{i+2l}]^-) = [\alpha_0, \ldots, \alpha_{2l}]^-$. Thus, by Lemmas 3.6 and 3.7, we have then that $\tau_{\mathscr{B}^-}^{\pm 1}(\mathscr{C}) \subset (\mathscr{C} \cup \mathscr{B})^3$, and by equation (1), we obtain that $\tau_{\mathscr{B}^-}^{\pm 1}(\mathscr{C}) \cup \tau_{\mathscr{C}}^{\pm 1}(\mathscr{B}^-) \subset (\mathscr{C} \cup \mathscr{B})^3$.

Part 2 To prove the rest of the cases, recall that the hyperelliptic involution ι is an element of $\mathrm{stab}_{pt}(\mathscr{C})$ and $\iota([\alpha_i, \ldots, \alpha_{i+2k}]^+) = [\alpha_i, \ldots, \alpha_{i+2k}]^-$ for all $\{\alpha_i, \ldots, \alpha_{i+2k}\} \subset \mathscr{C}$. Given that (as was shown in part 1) for all $1 \leq l \leq g-2$, the families of maximal closed chains $H_{\mathscr{C}}^+(\mathscr{C}_l)$ satisfy the conditions of Lemma 3.7, Lemma 3.6 implies that the same is true for the maximal closed chains $\iota H_{\mathscr{C}}^+(\mathscr{C}_l)$. Therefore,

$$\tau_{\iota H_{\mathscr{C}}^+(\mathscr{B}^-)}^{\pm 1}(\iota H_{\mathscr{C}}^+(\mathscr{C})) \cup \tau_{\iota H_{\mathscr{C}}^+(\mathscr{C})}^{\pm 1}(\iota H_{\mathscr{C}}^+(\mathscr{B}^-)) = \tau_{\mathscr{B}^+}^{\pm 1}(\mathscr{C}) \cup \tau_{\mathscr{C}}^{\pm 1}(\mathscr{B}^+) \subset (\mathscr{C} \cup \mathscr{B})^3,$$

as desired.

3.4. Proof of Claim 3: $\tau_\zeta^{\pm 1}(\mathscr{B}) \subset (\mathscr{C} \cup \mathscr{B})^4$. Recall $\zeta = [\alpha_0, \alpha_1, \alpha_2]^-$, and let $\gamma \in \mathscr{B}$. In the cases, where ζ is disjoint from γ, we have $\tau_\zeta^{\pm 1}(\gamma) = \gamma \in \mathscr{C} \cup \mathscr{B}$. So, we assume that $i(\gamma, \zeta) \neq 0$, which by construction implies

$$\gamma \in \{[\alpha_1, \ldots, \alpha_{2k+1}]^\pm, [\alpha_3, \ldots, \alpha_{2k+1}]^\pm, [\alpha_2, \ldots, \alpha_{2l}]^- : 1 \leq k \leq g-1, 2 \leq l \leq g-1\}.$$

In these cases, there exist subsets of $C_0 \subset \mathscr{C}$ and $\{\beta_0\} \subset \mathscr{B}$, such that $\gamma = \langle C_0 \cup \{\beta_0\} \rangle$ and β_0 is disjoint from ζ. Note that $\tau_\zeta^{\pm 1}(C_0) \subset (\mathscr{C} \cup \mathscr{B})^3$ by claim 2, and $\tau_\zeta^{\pm 1}(\beta_0) = \beta_0 \in \mathscr{C} \cup \mathscr{B}$ by construction. Therefore,

$$\tau_\zeta^{\pm 1}(\gamma) = \tau_\zeta^{\pm 1}(\langle C_0 \cup \beta_0 \rangle) = \langle \tau_\zeta^{\pm 1}(C_0) \cup \tau_\zeta^{\pm 1}(\beta_0) \rangle \in (\mathscr{C} \cup \mathscr{B})^4.$$

For a more detailed account on C_0 and β_0, see **[8]**. For some examples, see Figures 7 and 8.

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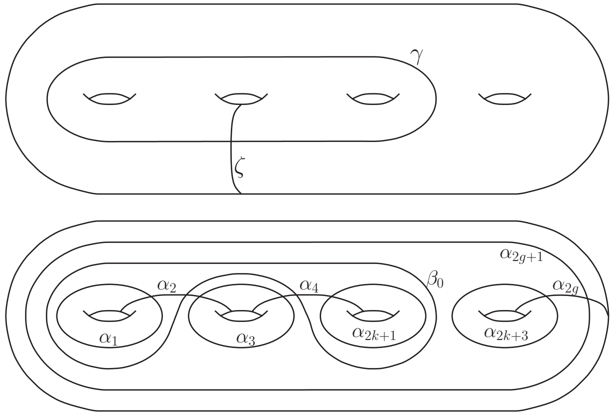

Figure 7. An example of $\gamma = [\alpha_1, \ldots, \alpha_{2k+1}]^+$, and the corresponding C_0 and β_0.

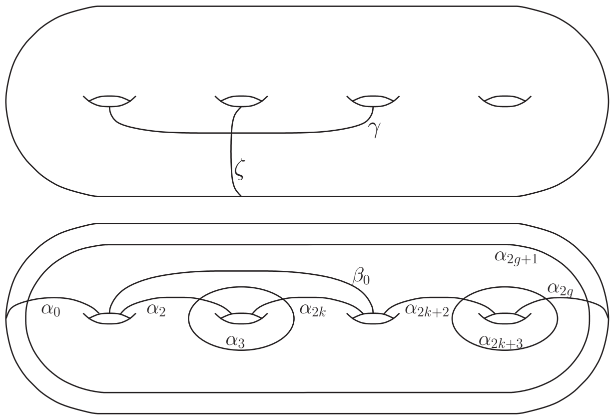

Figure 8. An example of $\gamma = [\alpha_2, \ldots, \alpha_{2k}]^-$, and the corresponding C_0 and β_0.

4. Exhaustion of $\mathcal{C}(S)$ for punctured surfaces. In this section, we suppose that $S = S_{g,n}$ with genus $g \geq 3$ and $n \geq 1$ punctures. In Section 4.1, we fix the principal set and give notation; in addition to a principal set of curves analogous to $\mathscr{C}$ and $\mathscr{B}$, we introduce some auxiliary curves to aid the exposition in Section 4.2, we also prove they are in specific expansions of the principal set, and state and prove several technical propositions; in Section 4.3, we prove the main theorem, pending the proof of a technical lemma; Sections 4.4–4.9 give the proofs of the claims for the technical lemma.

Note that the presence of punctures complicates the setting more than one might be led to believe. Examples of this is the presence of half-twists in the Humphries–Lickorish generators, and the presence of outer curves.

4.1. Statement of Theorem 4.3. The idea of the proof of the analogous result to Theorem 3.1 is the same as in the closed surface case. Using arguments similar to those of Theorem 3.1, we show that every nonseparating curve is in some expansion and then use that to prove the same for the separating curves.

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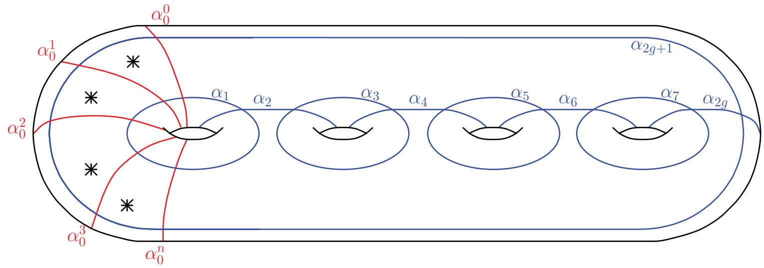

Figure 9. (Colour online) $\mathscr{C}_0$ in blue and $\mathscr{C}_f$ in red for $S_{5,4}$.

However, the presence of punctures induce several small but important changes, both in the principal set of curves that is used (which while analogous to the closed case, is not as symmetric and thus induces changes in the proofs), and in the manner auxiliary curves are used. For this reason, we first introduce the sets $\mathscr{C}$ and $\mathscr{B}_0$ whose union is the **principal set** and then we state in detail the theorem to prove.

Let $\mathscr{C}_0 = \{\alpha_1, \ldots, \alpha_{2g+1}\}$ be the chain depicted in Figure 9, and $\mathscr{C}_f = \{\alpha_0^0, \alpha_0^1, \ldots, \alpha_0^n\}$ be the multicurve also depicted in Figure 9, and $\mathscr{C} := \mathscr{C}_0 \cup \mathscr{C}_f$.

REMARK 4.1. Note that for $i \in \{0, \ldots, n-1\}$, we have that $S \backslash \{\alpha_0^i, \alpha_0^{i+1}\}$ has a connected component homeomorphic to a thrice punctured sphere. Also, note that, for each $j \in \{0, \ldots, n\}$, the set $C_j = \{\alpha_0^j, \alpha_1, \ldots, \alpha_{2g+1}\}$ is a maximal closed chain.

We turn now to adapting the notation used in the previous section. Consider the closed chain C_i. Then, we denote the curve α_0^i by α_0 to simplify notation when it is understood that $\alpha_0 \in C_i$. As such C_i has the subsets: $C_{i(o)} = \{\alpha_j \in C_i : j \text{ is odd}\}$ and $C_{i(e)} = \{\alpha_j \in C_i : j \text{ is even}\}$. These subsets are such that

- $S \backslash C_{i(e)}$ has two connected components, $S^+_{i(e)} = S_{0,i+g+1}$ and $S^-_{i(e)} = S_{0,n-i+g+1}$.
- $S \backslash C_{i(o)}$ has two connected components, $S^+_{i(o)} = S_{0,n+g+1}$ and $S^-_{i(o)} = S_{0,g+1}$.

Recalling that the subindices are modulo $2g+2$, we denote by $[\alpha_j, \ldots, \alpha_{j+2k}]^+$ for $0 < k < g-1$, the boundary component of a closed regular neighbourhood $N(\{\alpha_j, \ldots, \alpha_{j+2k}\})$, that is contained in either $S^+_{i(o)}$ or in $S^+_{i(e)}$. Analogously, $[\alpha_j, \ldots, \alpha_{j+2k}]^-$ denotes the boundary component of a closed regular neighbourhood $N(\{\alpha_j, \ldots, \alpha_{j+2k}\})$, that is contained in either $S^-_{i(o)}$ or in $S^-_{i(e)}$.

REMARK 4.2. Note that this notation is the same as in the closed surface case for the set $\mathscr{B}$; however, when $0 \in \{j, \ldots, j+2k\}$ (modulo $2g+2$), the curves $[\alpha_j, \ldots, \alpha_{j+2k}]^{\pm}$ depend on the choice of $i \in \{0, \ldots, n\}$ (recall α_0 stands for α_0^i).

Let $J = \{2l, \ldots, 2(l+k)\}$, for some $1 \leq k \leq g-1$, be a proper interval in the cyclic order modulo $2g+2$. Let also $\beta_J^{\pm} = [\alpha_{2l}, \ldots, \alpha_{2(l+k)}]^{\pm}$ (with $\alpha_0 = \alpha_0^1$ when necessary). See Figure 10 for examples. We define

$$\mathscr{B}_0 := \{\beta_J^{\pm} : J = \{2l, \ldots, 2(l+k)\}, 1 \leq k \leq g-1\}.$$

Now that we have the principal set of curves, we are able to state the punctured case version of Theorem A.

THEOREM 4.3. *Let S be an orientable surface with genus $g \geq 3$ and $n \geq 1$ punctures; let also $\mathscr{C}$ and $\mathscr{B}_0$ be defined as above. Then, $\bigcup_{i \in \mathbb{N}} (\mathscr{C} \cup \mathscr{B}_0)^i = \mathcal{C}(S)$.*

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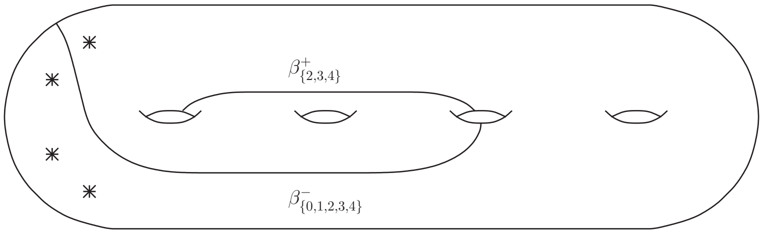

Figure 10. Examples of curves $\beta^{+}_{\{2,3,4\}}$ and $\beta^{-}_{\{0,1,2,3,4\}}$.

4.2. Auxiliary curves. We need for the proof some auxiliary curves and some technical results.

For $0 \leq i \leq n-2$, we define

$$\epsilon^{i,\,i+2} := \langle \mathscr{C}_0 \cup (\mathscr{C}_f \backslash \{\alpha_0^{i+1}\}) \rangle;$$

note that $\epsilon^{i,\,i+2} \in \mathscr{C}^1$; this can be seen using Figure 9 and removing α_0^{i+1} for the chosen i.

For $0 \leq i < j \leq n$ with $j - i > 2$, we define the curve

$$\epsilon^{i,j} := \langle \mathscr{C}_0 \cup (\mathscr{C}_f \backslash \{\alpha_0^k : i < k < j\}) \cup \{\epsilon^{k,\,k+2} : i \leq k \leq j-2\} \rangle;$$

note that $\epsilon^{i,j} \in \mathscr{C}^2$.

A way to visualise the curves $\epsilon^{i,j}$ is the following: If we label the punctures so that the ith puncture is the one delimited by α_0^{i-1} and α_0^i, then $\epsilon^{i,j}$ is the separating curve that bounds a disc containing the punctures with labels $i+1, \ldots, j$.

Then, we define the set

$$\mathscr{D} := \{\epsilon^{i,j} : j - i > 1\} \subset \mathscr{C}^2.$$

We now expand this set. Let $0 \leq i \leq j \leq n$. We define

$$C^{i,j} := \{\alpha_0^i, \ldots, \alpha_0^j, \alpha_1, \ldots, \alpha_{2g+1}\}, \quad \text{and}$$

$$E^{i,j} := \{\epsilon^{k,\,l} \in \mathscr{D} : 0 \leq k < l - 1 \leq i, \quad \text{or} \quad j \leq k < l - 1 \leq n\}.$$

Note that $C^{i,i} = C_i$, and $E^{i,j} = \mathscr{D} \backslash \{\epsilon^{k,\,l} : (\exists \gamma \in C^{i,j}) \quad i(\epsilon^{k,\,l}, \gamma) \neq 0\}$.

Let $k \in \{1, \ldots, g\}$, and define (see Figure 11)

$$\epsilon_k^{(i,j)} := \langle (C^{i,j} \backslash \{\alpha_{2k}\}) \cup E^{i,j} \rangle;$$

for $k = 0$, we take $\epsilon_0^{(i,j)} := \epsilon^{i,j}$, and for $k = -1$, we take $\epsilon_{-1}^{(i,j)} = \epsilon_g^{(i,j)}$.

