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GAUSSIAN PROCESSES WITH MARKOVIAN COVARIANCES

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ABSTRACT. We show that any Gaussian process can be derived in a simple manner from a Markov process if it has zero mean and covariance identical to the covariance of a real valued function of a temporally homogeneous Markov process.

Suppose that M_T, T being either the nonnegative integers or the nonnegative real numbers, is a temporally homogeneous Markov process on a measurable space (S, Σ) with initial distribution $P(\cdot)$ and transition probability function $P_t(\cdot, \cdot)$. Let f be a mapping of $T\times S$ into the real numbers which is square integrable with respect to the measure $P_t(a, \cdot)$ for each $t \in T$ and $a \in S$.

Suppose now that X_T is a real Gaussian process with zero expectations and covariance

$$\Gamma_{st} = \int_S P(du)\int_S f(s, v)P_s(u, dv)\int_S f(t-s, w)P_{t-s}(v, dw)$$

identical to the covariance of $f(t, M_t)$, $t \in T$. Let $X_T^*(\Sigma)$ be the generalized Gaussian random field (see [2]) on $T\times\Sigma$ with zero expectations and covariance function

$$\Gamma_{st}^*(U, V) = \int_S P(du)\int_U P_s(u, dv)\int_V P_{t-s}(v, dw)$$

identical to the covariance of the random field

$$I_U(M_t), \quad t \in T, \quad U \in \Sigma$$

where I_U is the indicator function of U. Then if $F(\Sigma)$ is the set of all functions mapping Σ into the real numbers we have the following

THEOREM. *Under the above conditions X_T^* is a Markov process on $F(\Sigma)$ and*

$$X_T = \int_S f(t, u)X_t^*(du)$$

where the integral is the Wiener-Ito stochastic integral.

Proof. Since conditioning and projections are the same for a Gaussian field (see [1] or [3]), it follows that X_T is a Markov process on $F(\Sigma)$ if for each s and

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t in T and $V\in\Sigma$, the projection $E_2(X^*_{s+t}(V)\mid X^*_s(U), U\in\Sigma)$ of X^*_{s+t} on the closed linear span of the functions $X^*_s(U)$, $U\in\Sigma$ is equal to the projection $E_2(X^*_{+t}(V)\mid X^*_r(U), r\leq s, U\in\Sigma)$ of X^*_{s+t} on the closed linear span of the functions $X^*_r(U)$, $r\leq s$ and $U\in\Sigma$. To do this we show that for each $V\in\Sigma$,

$$E_2(X^*_{s+t}(V)\mid X^*_s(U), U\in\Sigma)=\int_S P_t(u, V)X^*_s(du)$$
$$= E_2(X^*_{s+t}(V)\mid X^*_r(U)\ r\leq s, U\in\Sigma)$$

That is, we show that for each $r\leq s$, $U\in\Sigma$ and $V\in\Sigma$,

$$X^*_r(U)\perp X^*_{s+t}(V)-\int_S P_t(w, V)X^*_s(dw).$$

But

$$EX^*_r(U)\left[X^*_{s+t}(V)-\int_S P_t(w. V)X^*_s(dw)\right]$$
$$= EX^*_r(U)X^*_{s+t}(V)-\int_S P_t(w, V)EX^*_r(U)X^*_s(dw)$$
$$=\int_S P(du)\int_U P_r(u, dv)\int_V P_{s+t-r}(v, dw)-\int_S P_t(w, V)$$
$$\times\int_S P(du)\int_U P_r(u, dv)P_{s-r}(v, dw)$$
$$=\int_S P(du)\int_U P_r(u, dv)P_{s+t-r}(v, V)-\int_S P(du)\int_U P_r(u, dv)$$
$$\times\int_S P_{s-r}(v, dw)P_t(w, V)=\int_S P(du)\int_U P_r(u, dv)P_{s+t-r}(v, V)$$
$$-\int_S P(du)\int_U P_r(u, dv)P_{s+t-r}(v, V)=0$$

and so X^*_T is a Markov process on $F(\Sigma)$.

To complete the proof, we need only show that

$$X_t=\int_S f(t, u)X^*_t(du),\qquad t\in T.$$

Clearly both X_t and $\int_S f(t, u)X^*_t(du)$ are Gaussian with zero expectation. To show that they are equal in distribution we need only show that their covariances are

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equal. Since the covariance of X_t is Γ_s^t and since

$$E\int_S f(s, v)X_s^*(dv)\int_S f(t, w)X_t^*(dw)$$

$$= \int_S\int_S f(s, v)f(t, w)EX_s^*(dv)X_t^*(dw)$$

$$= \int_S\int_S f(s, v)f(t, w)\int_S P_s(du)P(u, dv)P_{t-s}(v, dw)$$

$$= \int_S P(du)\int_S f(s, v)P_s(u, dv)\int_S f(t, w)P_{t-s}(v, dw) = \Gamma_{st}$$

the theorem is proved.

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