Canad. Math. Bull. Vol. **21** (2), 1978

REGULAR PARTITIONS OF REGULAR GRAPHS

BY
PETER KLEINSCHMIDT

In the study of the combinatorial structure of edge-graphs of convex polytopes one may ask whether a given graph possesses a partition consisting of certain kinds of subgraphs.

In this paper we describe some special partitions of 3-valent and 4-valent graphs. These partitions can serve as examples for a type of partially ordered structures, called *polystromas*, which have recently been considered by Grünbaum [3].

1. **3-valent graphs.** A path of length 3, i.e. a path possessing 3 edges, in the edge-graph of a 3-polytope is called a *Z-path* provided any two adjacent edges of the path lie in one facet, but not all edges are contained in the same facet (a Z-path has the shape of a 'Z').

If the edge-graph of a 3-polytope P is the union of Z-paths, no two of which have an edge in common, we say that P possesses a *Z-partition* (see Fig. 1).

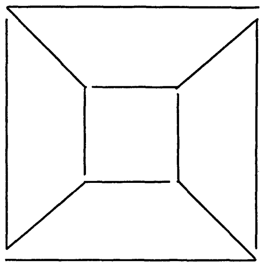

Figure 1

We can prove the following

THEOREM 1. *Every simple* 3-*polytope possesses a* Z-*partition.*

Proof of Theorem 1. Let P be a simple 3-polytope. Then all vertices of P have valency 3. From a theorem of Peterson (see [4], p. 186) it follows that the edge-graph of G of P possesses a 1-factor, i.e. there is a set of pairwise disjoint edges in G which contain all the vertices of P.

Consequently, the remaining edges form a set of disjoint circuits. It is well known that for a set of disjoint circuits on the sphere each circuit may be assigned an orientation so that the following is satisfied: for each region R

Received by the editors August 2, 1976 and, in revised form, October 2, 1976.

determined on the sphere by the circuits, the circuits that form the boundary of R have all the same orientation with respect to R (all clockwise, or else all counterclockwise). Orienting the circuits of G in this way, their edges become directed edges.

Now we describe how to construct the required Z-paths. We consider an edge E in G with vertices v and w, which belongs to the 1-factor of G. v and w lie in oriented circuits C_1 and C_2 respectively. In C_1 there is exactly one edge E_1 with endpoint v which points to the direction of v. Similarly, in C_2 there exists exactly one edge E_2 with endpoint w which points to the direction of w.

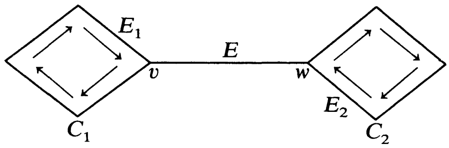

Figure 2

The edges E, E_1 and E_2 form a Z-path (see Fig. 2) whose central edge is E. Taking all edges of the 1-factor as central edges of Z-paths and obtaining the other edges as described for E_1 and E_2 we get a Z-partition of G.

COROLLARY 1. *If G is the edge-graph of a simple 3-polytope, then there exists a 2-to-1 map φ from the set of all the Z-partitions of P to the set of all the 1-factors of G.*
The relation between having a Z-partition and having a 1-factor was hinted in [3].

Proof of Corollary 1. As in the proof of Theorem 1, every 1-factor corresponds to some Z-partition, and by taking a different orientation of the said circuits one gets another Z-partition of P. For an arbitrary Z-partition $\mathfrak{A}$ of P, let $\varphi(\mathfrak{A})$ be the set of all the central edges of the members of $\mathfrak{A}$ (which are paths of length 3).

The central edges of two different members of $\mathfrak{A}$ are disjoint, because otherwise G would have a vertex of valency ≥ 4. Let G have v vertices and e edges, then $3v = 2e$, hence $e = 3(v/2)$, i.e. $\mathfrak{A}$ has precisely $v/2$ elements, therefore $\varphi(\mathfrak{A})$ consists of $v/2$ disjoint edges, hence $\varphi(\mathfrak{A})$ is a 1-factor of G. The rest is obvious.

COROLLARY 2. *Every simple 3-polytope possesses at least six different Z-partitions.*

Proof of Corollary 2. Let P be a simple 3-polytope, and G its edge-graph. G has a 1-factor by a Theorem of Peterson (see [4], p. 186), and since G is 3-connected it has at least three different 1-factors by a theorem of Beineke-Plummer ([1], see also [5], [6] and [8]), It follows by Corollary 1 that P has at least six different Z-partitions. As the tetrahedron has precisely six different Z-partitions, Corollary 2 is best possible. The proof of Theorem 1 can be used to establish the following general result.

COROLLARY 3. *If a* 3-*valent graph G has k different* 1-*factors, then G can be represented in at least* $2k$ *different ways as edge-disjoint union of paths of length* 3.

In this connection, we conjecture that Petersen's Theorem can be strengthened to say that every edge of a bridgeless 3-valent graph G belongs to some 1-factor of G.