Finally, we define

$$\mathscr{E} := \mathscr{D} \cup \{\epsilon_k^{(i,j)} : 0 \leq i \leq j \leq n, \;\; k \in \{1, \ldots, g\}\}.$$

Note that $\mathscr{E} \subset \mathscr{C}^3$ by construction. See Figure 12 for examples of curves in $\mathscr{E}$.

REMARK 4.4. Note that the sets $\mathscr{D}$ and $\mathscr{E}$ are only defined when $n \geq 2$. For this reason, if $n = 1$, we define $\mathscr{D} = \mathscr{E} = \varnothing$.

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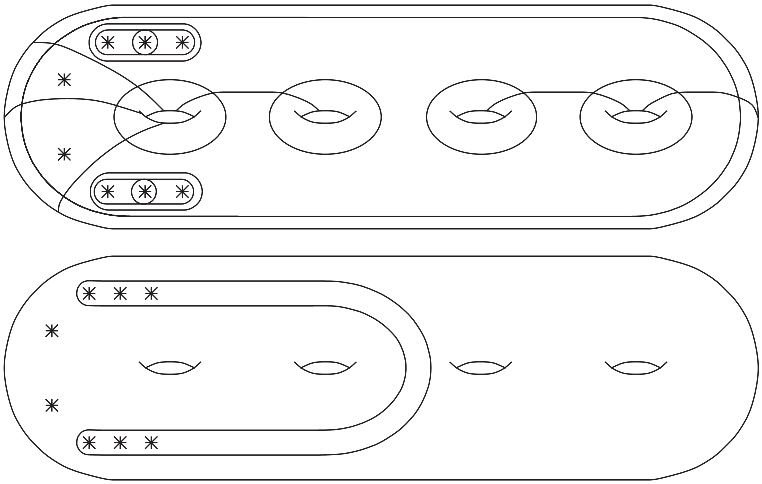

Figure 11. The curve $\epsilon_k^{(i,j)} := \langle (C^{i,j} \backslash \{\alpha_{2k}\}) \cup E^{i,j} \rangle$.

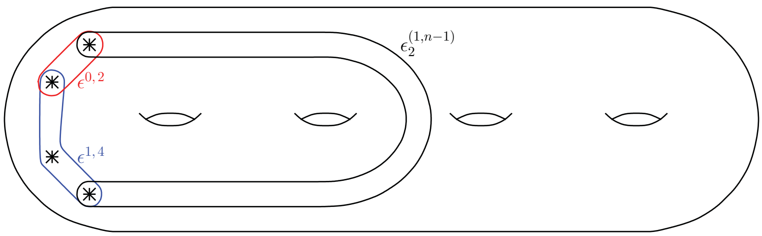

Figure 12. (Colour online) Examples of curves in $\mathscr{E}$.

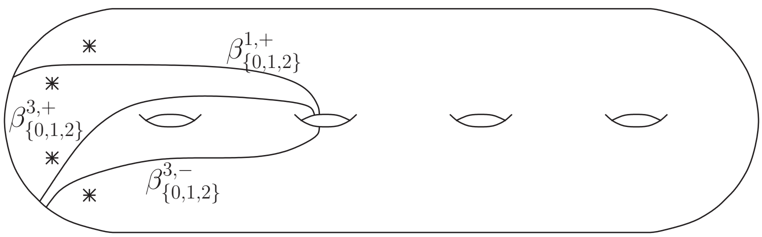

Figure 13. Examples of curves $\beta^{1,+}_{\{0,1,2\}}$, $\beta^{3,+}_{\{0,1,2\}}$, and $\beta^{3,-}_{\{0,1,2\}}$ in $S_{4,4}$.

Now, in the case where $1 \leq j \leq n-1$. We define the following curves (see Figure 13 for examples):

$$\beta^{j,+}_{\{0,1,2\}} := \langle (\mathscr{C}_f \backslash \{\alpha_0^k : k < j\}) \cup (\mathscr{C}_0 \backslash \{\alpha_3, \alpha_{2g+1}\}) \cup E^{j,n} \cup \{\epsilon_k^{(1,n-1)} : 2 \leq k \leq g\} \rangle;$$

$$\beta^{j,-}_{\{0,1,2\}} := \langle (\mathscr{C}_f \backslash \{\alpha_0^k : k > j\}) \cup (\mathscr{C}_0 \backslash \{\alpha_3, \alpha_{2g+1}\}) \cup E^{0,j} \cup \{\epsilon_k^{(1,n-1)} : 2 \leq k \leq g\} \rangle;$$

in the case where $j = n$, we define (see Figure 14)

$$\beta^{n,+}_{\{0,1,2\}} := \langle (\mathscr{C}_0 \backslash \{\alpha_3, \alpha_{2g+1}\}) \cup \{\alpha_0^n\} \cup \mathscr{D} \cup \{\beta^{\pm}_{\{4,\ldots,2g\}}\} \rangle;$$

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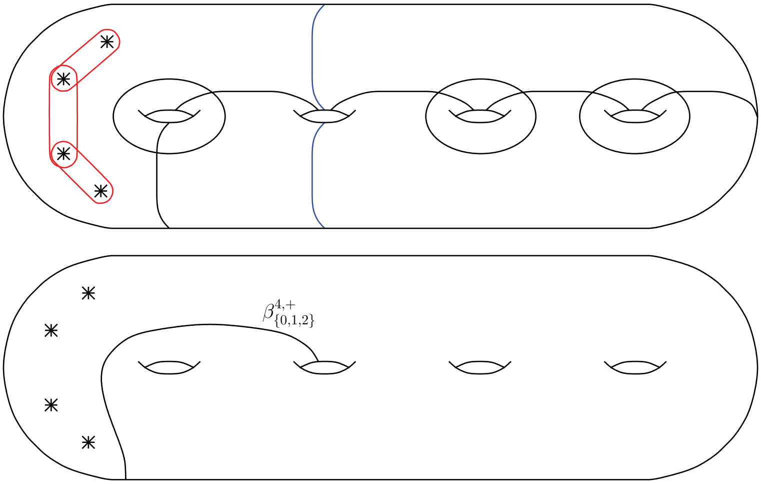

Figure 14. (Colour online) The curve $\beta^{4,+}_{\{0,1,2\}} := \langle(\mathscr{C}_0 \backslash \{\alpha_3, \alpha_{2g+1}\}) \cup \{\alpha_0^n\} \cup \mathscr{D} \cup \{\beta^{\pm}_{\{4,\ldots,2g\}}\}\rangle$.

$$\beta^{n,-}_{\{0,1,2\}} := \beta^{-}_{\{4,\ldots,2g\}} \qquad (\in \mathscr{B}_0).$$

Similarly, we define the following curves: in the case where $1 \leq j \leq n-1$, we define (see Figure 15 for examples)

$$\beta^{j,+}_{\{2g,2g+1,0\}} := \langle(\mathscr{C}_0 \backslash \{\alpha_1, \alpha_{2g-1}\}) \cup (\mathscr{C}_f \backslash \{\alpha_0^k : k < j\}) \cup E^{j,n} \cup \{\epsilon_k^{(1,n-1)} : 1 \leq k \leq g-1\}\rangle;$$

$$\beta^{j,-}_{\{2g,2g+1,0\}} := \langle(\mathscr{C}_0 \backslash \{\alpha_1, \alpha_{2g-1}\}) \cup (\mathscr{C}_f \backslash \{\alpha_0^k : k > j\}) \cup E^{0,j} \cup \{\epsilon_k^{(1,n-1)} : 1 \leq k \leq g-1\}\rangle;$$

in the case where $j = n$, we define (see Figure 16 for examples)

$$\beta^{n,+}_{\{2g,2g+1,0\}} := \langle(\mathscr{C}_0 \backslash \{\alpha_1, \alpha_{2g-1}\}) \cup \{\alpha_0^n\} \cup \mathscr{D} \cup \{\beta^{\pm}_{\{2,\ldots,2g-2\}}\}\rangle$$

$$\beta^{n,-}_{\{2g,2g+1,0\}} := \beta^{-}_{\{2,\ldots,2g-2\}}.$$

Note that $\beta^{j,\pm}_{\{0,1,2\}}, \beta^{j,\pm}_{\{2g,2g+1,0\}} \in \mathscr{C}^4$ for all $1 \leq j \leq n-1$, and $\beta^{n,\pm}_{\{0,1,2\}}, \beta^{n,\pm}_{\{2g,2g+1,0\}} \in (\mathscr{C} \cup \mathscr{B}_0)^4$. Then, we define the set

$$\mathscr{B}_T := \{\beta_J^{i,+}, \beta_J^{i,-} : J \in \{\{0,1,2\}, \{2g, 2g+1, 0\}\}, 1 \leq i \leq n\} \subset (\mathscr{C} \cup \mathscr{B}_0)^4.$$

The set $\mathscr{B}_0 \cup \mathscr{B}_T$ and the set $\mathscr{B}$ of the previous section are quite similar. However, $\mathscr{B}_0 \cup \mathscr{B}_T$ is not as symmetric as $\mathscr{B}$ and it has a sense of incompleteness, for example, that we are not including the boundary components of regular neighbourhoods of chains of odd length whose first curve has odd index. While some of these missing curves are not needed for the proof of Theorem 4.3, others are necessary in order to make the bootstrapping method of the previous section work in this context.

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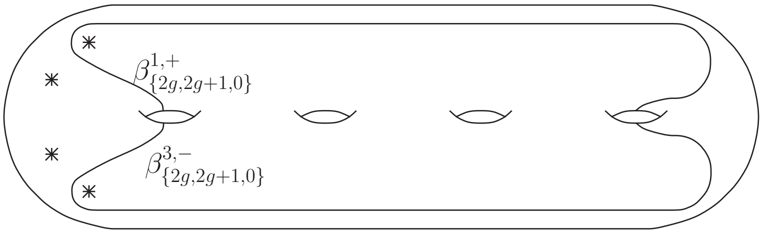

Figure 15. Examples of curves $\beta^{1,+}_{\{2g,2g+1,0\}}$ and $\beta^{3,-}_{\{2g,2g+1,0\}}$ in $S_{4,4}$.

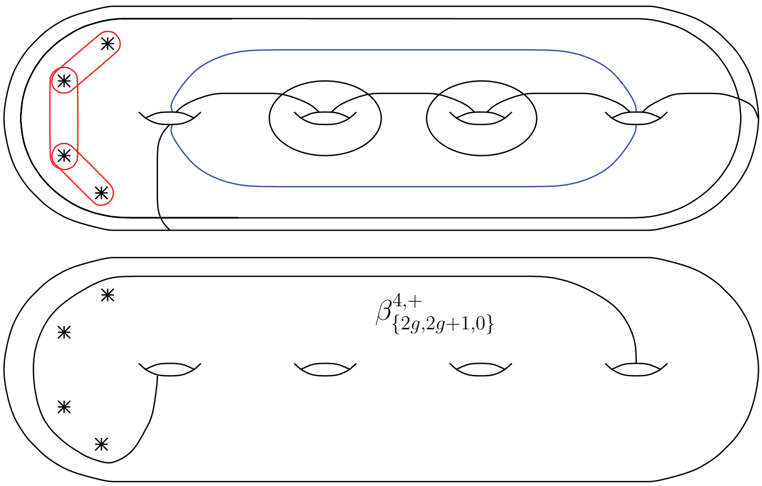

Figure 16. (Colour online) The curve $\beta^{4,+}_{\{2g,2g+1,0\}} := \langle (\mathscr{C}_0 \backslash \{\alpha_1, \alpha_{2g-1}\}) \cup \{\alpha_0^n\} \cup \mathscr{D} \cup \{\beta^{\pm}_{\{2,\ldots,2g-2\}}\} \rangle$.

Proposition 4.5. *Let $k \in \mathbb{Z}$. Then, $[\alpha_{k+1}, \ldots, \alpha_{k+(2g-1)}]^+ \in (\mathscr{C} \cup \mathscr{B}_0)^4$ for any choice of $i \in \{0, \ldots, n\}$ (with $\alpha_0 = \alpha_0^i$ when necessary).*

Proof. Suppose $n \geq 2$. If $k = 0$, $[\alpha_1, \ldots, \alpha_{2g-1}]^+ = \langle (\mathscr{C}_0 \backslash \{\alpha_{2g}\}) \cup \mathscr{D} \rangle$. We split the rest of the proof into two cases according to the parity of k.

If $k \neq 0$ is even, then $k = 2(l+2)$ for $l \in \{-1, \ldots, g-2\}$, and so $[\alpha_{k+1}, \ldots, \alpha_{k+(2g-1)}]^+ = [\alpha_{2l+5}, \ldots, \alpha_{2l+1}]^+$ (recall the subindices are modulo $2g+2$). Thus, if $i \in \{1, \ldots, n-1\}$, we get,

$$[\alpha_{2l+5}, \ldots, \alpha_{2l+1}]^+ = \langle \{\alpha_{2l+5}, \ldots, \alpha_{2l+1}\} \cup \{\alpha_{2l+3}\} \cup E^{i,i} \cup \{\epsilon^{(1,n-1)}_{l+1}, \epsilon^{(1,n-1)}_{l+2}\} \rangle \in (\mathscr{C} \cup \mathscr{B}_0)^4,$$

see Figure 17; if $i \in \{0, n\}$, we get

$$[\alpha_{2l+5}, \ldots, \alpha_{2l+1}]^+ = \langle \{\alpha_{2l+5}, \ldots, \alpha_{2l+1}\} \cup \{\alpha_{2l+3}\} \cup \mathscr{D} \rangle \in (\mathscr{C} \cup \mathscr{B}_0)^3.$$

If k is odd, then $k = 2l-1$ for some $0 \leq l \leq g$, and so $[\alpha_{k+1}, \ldots, \alpha_{k+(2g-1)}]^+ = [\alpha_{2l}, \ldots, \alpha_{2l-4}]^+$ (recall the subindices are modulo $2g+2$). Thus, if $i = 0$, we have that $[\alpha_{2l}, \ldots, \alpha_{2l-4}]^+ = \alpha_{2l-2}$; if $i = 1$, we have that $[\alpha_{2l}, \ldots, \alpha_{2l-4}]^+ \in \mathscr{B}_0$; if $i \in \{2, \ldots, n-$

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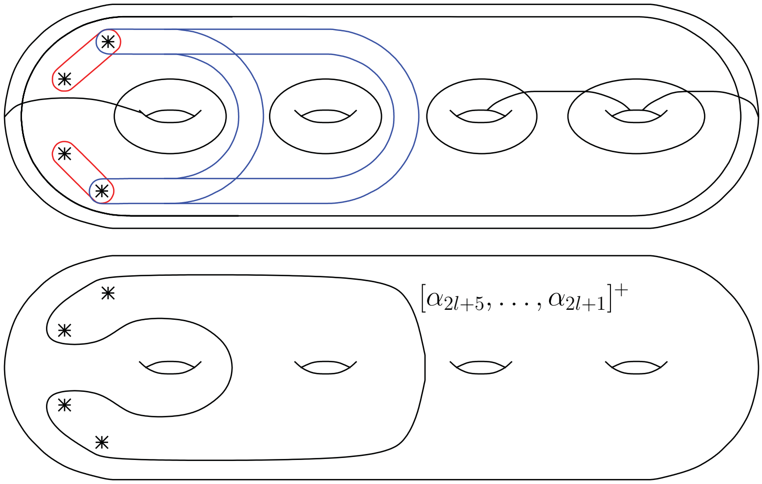

Figure 17. (Colour online) $[\alpha_{2l+5}, \ldots, \alpha_{2l+1}]^+ = \langle\{\alpha_{2l+5}, \ldots, \alpha_{2l+1}\} \cup \{\alpha_{2l+3}\} \cup E^{i,i} \cup \{\epsilon_{l+1}^{(1,n-1)}, \epsilon_{l+2}^{(1,n-1)}\}\rangle$ for $l = 0$ and $i = 2$, in $S_{4,4}$.

1}, we have (see Figure 18)

$$[\alpha_{2l}, \ldots, \alpha_{2l-4}]^+ = \langle\{\alpha_{2l}, \ldots, \alpha_{2l-4}\} \cup \{\alpha_{2l-2}\} \cup \{\alpha_0^j : j > i\} \cup E^{i,i} \cup \{\epsilon_{l-1}^{(1,n-1)}\}\rangle \in (\mathscr{C} \cup \mathscr{B}_0)^4;$$

and finally, if $i = n$, we have

$$[\alpha_{2l}, \ldots, \alpha_{2l-4}]^+ = \langle\{\alpha_{2l}, \ldots, \alpha_{2l-4}\} \cup \{\alpha_{2l-2}\} \cup \mathscr{D}\rangle \in (\mathscr{C} \cup \mathscr{B}_0)^3.$$

Now, if $n = 1$, we can uniquely determine the curves $[\alpha_{k+1}, \ldots, \alpha_{k+(2g-1)}]^+$ in the same way as above, recalling that in this instance $\mathscr{D} = \mathscr{E} = \varnothing$ and taking the cases $i = 0$ and $i = n$.

Therefore, for $i \in \{0, \ldots, n\}$, $k \in \mathbb{Z}$, $[\alpha_{k+1}, \ldots, \alpha_{k+(2g-1)}]^+ \in (\mathscr{C} \cup \mathscr{B}_0)^4$. □

Note that for $n = 1$, the curves $[\alpha_{k+1}, \ldots, \alpha_{k+(2g+1)}]^+$ are elements of $(\mathscr{C} \cup \mathscr{B}_0)^1$ for any $k \in \mathbb{Z}$ and any choice of $i \in \{0, 1\}$.

Now, we have the following proposition.

Proposition 4.6. *Let $k \in \mathbb{Z}$. Then, $[\alpha_k, \alpha_{k+1}, \alpha_{k+2}]^\pm \in (\mathscr{C} \cup \mathscr{B}_0)^6$ for any choice of $i \in \{0, \ldots, n\}$ (with $\alpha_0 = \alpha_0^i$ when necessary).*

Proof. We start by proving that for $k \in \mathbb{Z}$ even, $[\alpha_k, \alpha_{k+1}, \alpha_{k+2}]^\pm \in (\mathscr{C} \cup \mathscr{B}_0)^4$ (part 1); afterwards, we prove that for k odd, $[\alpha_k, \alpha_{k+1}, \alpha_{k+2}]^- \in (\mathscr{C} \cup \mathscr{B}_0)^4$ (part 2); finally, we prove that for k odd, $[\alpha_k, \alpha_{k+1}, \alpha_{k+2}]^+ \in (\mathscr{C} \cup \mathscr{B}_0)^6$ (part 3).

Part 1: If k is even, $[\alpha_k, \alpha_{k+1}, \alpha_{k+2}]^\pm \in (\mathscr{C} \cup \mathscr{B}_0)^4$ with the exception of $[\alpha_{2g}, \alpha_{2g+1}, \alpha_0^0]^-$, and $[\alpha_0^0, \alpha_1, \alpha_2]^-$ (we can verify that $[\alpha_{2g}, \alpha_{2g+1}, \alpha_0^0]^+ = \beta^+_{\{2,\ldots,2g-2\}}$ and $[\alpha_0^0, \alpha_1, \alpha_2]^+ = \beta^+_{\{4,\ldots,2g\}}$). This happens since $[\alpha_k, \alpha_{k+1}, \alpha_{k+2}]^\pm$ is an element of $\mathscr{B}_0$ or $\mathscr{B}_T$, for k even with the aforementioned exceptions. So, for the first exception, we have the following (see Figure 19)

$$[\alpha_{2g}, \alpha_{2g+1}, \alpha_0^0]^- = \langle(C_0 \backslash \{\alpha_1, \alpha_{2g-1}\}) \cup \mathscr{D} \cup \{\beta^\pm_{\{2,\ldots,2g-2\}}\}\rangle.$$

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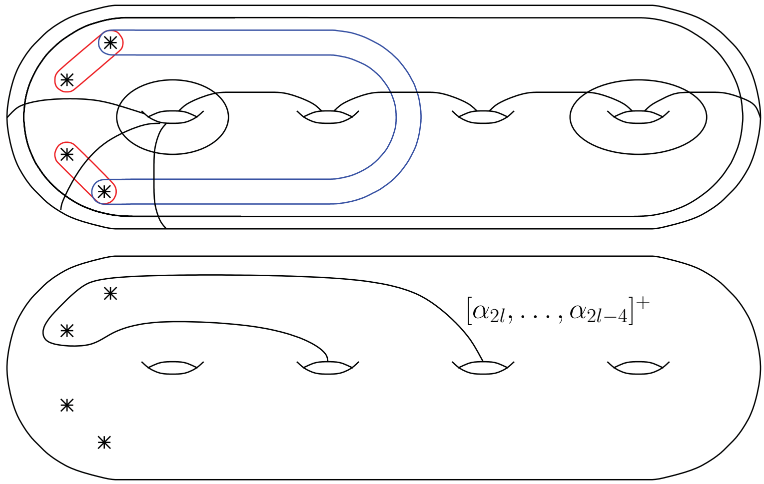

Figure 18. (Colour online) $[\alpha_{2l}, \ldots, \alpha_{2l-4}]^+ = \langle\{\alpha_{2l}, \ldots, \alpha_{2l-4}\} \cup \{\alpha_{2l-2}\} \cup \{\alpha_0^j : j > i\} \cup E^{i,i} \cup \{\epsilon_{l-1}^{(1,n-1)}\}\rangle$ for $l = 3$ and $i = 2$ in $S_{4,4}$.

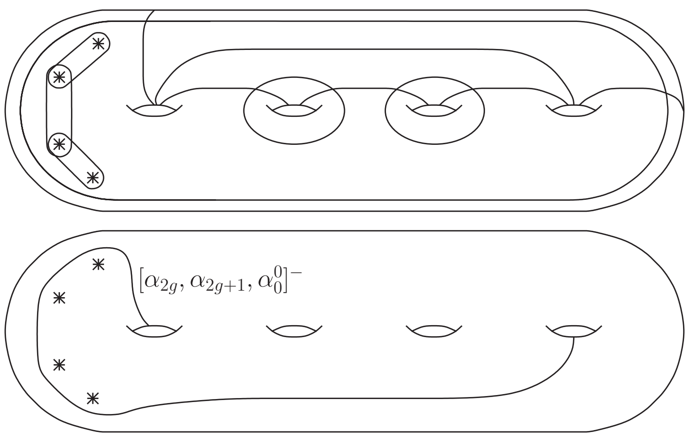

Figure 19. An illustration of $[\alpha_{2g}, \alpha_{2g+1}, \alpha_0^0]^- = \langle(C_0 \backslash \{\alpha_1, \alpha_{2g-1}\}) \cup \mathscr{D} \cup \{\beta^{\pm}_{\{2,\ldots,2g-2\}}\}\rangle$.

And, for the second exception, we have (see Figure 20)

$$[\alpha_0^0, \alpha_1, \alpha_2]^- = \langle(C_0 \backslash \{\alpha_3, \alpha_{2g+1}\}) \cup \mathscr{D} \cup \{\beta^{\pm}_{\{4,\ldots,2g\}}\}\rangle.$$

Therefore, for k even (with $\alpha_0 = \alpha_0^i$ when necessary), $[\alpha_k, \alpha_{k+1}, \alpha_{k+2}]^{\pm} \in (\mathscr{C} \cup \mathscr{B}_0)^4$.

Part 2: Here, we prove the case $n \geq 2$, leaving the analogous details of the case $n = 1$ to the reader (see [**8**]).

Let $i \in \{0, \ldots, n\}$. For k odd, we have to prove that each of the curves $[\alpha_k, \alpha_{k+1}, \alpha_{k+2}]^-$ is in $(\mathscr{C} \cup \mathscr{B}_0)^4$. Let $k \in \{3, 5, \ldots, 2g+1\} \backslash \{2g-1\}$, then (see

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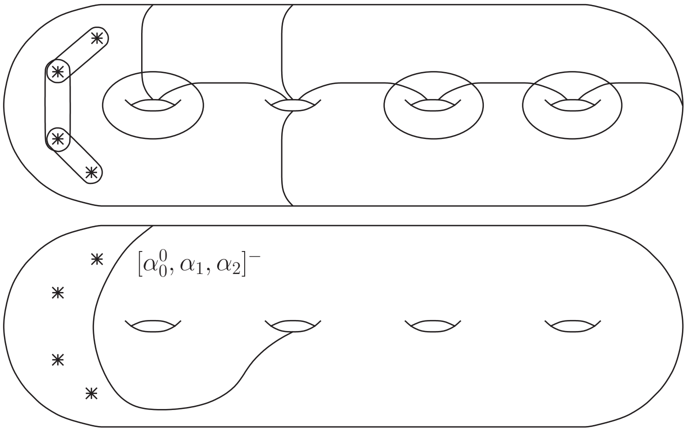

Figure 20. An illustration of $[\alpha_0^0, \alpha_1, \alpha_2]^- = \langle (C_0 \backslash \{\alpha_3, \alpha_{2g+1}\}) \cup \mathscr{D} \cup \{\beta^{\pm}_{\{4,\ldots,2g\}}\}\rangle$.

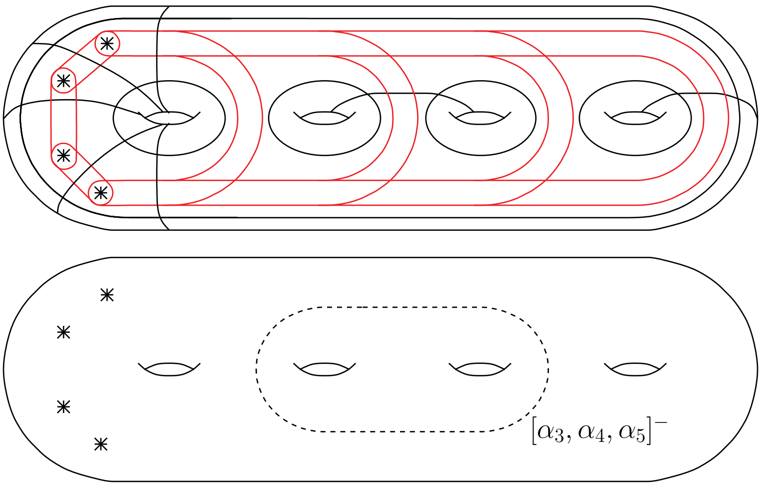

Figure 21. (Colour online) $[\alpha_k, \alpha_{k+1}, \alpha_{k+2}]^- = \langle (\mathscr{C} \backslash \{\alpha_{k-1}, \alpha_{k+3}\}) \cup \mathscr{E} \rangle$ for $k = 3$ in $S_{4,4}$.

Figure 21)

$$[\alpha_k, \alpha_{k+1}, \alpha_{k+2}]^- = \langle (\mathscr{C} \backslash \{\alpha_{k-1}, \alpha_{k+3}\}) \cup \mathscr{E} \rangle.$$

We also have

$$[\alpha_1, \alpha_2, \alpha_3]^- = \langle ((\mathscr{C}_0 \backslash \{\alpha_4\}) \cup \mathscr{E} \rangle,$$

$$[\alpha_{2g-1}, \alpha_{2g}, \alpha_{2g+1}]^- = \langle ((\mathscr{C}_0 \backslash \{\alpha_{2g-2}\}) \cup \mathscr{E} \rangle.$$

Therefore, for k odd (with $\alpha_0 = \alpha_0^i$ when necessary), $[\alpha_k, \alpha_{k+1}, \alpha_{k+2}]^-) \in (\mathscr{C} \cup \mathscr{B}_0)^4$.

Part 3: As above, we only prove the case $n \geq 2$; for the details of the case $n = 1$ see **[8]**.

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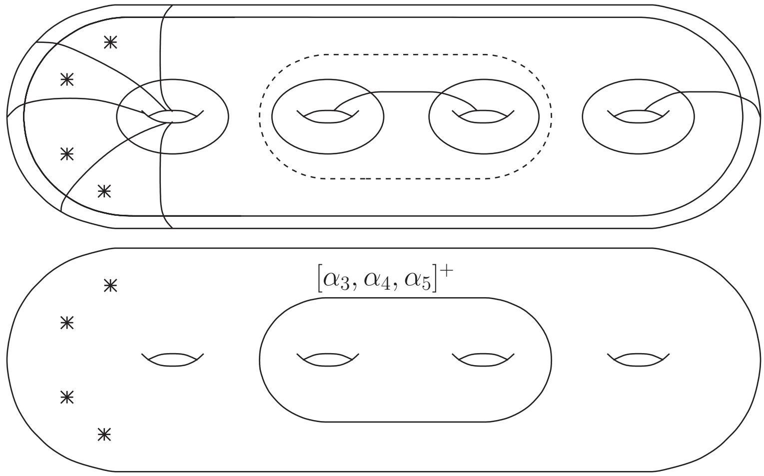

Figure 22. $[\alpha_k, \alpha_{k+1}, \alpha_{k+2}]^+ = \langle(\mathscr{C}\backslash\{\alpha_{k-1}, \alpha_{k+3}\}) \cup \{[\alpha_k, \alpha_{k+1}, \alpha_{k+2}]^-\}\rangle$ for $k = 3$ in $S_{4,4}$.