2. **4-valent graphs.** If the edges of a path of length 2 in the graph of a 3-polytope lie in one facet, we call this path a *V-path.* If the edges do not lie in a facet, we call it a *C-path.* In analogy to the above definitions we can define *V-partitions* and *C-partitions* of a graph (see Fig. 3)

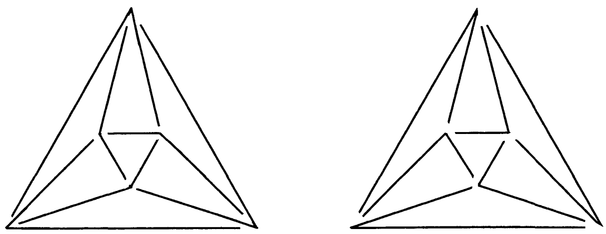

Figure 3

We obtain the following results:

THEOREM 2. *The graph of a polytope whose vertices are all* 4-*valent possesses a V-partition.*

THEOREM 3. *The graph of a polytope whose vertices are all* 4-*valent possesses a C-partition.*

Proof of Theorem 2. A 4-valent graph may be considered as the union of closed geodesic arcs (see [2], p. 239) no two of which have an edge in common. We may choose an arbitrary orientation for each of these arcs. Consequently, every vertex of the graph is contained in precisely two edges whose orientations point to the common vertex. These edges form a V-path and all V-paths obtained in this way build up a V-partition.

Proof of Theorem 3. As in the proof of Theorem 2 we consider the geodesic arcs which build up the graph.

From a theorem of Tait ([7], p. 133) it follows that the vertices of such a graph can be considered as 'bridges', i.e. one of the arcs meeting a vertex

https://doi.org/10.4153/CMB-1978-030-4 Published online by Cambridge University Press

passes 'above' the vertex and the other 'below'. Furthermore, if a geodesic arc passes above one vertex it will pass below the next vertex. Consequently, the two edges which meet at a vertex v and belong to a geodesic arc passing above v form a C-path and all C-paths obtained in this way yield a C-partition.

The proofs of Theorem 2 and Theorem 3 show that there are at least two distinct V- or C-partitions in each case.

Remarks. As a Z-path in a simple polytope corresponds to a Z-path in its dual polytope, it follows from Theorem 1 that every simplicial 3-polytope possesses a Z-partition.

We conjecture that every 3-polytope whose number of edges is divisible by 3 possesses a Z-partition.

Similarly, one might conjecture that there are analoga of Theorem 2 and Theorem 3 for polytopes with an even number of edges. For such an extension of Theorem 3 it has to be assumed that the considered graphs possess no 3-valent vertices.

It should be noticed that every 4-valent connected graph G having an even number of edges is the edge-disjoint union of paths of length two, since G is Eulerian. The same is true if G is a graph such that every connected component of G has an even number of edges and at most two vertices of odd valency.

It would be interesting to find analoga of our results for 5-valent polytopes or polytopes with non-regular graphs. Figure 4 shows a partition of the graph of an icosahedron consisting of paths of length 5, with the property that no consecutive edges of a path lie in one facet.

We conjecture that there is a 5-valent polytope which does not admit such a partition.

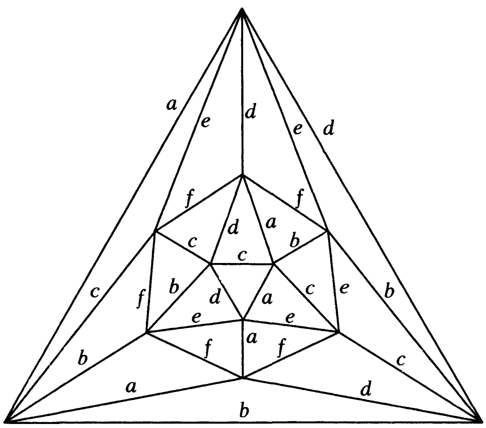

Figure 4
(edges belonging to the same Z-paths are marked by the same symbols)

References

1. Beineke-Plummer, *On the 1-factors of a non-separable graph*, J. Combinatorial Theory **2** (1967), 285–289.
2. B. Grünbaum, *Convex Polytopes*, New York 1967.
3. B. Grünbaum, *Regularity of graphs, complexes and designs*, to appear.
4. D. König, *Theorie der endlichen und unendlichen Graphen*, New York 1950.
5. W. Mader, *Über die Anzahl der von den 1-Faktoren eines Graphen überdeckten Ecken*, Math. Nachr. **56** (1973), 195–200.
6. W. Mader, *1-Faktoren von Graphen*, Math. Ann. **201** (1973), 269–282.
7. P. G. Tait, *Some elementary properties of closed plane curves*, Messenger Math. **(2) 6** (1877), 132–133.
8. J. Zaks, *On the 1-factors of n-connected graphs*, J. Combinatorial Theory **B** (1971), 169–181.

Mathematisches Institut der
Ruhr-Universitat
4630 Bochum. West Germany