Let $i \in \{0, \ldots, n\}$, k be odd. Similarly, to the previous part, we have to prove that $[\alpha_k, \alpha_{k+1}, \alpha_{k+2}]^+ \in (\mathscr{C} \cup \mathscr{B}_0)^6$. Let $k \in \{3, 5, \ldots, 2g-3\}$. Thus, (see Figure 22)

$$[\alpha_k, \alpha_{k+1}, \alpha_{k+2}]^+ = \langle(\mathscr{C}\backslash\{\alpha_{k-1}, \alpha_{k+3}\}) \cup \{[\alpha_k, \alpha_{k+1}, \alpha_{k+2}]^-\}\rangle.$$

We then have (see Figure 23)

$$[\alpha_{2g-1}, \alpha_{2g}, \alpha_{2g+1}]^+ = \left\langle (\mathscr{C}_0\backslash\{\alpha_{2g-2}\}) \cup \mathscr{D} \cup \left(\bigcup_{l\in\{1,\ldots,g-1\}} \epsilon_l^{(1,n-1)} \right) \cup \{[\alpha_{2g-1}, \alpha_{2g}, \alpha_{2g+1}]^-\} \right\rangle,$$

$$[\alpha_1, \alpha_2, \alpha_3]^+ = \left\langle (\mathscr{C}_0\backslash\{\alpha_4\}) \cup \mathscr{D} \cup \left(\bigcup_{l\in\{2,\ldots,g\}} \{\epsilon_l^{(1,n-1)}\} \right) \cup \{[\alpha_1, \alpha_2, \alpha_3]^-\} \right\rangle.$$

For $i \in \{1, \ldots, n-1\}$, we get (see Figure 24)

$$[\alpha_{2g+1}, \alpha_0^i, \alpha_1]^+ = \left\langle (C_i\backslash\{\alpha_2, \alpha_{2g}\}) \cup E^{i,i} \cup \left(\bigcup_{l\in\{1,\ldots,g\}} \{\epsilon_l^{(1,n-1)}\} \right) \cup \{[\alpha_{2g+1}, \alpha_0, \alpha_1]^-\} \right\rangle;$$

for $i \in \{0, n\}$, to prove the result for $[\alpha_{2g+1}, \alpha_0^i, \alpha_1]^+$, we need the auxiliary curve (see Figure 25)

$$[\alpha_3, \ldots, \alpha_{2g-1}]^+ = \langle(\mathscr{C}\backslash\{\alpha_2, \alpha_{2g}\}) \cup \{[\alpha_{2g+1}, \alpha_0^i, \alpha_1]^-\}\rangle \in (\mathscr{C} \cup \mathscr{B}_0)^5;$$

and so (see Figure 26)

$$[\alpha_{2g+1}, \alpha_0^i, \alpha_1]^+ = \langle(C_i\backslash\{\alpha_2, \alpha_{2g}\}) \cup \mathscr{D} \cup \{[\alpha_{2g+1}, \alpha_0^i, \alpha_1]^-, [\alpha_3, \ldots, \alpha_{2g-1}]^+\}\rangle.$$

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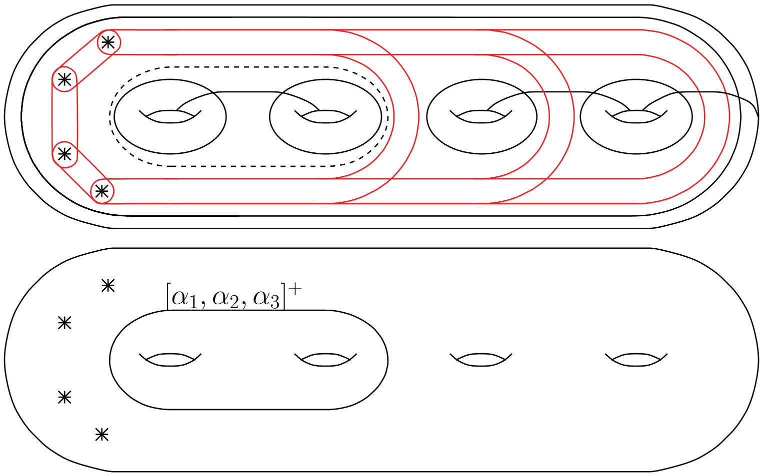

Figure 23. (Colour online)
$[\alpha_1, \alpha_2, \alpha_3]^+ = \left\langle (\mathscr{C}_0 \backslash \{\alpha_4\}) \cup \mathscr{D} \cup \left(\bigcup_{l \in \{2,\ldots,g\}} \{\epsilon_l^{(1,n-1)}\} \right) \cup \{[\alpha_1, \alpha_2, \alpha_3]^-\} \right\rangle$ in $S_{4,4}$.

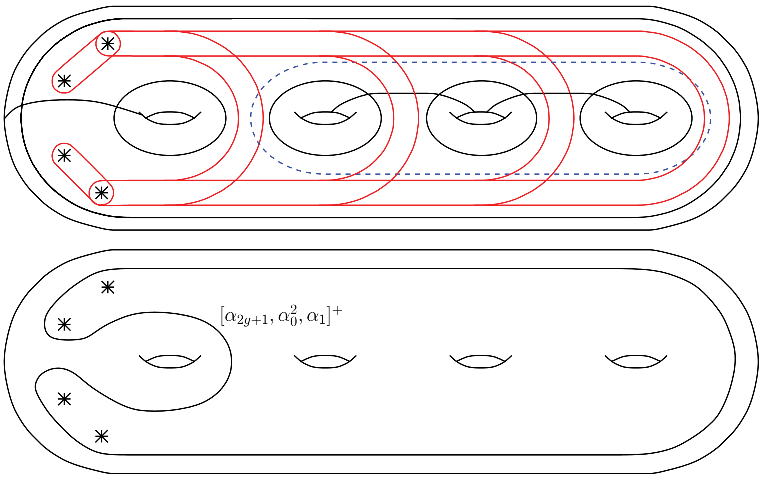

Figure 24. (Colour online) The curve $[\alpha_{2g+1}, \alpha_0^2, \alpha_1]^+$
$= \left\langle (C_i \backslash \{\alpha_2, \alpha_{2g}\}) \cup E^{i,i} \cup \left(\bigcup_{l \in \{1,\ldots,g\}} \{\epsilon_l^{(1,n-1)}\} \right) \cup \{[\alpha_{2g+1}, \alpha_0, \alpha_1]^-\} \right\rangle$.

Therefore, for $k \in \mathbb{Z}$ (with $\alpha_0 = \alpha_0^i$ when necessary), $[\alpha_k, \alpha_{k+1}, \alpha_{k+2}]^{\pm} \in (\mathscr{C} \cup \mathscr{B}_0)^6$. □

Finally, we define the set of auxiliary curves $\mathscr{B}$. Note that by construction, $\mathscr{B} \subset (\mathscr{C} \cup \mathscr{B}_0)^6$.

$$\mathscr{B} := \mathscr{B}_0 \cup \mathscr{B}_T \cup \left(\bigcup_{\substack{i \in \{0,\ldots,n\} \\ k \in \mathbb{Z}}} \{[\alpha_{k+1}, \ldots, \alpha_{k+(2g-1)}]^+, [\alpha_k, \alpha_{k+1}, \alpha_{k+2}]^{\pm}\} \right).$$

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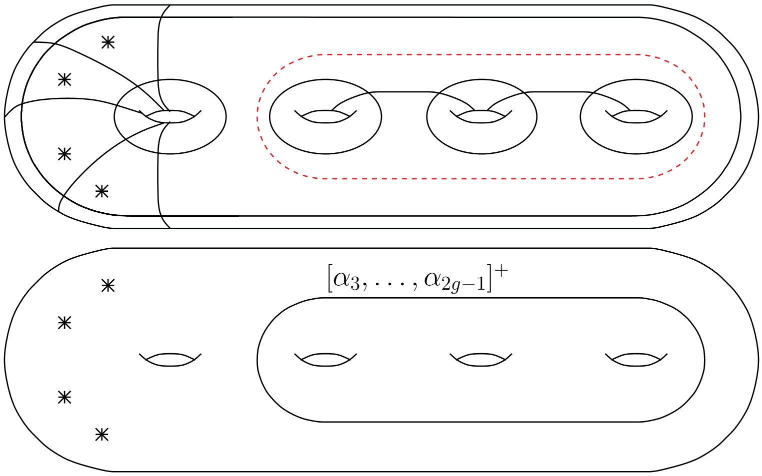

Figure 25. (Colour online) $[\alpha_3, \dots, \alpha_{2g-1}]^+ = \langle (\mathscr{C} \backslash \{\alpha_2, \alpha_{2g}\}) \cup \{[\alpha_{2g+1}, \alpha_0^i, \alpha_1]^-\} \rangle$ for $j = 3$ in $S_{4,4}$.

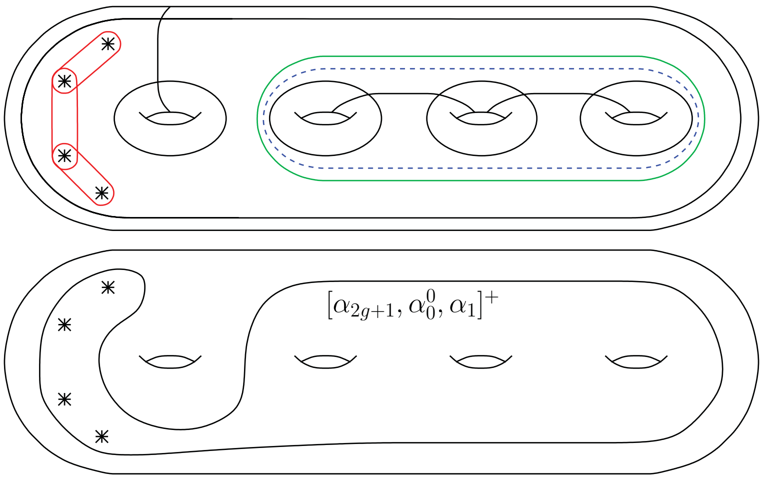

Figure 26. (Colour online) $[\alpha_{2g+1}, \alpha_0^i, \alpha_1]^+ = \langle (C_i \backslash \{\alpha_2, \alpha_{2g}\}) \cup \mathscr{D} \cup \{ [\alpha_{2g+1}, \alpha_0^i, \alpha_1]^-, [\alpha_3, \dots, \alpha_{2g-1}]^+ \} \rangle$ for $j = 3$ in $S_{4,4}$.

4.3. Proof of Theorem 4.3. Having defined the principal set $\mathscr{C} \cup \mathscr{B}_0$ and constructed the auxiliary sets $\mathscr{D} \subset \mathscr{E}$ and $\mathscr{B}_T \subset \mathscr{B}$, we state some results to ease the proofs of the following section, as well as give necessary notation and the proof of Theorem 4.3.

PROPOSITION 4.7. *Let* $h \in \mathrm{Mod}^*(S)$ *and* $Y \subset \mathcal{C}(S)$. *Then,* $h(Y^k) = (h(Y))^k$ *for all* $k \in \mathbb{N}$. *In particular, if* $h(Y) \subset (\mathscr{C} \cup \mathscr{B}_0)^k$ *for some* $k \in \mathbb{Z}$, *then* $h(Y^m) \subset (\mathscr{C} \cup \mathscr{B}_0)^{k+m}$.

Proof. Given that $\mathrm{Mod}^*(S)$ acts by automorphisms on $\mathcal{C}(S)$, it follows from the definition of rigid expansions that $h(Y^1) \subset (h(Y))^1$. Let $\gamma \in (h(Y))^1$. If $\gamma \in h(Y)$, then it is the image of a curve in Y and thus it is in the image of $h(Y^1)$.

If $\gamma \in (h(Y))^1 \backslash h(Y)$, then there exists $C \subset h(Y)$ with $\gamma = \langle C \rangle$. Since $C \subset h(Y)$, there exists $B \subset Y$ with $C = h(B)$. Hence, we have that $\gamma = \langle h(B) \rangle$. If $\beta = h^{-1}(\gamma)$,

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this implies that $\beta = \langle B \rangle$. Recalling that $B \subset Y$, we deduce that $\beta \in Y^1$. Therefore, $\gamma = h(\beta) \in h(Y^1)$ and so $h(Y^1) = (h(Y))^1$. The result follows by induction. □

As a consequence of this proposition, since $\mathscr{C} \cup \mathscr{E} \cup \mathscr{B} \subset (\mathscr{C} \cup \mathscr{B}_0)^6$, we have the following corollary.

COROLLARY 4.8. *Let* $h \in \text{Mod}^*(S)$. *If* $h(\mathscr{C} \cup \mathscr{B}_0) \subset (\mathscr{C} \cup \mathscr{B}_0)^k$ *for some* $k \in \mathbb{Z}$, *then* $h(\mathscr{C} \cup \mathscr{E} \cup \mathscr{B}) \subset (\mathscr{C} \cup \mathscr{B}_0)^{k+6}$.

An *outer curve* α is a separating curve such that cutting along α one of the resulting connected components is homeomorphic to a thrice-punctured sphere. When α is an outer curve, $\beta \in \mathcal{C}(S)$, and $A, B \subset \mathcal{C}(S)$, we denote by $\eta_\alpha(\beta)$ the half-twist of β along α and $\eta_A(B) = \bigcup_{\gamma \in A} \eta_\gamma(B)$.

We must recall that the half-twist η_α is defined if and only if α is an outer curve, and there is exactly one half-twist along α if $S \not\cong S_{0,4}$. Let $\mathscr{H} := \{\epsilon^{i-2,i} \in \mathscr{D} : 2 \leq i \leq n\}$, then we can state the following lemma.

LEMMA 4.9. *Let* $\zeta = \beta^+_{\{2g-2,2g-1,2g\}}$, *and* $\mathscr{H}$ *as above. Then,* $\tau^{\pm 1}_{\mathscr{C} \cup \{\zeta\}}(\mathscr{C} \cup \mathscr{E} \cup \mathscr{B}) \cup \eta^{\pm 1}_{\mathscr{H}}(\mathscr{C} \cup \mathscr{E} \cup \mathscr{B}) \subset (\mathscr{C} \cup \mathscr{B}_0)^{18}$.

Assuming this lemma (for which we give a proof in the following subsections), we can proceed to prove Theorem 4.3 as follows.

Proof of Theorem 4.3. Let $\mathscr{G} = (\mathscr{C} \backslash \{\alpha_{2g+1}\}) \cup \{\zeta\}$. Recalling Lickorish–Humphries Theorem (see [**10**, **16**] and Section 4 of [**5**]), we have that Mod(S) is generated by the Dehn twists along $\mathscr{G}$ if $n \leq 1$ and by the Dehn twists along $\mathscr{G}$ and the half twists along $\mathscr{H}$ if $2 \leq n$. By Lemma 4.9, we know that $\tau^{\pm 1}_{\mathscr{G}}(\mathscr{C} \cup \mathscr{E} \cup \mathscr{B}) \cup \eta^{\pm 1}_{\mathscr{H}}(\mathscr{C} \cup \mathscr{E} \cup \mathscr{B}) \subset (\mathscr{C} \cup \mathscr{B}_0)^{18}$.

Let us denote by f_γ the Dehn twist along γ if $\gamma \in \mathscr{G}$ or the half-twist along γ if $\gamma \in \mathscr{H}$.

Now let γ be either a nonseparating curve or an outer curve, and α an element in $\mathscr{G}$ or $\mathscr{H}$, respectively. There exists $h \in \text{Mod}(S)$ such that $\gamma = h(\alpha)$. Thus, by an iterated used of Lemma 4.9, there are some curves $\gamma_1, \ldots, \gamma_l \in \mathscr{G} \cup \mathscr{H}$ (not necessarily different), such that

$$\gamma = f_{\gamma_1} \circ \cdots \circ f_{\gamma_l}(\alpha) \in (\mathscr{C} \cup \mathscr{B}_0)^{18l}.$$

So, every nonseparating curve and every outer curve is an element of $\bigcup_{i \in \mathbb{N}} (\mathscr{C} \cup \mathscr{B}_0)^i$.

Let γ be a nonouter separating curve. We can always find sets F_1 and F_2 containing only nonseparating and outer curves, such that $\gamma = \langle F_1 \cup F_2 \rangle$, see Figure 27. By the previous case, $F_1 \cup F_2 \subset (\mathscr{C} \cup \mathscr{B}_0)^k$ for some $k \in \mathbb{N}$; thus $\gamma \in (\mathscr{C} \cup \mathscr{B}_0)^{k+1}$. Therefore, $\mathcal{C}(S) = \bigcup_{i \in \mathbb{N}} (\mathscr{C} \cup \mathscr{B}_0)^i$. □

Now, to prove Lemma 4.9, due to Corollary 4.8, we only need to prove that $\tau^{\pm 1}_{\mathscr{C} \cup \{\zeta\}}(\mathscr{C} \cup \mathscr{B}_0) \cup \eta^{\pm 1}_{\mathscr{H}}(\mathscr{C} \cup \mathscr{B}_0) \subset (\mathscr{C} \cup \mathscr{B}_0)^{12}$. For this, we divide the proof into the following claims.

Claim 1: $\tau^{\pm 1}_{\mathscr{C}}(\mathscr{C}) \subset (\mathscr{C} \cup \mathscr{B}_0)^8$
Claim 2: $\tau^{\pm 1}_{\mathscr{C}}(\mathscr{C} \cup \mathscr{B}_0) \subset (\mathscr{C} \cup \mathscr{B}_0)^{12}$
Claim 3: $\tau^{\pm 1}_{\zeta}(\mathscr{C}) \subset (\mathscr{C} \cup \mathscr{B}_0)^8$
Claim 4: $\tau^{\pm 1}_{\zeta}(\mathscr{C} \cup \mathscr{B}_0) \subset (\mathscr{C} \cup \mathscr{B}_0)^{10}$

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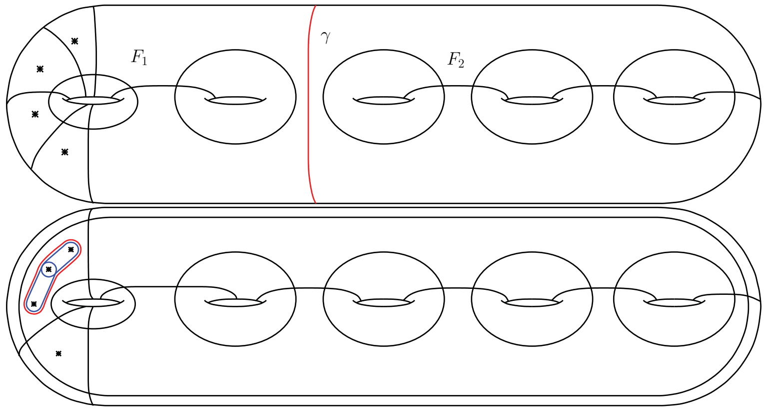

Figure 27. (Colour online) Above, a separating curve γ with every connected component of $S\backslash\{\gamma\}$ of positive genus; below, a separating curve γ with a connected component of $S\backslash\{\gamma\}$ of genus zero, the set F_1 in blue, the set F_2 in black, and γ in red.

Claim 5: $\eta_{\mathscr{H}}^{\pm 1}(\mathscr{C}) \subset (\mathscr{C} \cup \mathscr{B}_0)^7$
Claim 6: $\eta_{\mathscr{H}}^{\pm 1}(\mathscr{C} \cup \mathscr{B}_0) \subset (\mathscr{C} \cup \mathscr{B}_0)^{11}$

4.4. Proof of Claim 1: $\tau_{\mathscr{C}}^{\pm 1}(\mathscr{C}) \subset (\mathscr{C} \cup \mathscr{B}_0)^8$**.** Let $\alpha_j, \alpha_k \in \mathscr{C}$ (taking $\alpha_0 = \alpha_0^i$ when necessary). If $|k-j| > 1$, $i(\alpha_j, \alpha_k) = 0$ and so we have that $\tau_{\alpha_k}^{\pm 1}(\alpha_j) = \alpha_j \in \mathscr{C}$. We then only need to prove for the case when $|k-j| = 1$.

In contrast to the closed surface case, the setwise stabiliser of C_i inside $\mathrm{Mod}^*(S)$, does not act transitively on C_i (it has two orbits). So, here we first prove that $\tau_{\alpha_{2g}}^{\pm 1}(\alpha_{2g-1})$ and $\tau_{\alpha_{2g-1}}^{\pm 1}(\alpha_{2g-2})$ are elements of $(\mathscr{C} \cup \mathscr{B}_0)^8$, then we use the action of a subgroup of $\mathrm{Mod}^*(S)$ to prove Claim 1.

Let $A \subset \mathcal{C}(S)$. We define $E(A) := \{\epsilon \in \mathscr{E} : i(\epsilon, \delta) = 0 \text{ for all } \delta \in A\}$.

Following **[2]**, as in Section 3.2, we prove first that $\tau_{\alpha_{2g}}^{\pm 1}(\alpha_{2g-1}) \in (\mathscr{C} \cup \mathscr{B}_0)^8$.

LEMMA 4.10. $\tau_{\alpha_{2g}}^{\pm 1}(\alpha_{2g-1}) \in (\mathscr{C} \cup \mathscr{B}_0)^8$.

Proof. Taking the set

$$\begin{aligned} C_{1^+} = \{&\alpha_{2g+1}, \alpha_1, \alpha_2, \ldots, \alpha_{2g-4}, \alpha_{2g-2}, [\alpha_{2g-3}, \alpha_{2g-2}, \alpha_{2g-1}]^+, \\ &[\alpha_{2g-4}, \alpha_{2g-3}, \alpha_{2g-2}]^+, [\alpha_1, \ldots, \alpha_{2g-1}]^+\}, \end{aligned}$$

then, by Proposition 4.6, we get that $\gamma_+ := \langle C_{1^+} \cup E(C_{1^+}) \rangle \in (\mathscr{C} \cup \mathscr{B}_0)^7$. See Figure 28.

Letting

$$C'_{1^+} = \{\alpha_1, \ldots, \alpha_{2g-3}, [\alpha_{2g-2}, \alpha_{2g-1}, \alpha_{2g}]^+, [\alpha_{2g-2}, \alpha_{2g-1}, \alpha_{2g}]^-, \gamma_+\},$$

we then have that $\tau_{\alpha_{2g}}(\alpha_{2g-1}) = \langle C'_{1^+} \cup E(C'_{1^+}) \rangle \in (\mathscr{C} \cup \mathscr{B}_0)^8$ (see Figure 29).

As in Section 3.2, let C_- be the set obtained by substituting $[\alpha_{2g-4}, \alpha_{2g-3}, \alpha_{2g-2}]^+ \in C_+$ for $[\alpha_{2g-4}, \alpha_{2g-3}, \alpha_{2g-2}]^-$, $\gamma_- = \langle C_- \cup E(C_-) \rangle$, and C'_- be the set obtained by substituting γ_+ in C'_+ for γ_-. As before, we have that $\tau_{\alpha_{2g}}^{-1}(\alpha_{2g-1}) = \langle C'_- \cup E(C'_-) \rangle \in (\mathscr{C} \cup \mathscr{B}_0)^8$. □

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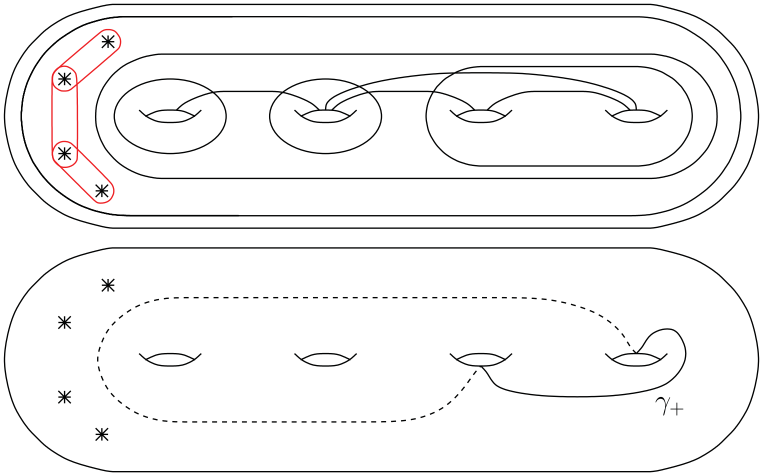

Figure 28. (Colour online) Examples of C_{1^+} and $E(C_{1^+})$ above, and the corresponding curve γ_+ below.

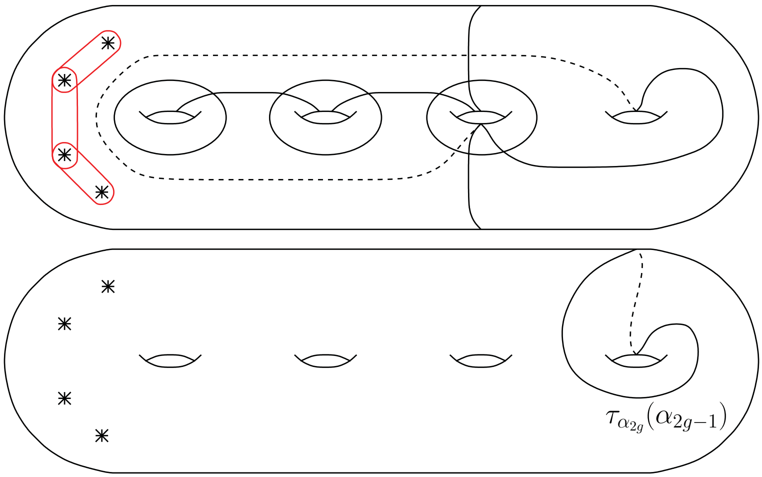

Figure 29. (Colour online) Examples of C'_{1^+} and $E(C'_{1^+})$ above, and $\tau_{\alpha_{2g}}(\alpha_{2g-1})$ below.

To prove that $\tau^{\pm 1}_{\alpha_{2g-1}}(\alpha_{2g-2}) \in (\mathscr{C} \cup \mathscr{B}_0)^8$, we cannot proceed as in the closed case. This is due to the fact that for any choice of i, there are nonhomeomorphic connected components of $S \backslash C_i$. This implies it is possible that there is no homeomorphism that leaves C_i invariant but sends α_{2g} into α_{2g-1}. So, we first prove a proposition for an auxiliary curve used only in this proof, and then follow a method similar to Lemma 4.10.

Proposition 4.11. *Let $k \in \mathbb{Z}$. Then, we have that $[\alpha_{2k}, \ldots, \alpha_{2k+(2g-2)}]^- \in (\mathscr{C} \cup \mathscr{B}_0)^2$ for any choice of $i \in \{0, \ldots, n\}$ (with $\alpha_0 = \alpha_0^i$ when necessary).*

Proof. If $2k \neq 2$ (modulo $2g+2$) and $i \neq n$, then we have (see Figure 30)

$$[\alpha_{2k}, \ldots, \alpha_{2k+(2g-2)}]^- = \langle \{\alpha_0^j : 0 \leq j \leq i\} \cup \{\alpha_{2k}, \ldots, \alpha_{2k+(2g-2)}\} \cup \{\alpha_{2k+2g}\} \cup \{\epsilon^{j,k} : i \leq j < k \leq n\} \rangle.$$

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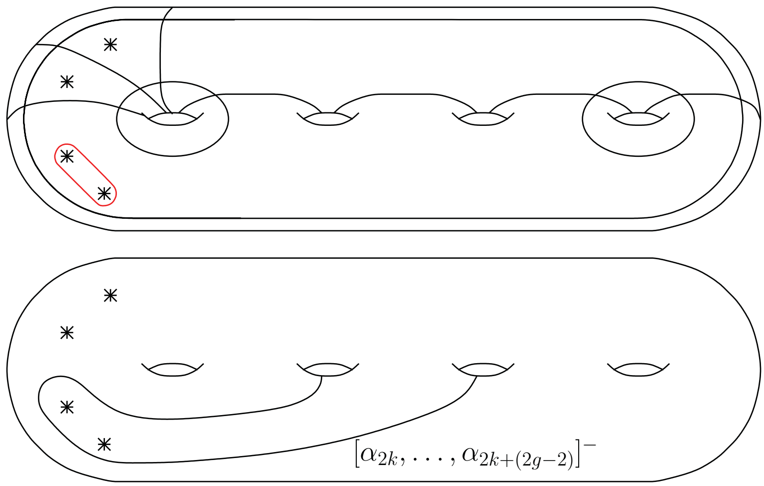

Figure 30. (Colour online) The curve $[\alpha_{2k}, \ldots, \alpha_{2k+(2g-2)}]^- = \langle \{\alpha_0^j : 0 \leq j \leq i\} \cup \{\alpha_{2k}, \ldots, \alpha_{2k+(2g-2)}\} \cup \{\alpha_{2k+2g}\} \cup \{\epsilon^{j,k} : i \leq j < k \leq n\} \rangle$.

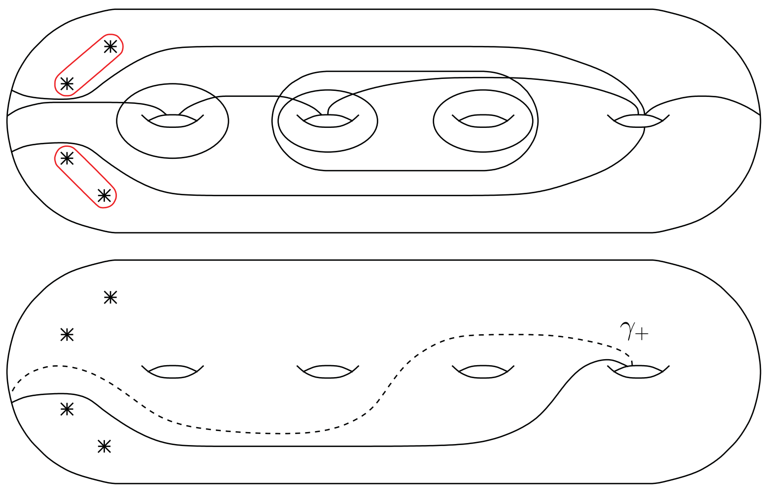

Figure 31. (Colour online) The curve $\gamma_+ := \langle C_{2^+} \cup E(C_{2^+}) \rangle$.

If $2k \neq 2$ (modulo $2g+2$) and $i = n$, then we have $[\alpha_{2k}, \ldots, \alpha_{2k+(2g-2)}]^- = \alpha_{2k+2g}$. If $2k = 2$ (modulo $2g+2$), then $[\alpha_{2k}, \ldots, \alpha_{2k+(2g-2)}]^- = \beta^-_{\{2,\ldots,2g-2\}} = \alpha_0^n$. □

Now, we prove:

LEMMA 4.12. $\tau^{\pm 1}_{\alpha_{2g-1}}(\alpha_{2g-2}) \in (\mathscr{C} \cup \mathscr{B}_0)^8$.

Proof. Let $i \in \{0, \ldots, n\}$. Taking the set

$$C_{2^+} = \{\alpha_{2g}, \alpha_0^i, \alpha_1, \ldots, \alpha_{2g-5}, \alpha_{2g-3}, [\alpha_{2g-4}, \alpha_{2g-3}, \alpha_{2g-2}]^+, [\alpha_{2g-5}, \alpha_{2g-4}, \alpha_{2g-3}]^+, [\alpha_0^i, \ldots, \alpha_{2g-2}]^\pm\},$$

then, by Lemma 4.6, we get that $\gamma_+ := \langle C_{2^+} \cup E(C_{2^+}) \rangle \in (\mathscr{C} \cup \mathscr{B}_0)^7$ (see Figure 31).

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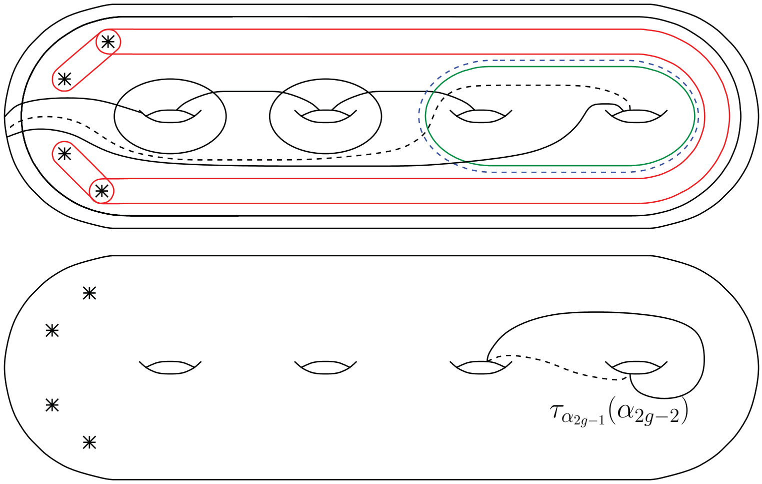

Figure 32. (Colour online) The curve $\tau_{\alpha_{2g-1}}(\alpha_{2g-2}) = \langle C'_{2+} \cup E(C'_{2+})\rangle \in (\mathscr{C} \cup \mathscr{B}_0)^8$.

Letting

$$C'_{2+} = \{\alpha_0^i, \ldots, \alpha_{2g-4}, [\alpha_{2g-3}, \alpha_{2g-2}, \alpha_{2g-1}]^+, [\alpha_{2g-3}, \alpha_{2g-2}, \alpha_{2g-1}]^-, \gamma_+\},$$

we then have that $\tau_{\alpha_{2g-1}}(\alpha_{2g-2}) = \langle C'_{2+} \cup E(C'_{2+})\rangle \in (\mathscr{C} \cup \mathscr{B}_0)^8$ (see Figure 32).

As in the case for $\tau^{-1}_{\alpha_{2g}}(\alpha_{2g-1})$ of Lemma 4.10, substituting the analogous curves, we also have that $\tau^{-1}_{\alpha_{2g-1}}(\alpha_{2g-2}) \in (\mathscr{C} \cup \mathscr{B}_0)^8$. □

Now, let h_i be the mapping class obtained by cutting S along C_i and rotating the resulting (sometimes punctured) discs so that $h_i(\alpha_i) = \alpha_{i+2}$ (with $\alpha_0 = \alpha_0^i$). See Figure 33.

Remark 4.13. Note that $h_i(S^+_{i(o)}) = S^+_{i(o)}$, $h_i(S^-_{i(o)}) = S^-_{i(o)}$, $h_i(S^+_{i(e)}) = S^+_{i(e)}$ and $h_i(S^-_{i(e)}) = S^-_{i(e)}$. Also, $h_i \in \mathrm{stab}_{pt}(E^{i,i})$ and if $k \in \{1, \ldots, g-1\}$, $h_i(\epsilon_k^{(1,n-1)}) = \epsilon_{k+1}^{(1,n-1)}$.

Proof of Claim 1: Using the same arguments as in Section 3.2 and Lemmas 4.5, 4.6, and 4.11, we can precompose by appropriate elements of the group $\langle h_i\rangle$ on C_i to translate of the elements of C_{j+}, C'_{j+} and the corresponding sets for the negative exponents of the Dehn twists, and following the procedure for $\tau^{\pm 1}_{\alpha_{2g}}(\alpha_{2g-1}), \tau^{\pm 1}_{\alpha_{2g-1}}(\alpha_{2g-2}) \in (\mathscr{C} \cup \mathscr{B}_0)^8$, we have that $\tau^{\pm 1}_{\alpha_j}(\alpha_{j-1}) \in (\mathscr{C} \cup \mathscr{B}_0)^8$. Therefore, $\tau^{\pm 1}_{\mathscr{C}}(\mathscr{C}) \subset (\mathscr{C} \cup \mathscr{B}_0)^8$. □

4.5. Proof of Claim 2: $\tau^{\pm 1}_{\mathscr{C}}(\mathscr{C} \cup \mathscr{B}_0) \subset (\mathscr{C} \cup \mathscr{B}_0)^{12}$. Let $\alpha \in \mathscr{C}$ and $\beta \in \mathscr{B}_0$; if $i(\alpha, \beta) = 0$, we have that $\tau^{\pm 1}_\alpha(\beta) = \beta \in \mathscr{C} \cup \mathscr{B}_0$. So, we assume this is not the case. Now, to prove the claim, we first see that every curve in $\mathscr{B}_0$ can be taken to be a curve uniquely determined by a set $C \cup E \cup B$ such that $C \subset \mathscr{C}$, $E \subset \mathscr{E}$, $B \subset \mathscr{B}$, and with every element in B disjoint from α (see Figure 34 for an example, and see **[8]** for a detailed account).

Since $\tau^{\pm 1}_\alpha$ are mapping classes, we get that $\tau^{\pm 1}_\alpha(\beta) = \langle \tau^{\pm 1}_\alpha(C \cup E \cup B)\rangle = \langle \tau^{\pm 1}_\alpha(C) \cup \tau^{\pm 1}_\alpha(E) \cup B)\rangle$. Using the result in Claim 1, we get that $\tau^{\pm 1}_\alpha(C) \subset (\mathscr{C} \cup \mathscr{B}_0)^8$, and by Proposition 4.7 $\tau^{\pm 1}_\alpha(E) \subset (\mathscr{C} \cup \mathscr{B}_0)^{11}$. Therefore, $\tau^{\pm 1}_\alpha(\beta) \in (\mathscr{C} \cup \mathscr{B}_0)^{12}$.

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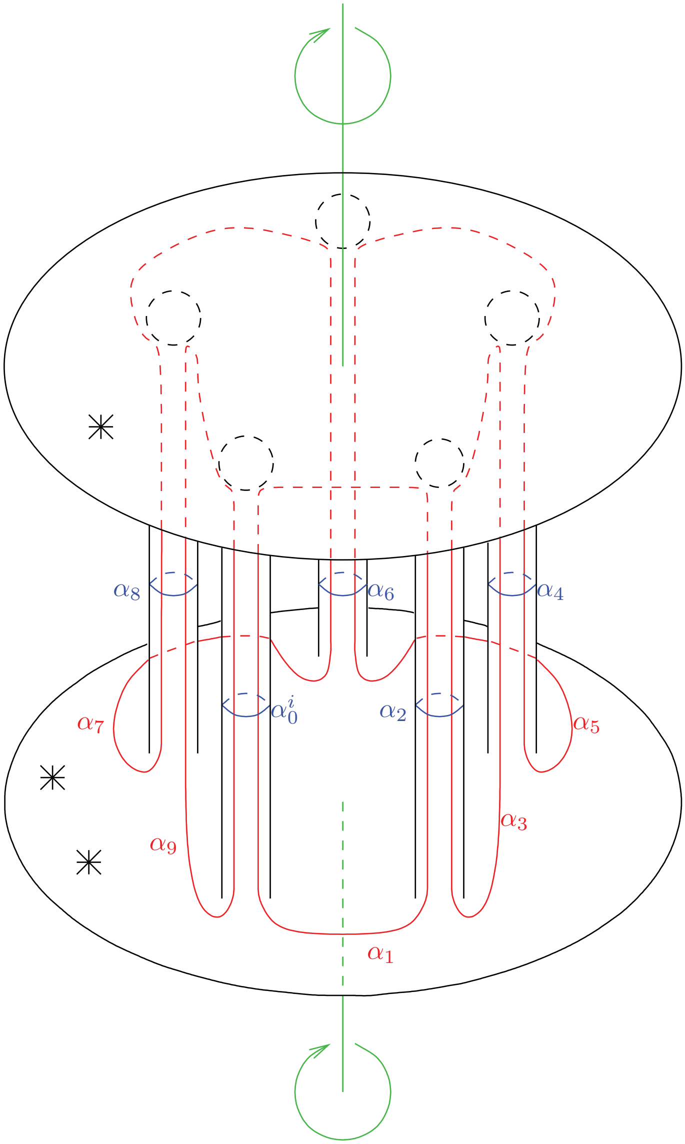

Figure 33. (Colour online) An example of h_i in $S_{4,3}$, for which $h_i(S^+_{i(o)}) = S^+_{i(o)}$, $h_i(S^-_{i(o)}) = S^-_{i(o)}$, $h_i(S^+_{i(e)}) = S^+_{i(e)}$ and $h_i(S^-_{i(e)}) = S^-_{i(e)}$. **In particular**, $h_i(\alpha_j) = \alpha_{j+2}$. Also, $h_i \in \text{stab}_{pt}(E^{i,i})$ and if $k \in \{1, \ldots, g-1\}$, $h_i(\epsilon_k^{(1,n-1)}) = \epsilon_{k+1}^{(1,n-1)}$.

4.6. Proof of Claim 3: $\tau_\zeta^{\pm1}(\mathscr{C}) \subset (\mathscr{C} \cup \mathscr{B}_0)^8$. Recall $\zeta = \beta^+_{\{2g-2,2g-1,2g\}}$, which is disjoint from every element in $\mathscr{C}\backslash\{\alpha_{2g-3}, \alpha_{2g+1}\}$. This implies we only need to prove that $\tau_\zeta^{\pm1}(\alpha_{2g-3}), \tau_\zeta^{\pm1}(\alpha_{2g+1}) \in (\mathscr{C} \cup \mathscr{B}_0)^8$.

We proceed as in Claim 1, using the following ordered maximal closed chain:

$$\gamma_0 = \alpha_0^1, \gamma_1 = \alpha_1, \ldots, \gamma_{2g-5} = \alpha_{2g-5}, \gamma_{2g-4} = \beta^-_{\{2g-4,2g-3,2g-2\}},$$

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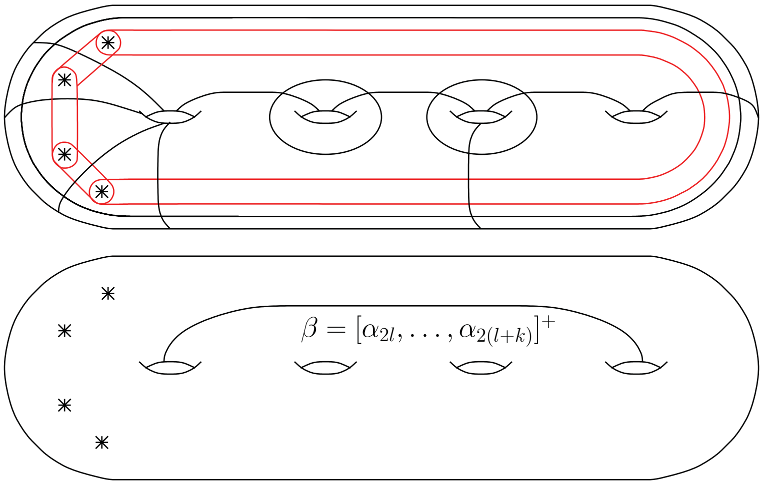

Figure 34. (Colour online) An example of the sets B, C and E, for $\beta = [\alpha_{2l}, \ldots, \alpha_{2(l+k)}]^+$ and α either α_1 or α_{2g-1}.

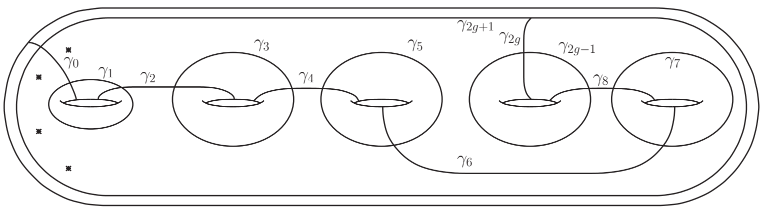

Figure 35. The ordered maximal closed chain for $S = S_{5,4}$, which is used for the case of $\tau_\zeta^{\pm 1}(\alpha_{2g-3})$.

$$\gamma_{2g-3} = \alpha_{2g-1}, \gamma_{2g-2} = \alpha_{2g-2}, \gamma_{2g-1} = \alpha_{2g-3}, \gamma_{2g} = \zeta, \gamma_{2g+1} = \alpha_{2g+1},$$

(see Figure 35) and proving first that $\tau_\zeta(\alpha_{2g-3}) \in (\mathscr{C} \cup \mathscr{B}_0)^8$.

Using the set

$$\begin{aligned} C_{1^+} &= \{\alpha_{2g+1}, \alpha_1, \alpha_2, \ldots, \alpha_{2g-5}, \beta^-_{\{2g-4,2g-3,2g-2\}}, \alpha_{2g-2}, \\ &\quad [\alpha_{2g-3}, \alpha_{2g-2}, \alpha_{2g-1}]^+, \beta^+_{\{2,\ldots,2g-4\}}, [\alpha_1, \ldots, \alpha_{2g-1}]^+\} \\ &\quad \text{for genus } g \geq 4, \\ &= \{\alpha_7, \alpha_1, \alpha_2, \beta^-_{\{2,3,4\}}, \alpha_4, [\alpha_3, \alpha_4, \alpha_5]^+, [\alpha_1, \ldots, \alpha_5]^+\} \\ &\quad \text{for genus } g = 3, \end{aligned}$$

and Propositions 4.5 and 4.6, we obtain the curve $\gamma_+ = \langle C_{1^+} \cup \mathscr{D} \rangle \in (\mathscr{C} \cup \mathscr{B}_0)^7$. Then, using the set

$$C'_{1^+} = \{\alpha_1, \ldots, \alpha_{2g-5}, \beta^-_{\{2g-4,2g-3,2g-2\}}, \alpha_{2g-2}, \alpha_{2g-3}, \zeta, \alpha_{2g}\},$$

we obtain the curve $\gamma'_+ = \langle \mathscr{C}_f \cup C'_{1^+} \rangle \in (\mathscr{C} \cup \mathscr{B}_0)^1$. Finally, using the set

$$C''_{1^+} = \{\alpha_1, \ldots, \alpha_{2g-5}, \beta^-_{\{2g-4,2g-3,2g-2\}}, \alpha_{2g-1}, \gamma'_+, \alpha_{2g}, \gamma_+\},$$

we obtain that $\tau_\zeta(\alpha_{2g-3}) = \langle C''_{1^+} \cup \mathscr{C}_f \rangle \in (\mathscr{C} \cup \mathscr{B}_0)^8$.

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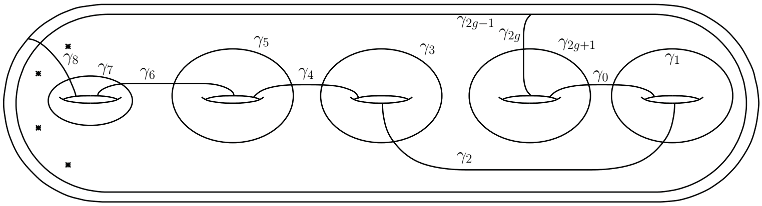

Figure 36. The ordered maximal closed chain for $S = S_{5,4}$, which is used for the case of $\tau_\zeta^{\pm 1}(\alpha_{2g+1})$.

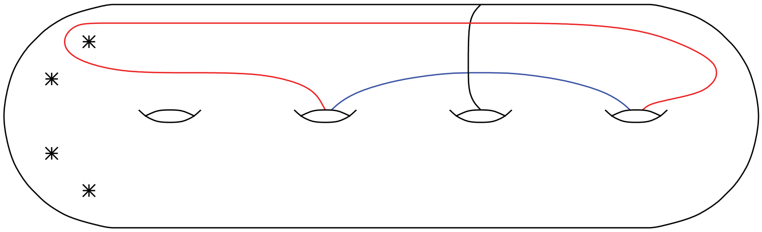

Figure 37. (Colour online) Examples of β for the first case (in blue), the second case (in red), and how they intersect ζ.

For $\tau_\zeta^{-1}(\alpha_{2g-3})$ we proceed analogously, substituting the appropriate curves, getting that $\tau_\zeta^{-1}(\alpha_{2g-3}) \in (\mathscr{C} \cup \mathscr{B}_0)^8$.

To prove that $\tau_\zeta^{\pm 1}(\alpha_{2g+1}) \in (\mathscr{C} \cup \mathscr{B}_0)^8$ we proceed analogously, using the ordered maximal closed chain

$$\gamma_0 = \alpha_{2g-2}, \gamma_1 = \alpha_{2g-1}, \gamma_2 = \beta^-_{\{2g-4,2g-3,2g-2\}}, \gamma_3 = \alpha_{2g-5},$$

$$\gamma_4 = \alpha_{2g-6}, \ldots, \gamma_{2g-4} = \alpha_2, \gamma_{2g-3} = \alpha_1, \gamma_{2g-2} = \alpha_0^1, \gamma_{2g-1} = \alpha_{2g+1}, \\ \gamma_{2g} = \zeta, \gamma_{2g+1} = \alpha_{2g-3}$$

(see Figure 36). Note that this closed chain as a set, is the same closed chain as the previous case but with the order reversed.

We have then $\tau_\zeta^{\pm 1}(\alpha_{2g+1}) \in (\mathscr{C} \cup \mathscr{B}_0)^8$.

4.7. Proof of Claim 4: $\tau_\zeta^{\pm 1}(\mathscr{C} \cup \mathscr{B}_0) \subset (\mathscr{C} \cup \mathscr{B}_0)^{10}$. Given that ζ is disjoint from every curve $\beta \in \mathscr{B}_0$ of the form $\beta^-_{\{2l,\ldots,2(l+k)\}}$ for some $l \in \mathbb{N}$ and $k \in \mathbb{Z}^+$, it follows that $\tau_\zeta^{\pm 1}(\beta^-_{\{2l,\ldots,2(l+k)\}}) = \beta^-_{\{2l,\ldots,2(l+k)\}} \in \mathscr{B}_0$. So, we need to prove the result for $\beta = \beta^+_{\{2l,\ldots,2(l+k)\}}$; we do so dividing into several cases in the following way (See Figure 37 for examples):

1. β is of the form $\beta^+_{\{2l,\ldots,2(l+k)\}}$ for $l < l + k$.
2. β is of the form $\beta^+_{\{2l,\ldots,2(l+k)\}}$ for $l > l + k$.

To prove the claim, as in Claim 2 in Section 4.5, we just have to remember that every curve in $\mathscr{B}_0$ can be taken to be a curve uniquely determined by a set $C \cup E \cup B$ such that $C \subset \mathscr{C}$, $E \subset \mathscr{E}$, $B \subset \mathscr{B}$, and with every element in B either disjoint from ζ, or corresponding to Case 1. See Figures 38 and 39 for examples, and see [8] for a detailed account of these sets.

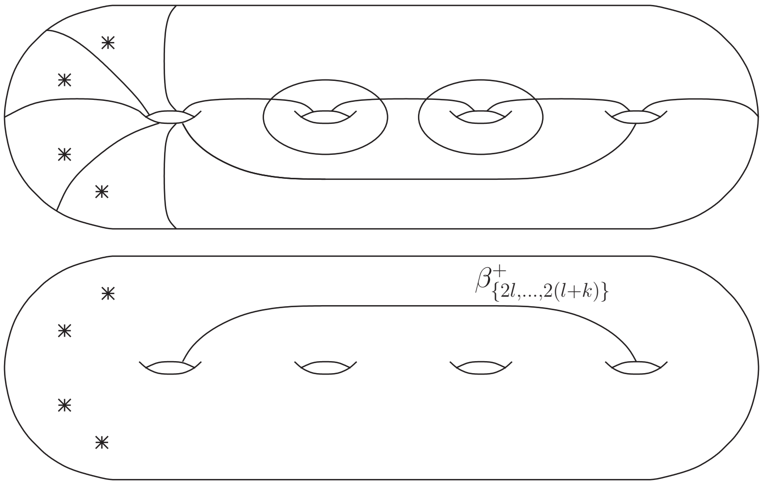

Figure 38. Example of $\beta = \beta^+_{\{2l,\ldots,2(l+k)\}}$ for the case $l < l+k$ with $l = 1$ and $k = 2$.

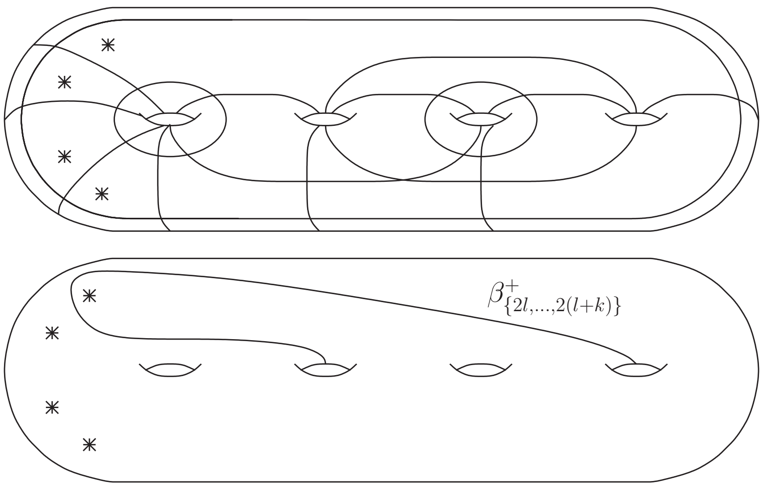

Figure 39. Example of $\beta = \beta^+_{\{2l,\ldots,2(l+k)\}}$ for the case $l > l+k$ with $l = g$.

4.8. Proof of Claim 5: $\eta^{\pm 1}_{\mathscr{H}}(\mathscr{C}) \subset (\mathscr{C} \cup \mathscr{B}_0)^7$. Recall $\mathscr{H} := \{\epsilon^{i-2,i} \in \mathscr{D} : 2 \leq i \leq n\}$. Given that $i(\alpha, \epsilon^{i-2,\,i}) = 0$ for all $\alpha \in \mathscr{C} \backslash \{\alpha_0^{i-1}\}$, to prove the claim, we just need to prove that $\eta^{\pm 1}_{\epsilon^{i-2,\,i}}(\alpha_0^{i-1}) \in (\mathscr{C} \cup \mathscr{B}_0)^7$. We do so following [**2**].

Let $i \in \{1, \ldots, n\}$; we define (see Figure 40)

$$\beta^i = \langle \{\alpha_1\} \cup \{\alpha_3, \ldots, \alpha_{2g+1}\} \cup \{\beta^{\pm}_{\{4,\ldots,2g\}}\} \cup \{\beta^{j,+}_{\{0,1,2\}} : j < i\} \cup \{\beta^{k,-}_{\{0,1,2\}} : i \leq k\}\rangle \in (\mathscr{C} \cup \mathscr{B}_0)^5.$$

Then, for $i \in \{1, \ldots, n\}$, we define (see Figure 41)

$$\gamma^i_+ = \langle \{\alpha_0^0, \alpha_1, \alpha_2\} \cup \{\alpha_4, \ldots, \alpha_{2g}\} \cup \{\beta^{\pm}_{\{4,\ldots,2g\}}\} \cup \{\beta^j : i \neq j\}\rangle \in (\mathscr{C} \cup \mathscr{B}_0)^6,$$
$$\gamma^i_- = \langle \{\alpha_0^n, \alpha_1, \alpha_2\} \cup \{\alpha_4, \ldots, \alpha_{2g}\} \cup \{\beta^{\pm}_{\{4,\ldots,2g\}}\} \cup \{\beta^j : i \neq j\}\rangle \in (\mathscr{C} \cup \mathscr{B}_0)^6.$$

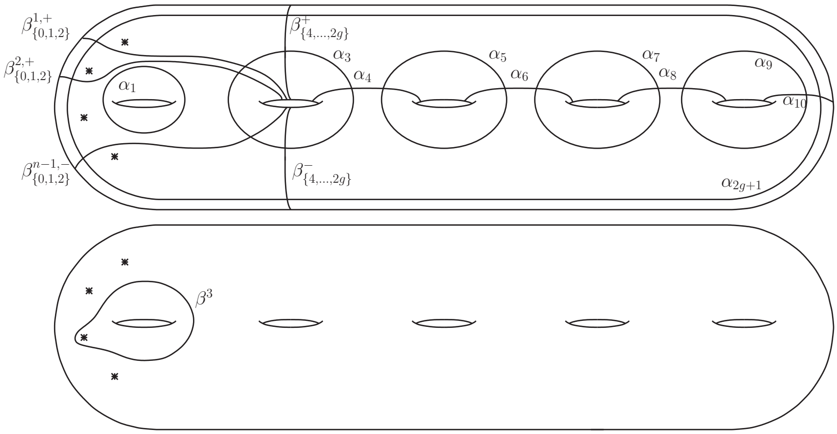

Figure 40. Above, the curves that uniquely determine β^3; below, the curve β^3.

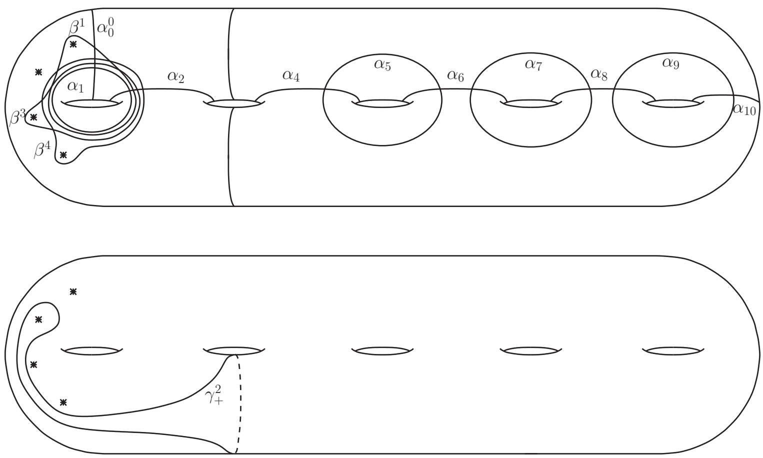

Figure 41. Above, the curves that uniquely determine γ_+^2; below, the curve γ_+^2.

Finally, for $i \in \{2, \ldots, n\}$, we get that $\eta_{\epsilon^{i-2,\,i}}(\alpha_0^{i-1}) = \langle (\mathscr{C}\backslash\{\alpha_{2g+1}, \alpha_0^{i-1}, \alpha_1\}) \cup \{\gamma_+^{i-1}\}\rangle$ and $\eta^{-1}_{\epsilon^{i-2,\,i}}(\alpha_0^{i-1}) = \langle (\mathscr{C}\backslash\{\alpha_{2g+1}, \alpha_0^{i-1}, \alpha_1\}) \cup \{\gamma_-^{i}\}\rangle$ (see Figure 42). Therefore, $\eta^{\pm 1}_{\mathscr{H}}(\mathscr{C}) \subset (\mathscr{C} \cup \mathscr{B}_0)^7$.

4.9. Proof of Claim 6: $\eta^{\pm 1}_{\mathscr{H}}(\mathscr{C} \cup \mathscr{B}_0) \subset (\mathscr{C} \cup \mathscr{B}_0)^{11}$. Let $\beta \in \mathscr{B}_0$, as such it is of the form $\beta^{\pm}_{\{2l,\ldots,2(l+k)\}}$ for some $l \in \mathbb{N}$ and some $k \in \mathbb{Z}^+$.

If $0 < l < l + k$, then β and $\epsilon^{i,\,i+2}$ are disjoint for $i \in \{0, \ldots, n-2\}$. This implies that $\eta^{\pm 1}_{\epsilon^{i,\,i+2}}(\beta) = \beta$.

If $l = 0$ and $k \in \{2, \ldots, g-1\}$, either we have that (see Figure 43)

$$\beta = \left\langle (\mathscr{C}\backslash\{\alpha_0^0, \alpha_{2g+1}, \alpha_{2k+1}\}) \cup \left(\bigcup_{l \in \{k+1,\ldots,g\}} \epsilon_l^{(1,n-1)} \right) \right\rangle,$$

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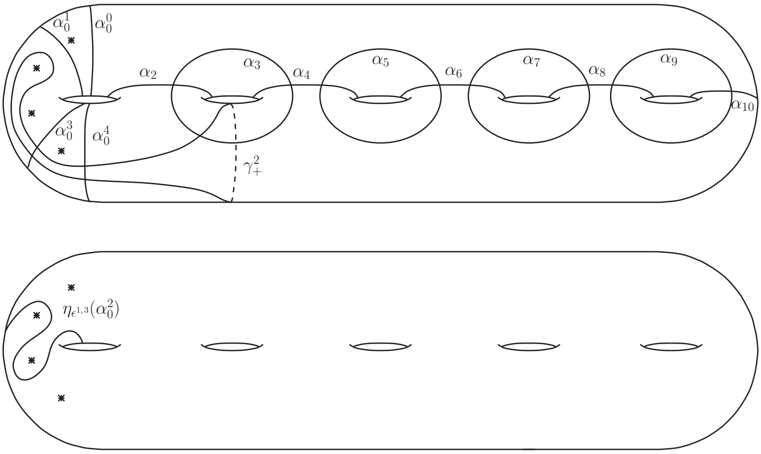

Figure 42. Above, the curves that uniquely determine $\eta_{\epsilon^{1,3}}(\alpha_0^2)$; below the curve $\eta_{\epsilon^{1,3}}(\alpha_0^2)$.

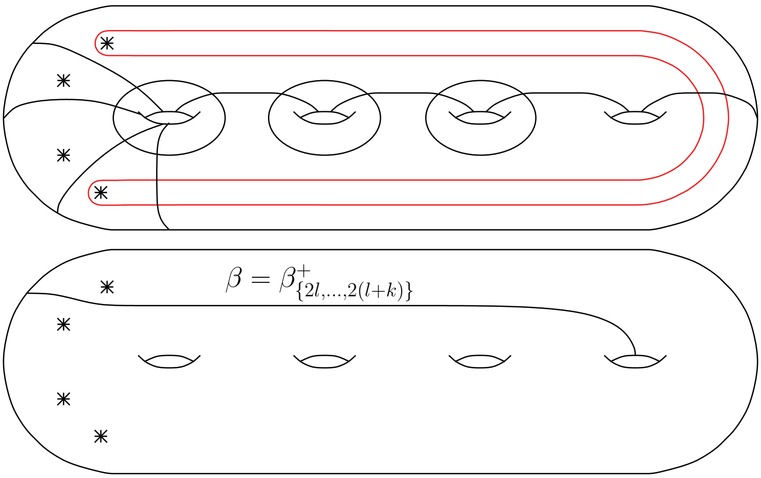

Figure 43. (Colour online) Examples of $\beta = \beta^+_{\{2l,\ldots,2(l+k)\}}$ with $l = 0$ and $k = 3$. The elements in $\mathscr{E}$ are coloured red.

or we have that (see Figure 44)

$$\beta = \left\langle (C_1 \backslash \{\alpha_{2k+1}, \alpha_{2g+1}\}) \cup \{\alpha_0^0\} \cup E^{1,1} \cup \left(\bigcup_{l \in \{k+1,\ldots,g\}} \epsilon_l^{(1,n-1)} \right) \right\rangle.$$

Using Claim 5 and Proposition 4.7, we obtain that $\eta^{\pm 1}_{\epsilon^{i,\,i+2}}(\beta) \in (\mathscr{C} \cup \mathscr{B}_0)^{11}$ for $i \in \{0, \ldots, n-2\}$.

If β is of the form $\beta^{\pm}_{\{2l,\ldots,2(l+k)\}}$ for $l > l + k$, following the proof of Claim 2, we have that $\beta = \langle C \cup E \cup B \rangle$ with $C \subset \mathscr{C}$, $E \subset \mathscr{E}$ and B a singleton of a curve disjoint from $\epsilon^{i,\,i+2}$. Using the result from Claim 5, we get that $\eta^{\pm 1}_{\epsilon^{i,\,i+2}}(C) \subset (\mathscr{C} \cup \mathscr{B}_0)^7$, and

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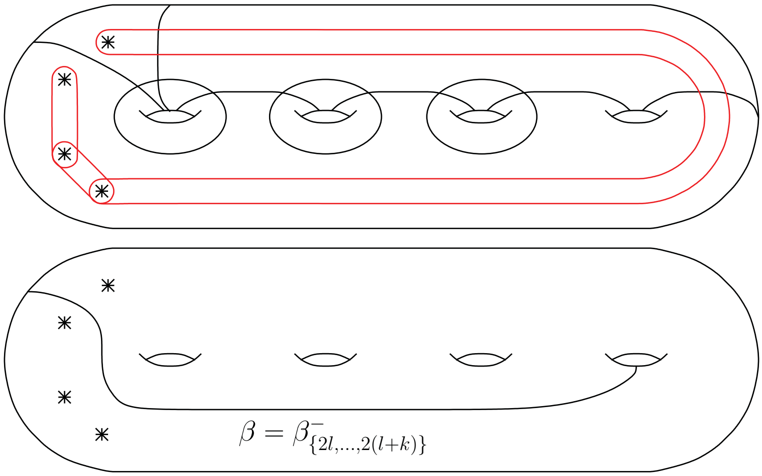

Figure 44. (Colour online) Examples of $\beta = \beta^{-}_{\{2l,\ldots,2(l+k)\}}$ with $l = 0$ and $k = 3$. The elements in $\mathscr{E}$ are coloured red.

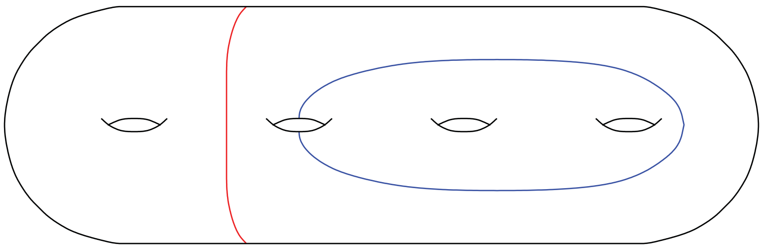

Figure 45. (Colour online) Examples of $\sigma_{\{0,1\}}$ in red, and $\sigma_{\{4,5,6,7\}}$ in blue.

applying Proposition 4.7, we obtain that $\eta^{\pm1}_{\epsilon_{i,\,i+2}}(E) \subset (\mathscr{C} \cup \mathscr{B}_0)^{10}$. Finally, this implies that $\eta^{\pm1}_{\epsilon_{i,\,i+2}}(\beta) \in (\mathscr{C} \cup \mathscr{B}_0)^{11}$, for $i \in \{0, \ldots, n-2\}$.

5. Rigid sets. In this section, we suppose $S = S_{g,n}$ with genus $g \geq 3$, and $n \geq 0$ punctures. Here, we reintroduce the finite rigid set from **[1]** (see Sections 5.1 for the closed surface case and 5.2 for the punctured surface case) and prove Theorem B.

5.1. $\mathfrak{X}(S)$ for closed surfaces. Let $\mathscr{C}$ and $\mathscr{B}$ be as in Section 3, and J be a subinterval (modulo $2g + 2$) of $\{0, \ldots, 2g + 1\}$ such that $|J| < 2g - 1$.

If $J = \{j, \ldots, j + 2k - 1\}$ for some $j \in \mathbb{N}$ and $k \in \mathbb{Z}^+$, we get the following curve:

$$\sigma_J := \langle \{\alpha_j, \ldots, \alpha_{j+2k-1}\} \cup \{\alpha_{j+2k+1}, \ldots, \alpha_{j-2}\} \rangle.$$

For examples, see Figure 45.

This way, we define the following set:

$$\mathscr{S} := \{\sigma_J : J \textit{ is an interval of the form } \{j, \ldots, j + 2k - 1\} \text{ for some } k \in \mathbb{Z}^+\}.$$

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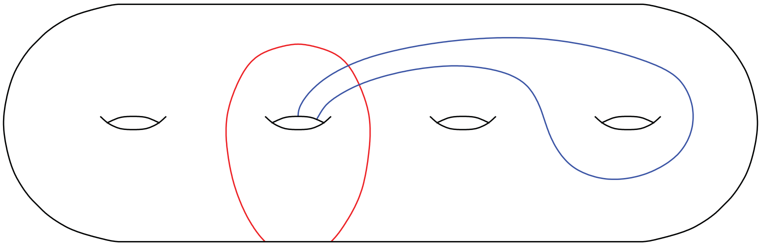

Figure 46. (Colour online) Examples of $\mu_{3,\{0,1,2\}}$ in red, and $\sigma_{7,\{4,5,6\}}$ in blue.

If $J = \{j, \ldots, j+2k\}$ for some $j \in \mathbb{N}$ and $k \in \mathbb{Z}^+$, let we get the following curves:

$$\mu^+_{j+2k+1,J} := \langle \{\beta^+_J, \alpha_{j+2k+1}\} \cup \{\alpha_j, \ldots, \alpha_{j+2k-1}\} \cup \{\beta^-_{\{j+2,\ldots,j+2k+2\}}\} \cup \{\alpha_{j+2k+3}, \ldots, \alpha_{j-2}\}\rangle,$$

$$\mu^-_{j+2k+1,J} := \langle \{\beta^-_J, \alpha_{j+2k+1}\} \cup \{\alpha_j, \ldots, \alpha_{j+2k-1}\} \cup \{\beta^+_{\{j+2,\ldots,j+2k+2\}}\} \cup \{\alpha_{j+2k+3}, \ldots, \alpha_{j-2}\}\rangle.$$

For examples, see Figure 46.

This way, we define the following set:

$$\mathscr{A} := \{\mu^{\pm}_{i,J} : J = \{j, \ldots, j+2k\} \text{ for some } k \in \mathbb{Z}^+, i = j+2k+1\}.$$

Finally, we have the set

$$\mathfrak{X}(S) := \mathscr{C} \cup \mathscr{B} \cup \mathscr{S} \cup \mathscr{A}.$$

Recall that, as was mentioned in Section 1, this set was proved to be rigid in [**1**], and by construction has trivial pointwise stabiliser in $\mathrm{Mod}^*(S)$.

5.2. $\mathfrak{X}(S)$ for punctured surfaces. Let $\mathscr{C}$, $\mathscr{B}_0$, $\mathscr{D}$ and $\mathscr{B}_T$ be as in Section 4.

For $0 \leq i \leq j \leq n$, we denote by $N^{i,j}_1$ and $N^{i,j}_{2g+1}$ closed regular neighbourhoods of the chains $\{\alpha^i_0, \alpha^j_0, \alpha_1\}$ and $\{\alpha^i_0, \alpha^j_0, \alpha_{2g+1}\}$, respectively.

Note that $N^{i,j}_1$ is a two-holed torus if $j - i \geq 1$ (one of the boundary components will be peripheral in S if $j - i = 1$). Also, $S \backslash N^{i,j}_1$ is the disjoint union of a subsurface homeomorphic to an at least once-punctured open disc, and a subsurface homeomorphic to $S_{g-1,n-(j-1)+1}$.

If $j - i > 1$, one of the boundary components of $N^{i,j}_1$ is the curve $\epsilon^{i,j}$. On the other hand, for $0 \leq i \leq j \leq n$, we denote by $\sigma^{i,j}_1$ the boundary component of $N^{i,j}_1$ such that one of the connected components of $S \backslash \{\sigma^{i,j}_1\}$ is homeomorphic to $S_{1,j-i+1}$.

We denote by $\sigma^{i,j}_{2g+1}$ the analogous boundary curves of $N^{i,j}_{2g+1}$ (whenever they are essential).

Then, we define

$$\mathscr{S}_T := \{\sigma^{i,j}_l : l \in \{1, 2g+1\}, 0 \leq i \leq j \leq n\}.$$

Now, let J be a subinterval of $\{0, \ldots, 2g+1\}$ (modulo $2g+2$) such that $|J| \leq 2g$.

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If $J = \{i, \ldots, i + 2k - 1\}$ for some $k \in \mathbb{Z}^+$, let $\sigma_J = [\alpha_i, \ldots, \alpha_{i+2k-1}]$ (with $\alpha_0 = \alpha_0^1$ if necessary). We define

$$\mathscr{S}_0 := \{\sigma_J : J = \{i, \ldots, i + 2k - 1\}, k \in \mathbb{Z}^+\}.$$

If $J = \{2l, \ldots, 2(l + k)\}$, for some $k \in \mathbb{Z}^+$, and $j = 2(l + k) + 1$, then $i(\alpha_j, \beta_J^+) = i(\alpha_j, \beta_J^-) = 1$. Let $\mu_{j,J}^+$ be the boundary curve of a regular neighbourhood of $\{\alpha_j, \beta_J^+\}$. Analogously, let $\mu_{j,J}^-$ be the boundary curve of a regular neighbourhood of $\{\alpha_j, \beta_J^-\}$. We define

$$\mathscr{A} := \{\mu_{j,J}^\pm : J = \{2l, \ldots, 2(l + k)\}, k \in \mathbb{Z}^+, j = 2(l + k) + 1\}.$$

Therefore, we define

$$\mathfrak{X} := \mathscr{C} \cup \mathscr{D} \cup \mathscr{S}_T \cup \mathscr{S}_0 \cup \mathscr{B}_T \cup \mathscr{B}_0 \cup \mathscr{A}.$$

Recall that, as was mentioned in Section 1, this set was proved to be rigid in **[1]**, and by construction has trivial pointwise stabiliser in $\mathrm{Mod}^*(S)$.

5.3. Proof of Theorem B. The set $\mathfrak{X}(S)$ is studied in **[1]** and **[2]**, and it is proven to be a finite rigid set of $\mathcal{C}(S)$ (Theorems 5.1 and 6.1 in **[1]**). Also, by construction, we have that the principal sets used in Sections 3 and 4 ($\mathscr{C} \cup \mathscr{B}$ for closed surfaces, $\mathscr{C} \cup \mathscr{B}_0$ for punctured surfaces) are contained in their respective $\mathfrak{X}(S)$, which gives us the proof of Theorem B.

Proof of Theorem B. Since $\mathscr{C} \cup \mathscr{B} \subset \mathfrak{X}(S)$ for S closed (and $\mathscr{C} \cup \mathscr{B}_0 \subset \mathfrak{X}(S)$ for S a punctured surface), we have that $(\mathscr{C} \cup \mathscr{B})^k \subset \mathfrak{X}(S)^k$ for any $k \in \mathbb{N}$ (analogously $(\mathscr{C} \cup \mathscr{B}_0)^k \subset \mathfrak{X}(S)^k$ for any $k \in \mathbb{N}$). This implies that $\bigcup_{i\in\mathbb{N}}(\mathscr{C} \cup \mathscr{B})^i = \bigcup_{i\in\mathbb{N}} \mathfrak{X}(S)^i$ (analogously $\bigcup_{i\in\mathbb{N}}(\mathscr{C} \cup \mathscr{B}_0)^i = \bigcup_{i\in\mathbb{N}} \mathfrak{X}(S)^i$).

This coupled with Theorem 3.1 and 4.3 gives us the desired result. □

Recalling that in Proposition 3.5 in **[2]**, Aramayona and Leininger prove that the rigid expansions of rigid set are themselves rigid, note that Theorem B gives and alternative proof of Theorem 1.1 in **[2]**.

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