MINIMALLY STRONG DIGRAPHS

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(Received 23rd December 1968)

Dirac (**2**) and Plummer (**5**) independently investigated the structure of minimally 2-connected graphs G, which are characterized by the property that for any line x of G, $G-x$ is not 2-connected. In this paper we investigate an analogous class of strongly connected digraphs D such that for any arc x, $D-x$ is not strong. Not surprisingly, these digraphs have much in common with the minimally 2-connected graphs, and a number of theorems similar to those in (**2**) and (**5**) are proved, notably our Theorems 9 and 12.

Unless otherwise noted, all definitions for digraphs given here are from (**4**). Definitions for graphs are not given, and can be found in (**3**). A *digraph* is an ordered pair $D = (V, X)$ where V is a finite set of *points* and X is a set of ordered pairs of distinct points; elements of X are called *arcs*. If $x = uv$ is an arc then x *joins* u to v: we also say that u is *adjacent to* v and that v is *adjacent from* u. Following Berge (**1**) the set of points adjacent from u is denoted Γu and the set of points adjacent to u is $\Gamma^{-1}u$. We call $|\Gamma^{-1}u|$ the *indegree* id(u) and $|\Gamma u|$ the *outdegree* od(u). The *degree* of u is id(u)+od(u). A *symmetric pair* (uv) is a pair of arcs uv and vu. An *oriented digraph* has no symmetric pairs. With each digraph D we can associate the "underlying" graph $G = G(D)$ by letting G have the same point set as D and joining u and v by a line if D has at least one of the arcs uv or vu.

A *semiwalk* (called "semisequence" in (**4**)) is a sequence of points and arcs $u_0x_0u_1x_1 \dots x_{n-1}u_n$ such that for each x_i either $x_i = u_iu_{i+1}$ or $x_i = u_{i+1}u_i$; a semiwalk is *spanning* if it contains all the points of D, and *closed* if $u_0 = u_n$. If all the points (and hence all the arcs) of a semiwalk are distinct we have a *semipath*. A semiwalk for which $u_0 = u_n$ but all other points are distinct is a *semicycle*. A *walk* from u_0 to u_n (a u_0-u_n walk) is a semiwalk $u_0x_0 \dots x_{n-1}u_n$ in which, for each i, $x_i = u_iu_{i+1}$; *path* and *cycle* are defined analogously. It is clear that any u_0-u_n walk contains a u_0-u_n path. A *pseudocycle* is a semicycle consisting of a path from u_0 to u_n together with the arc u_0u_n. A cycle with three points is a *triangle*. If D has a symmetric pair (uv) and W is a walk containing either uv or vu we will say that W contains (uv).

A digraph D is *weakly connected*, or *weak*, if $G(D)$ is connected; it is *unilaterally connected*, or *unilateral*, if for each pair u, v of points either there is a walk from u to v or there is a walk from v to u; it is *strongly connected*, or *strong*, if for each pair u, v of points there is both a walk from u to v and a walk

† Research supported by the Chemical Information Division of the National Science Foundation under Grant Number N-566.

from v to u. A digraph D with at least 3 points is a *block* if $G(D)$ is a 2-connected graph; if D is not a block then a *cutpoint* of D is a cutpoint of $G(D)$. An arc x of D is *basic* if there are points w_1 and w_2 such that every w_1-w_2 walk contains x; in particular, $x = uv$ is basic if and only if every $u-v$ walk contains x.

A digraph D is *minimally strong* if for each $x \in X$, $D-x$ is not strong. It is clear that a digraph is minimally strong only if each of its blocks is, so we need essentially concern ourselves only with investigating minimally strong blocks. We will first develop some basic results about strong and minimally strong digraphs. Theorems 1 and 2 appeared first in (**4**).

Theorem 1. *A digraph is unilateral if and only if it has a spanning walk, and strong if and only if it has a closed spanning walk.*

Theorem 2. *If D is a strong digraph and w is a point of D for which $D-w$ is not unilateral, then there are two points u and v in D such that each $u-v$ walk and each $v-u$ walk contains w.*

If w separates u and v as in Theorem 2 then, following (**4**), we say that w is 3-between u and v.

Theorem 3. *A strong block D with at least four points has at least two points u_1 and u_2 such that each $D-u_i$ is unilateral.*

Proof. Since D is a block it has no cutpoint; thus for each v, $D-v$ is weak. Suppose that there is some v for which $D-v$ is strictly weak, and let v be 3-between w_1 and w_2. If W is a shortest closed spanning walk in D then each w_i lies on a cycle Z_i which is a subwalk of W; since v can appear only once on a cycle, the cycles Z_1 and Z_2 are distinct. Each Z_i has a point u_i which appears only once in W, for otherwise a shorter closed spanning walk could be obtained from w by simply ignoring Z_i. But then clearly each $D-u_i$ has a spanning walk, and is thus unilateral.

As a corollary to the theorem we obtain a result from (**4**).

Corollary 3a. *Any strong block D with at least four points has at least four arcs x_i such that each $D-x_i$ is unilateral.*

Lemma 4. *A digraph is minimally strong if and only if each arc is basic.*

Corollary 4a. *No minimally strong digraph contains a pseudocycle.*

Corollary 4b. *If D is minimally strong then so is every strong subdigraph of D.*

The next corollary follows from this lemma and the theorem of Robbins (**6**) that for any 2-connected graph G there is a strong block D such that $G = G(D)$.

Corollary 4c. *If G is a minimally 2-connected graph there is a minimally strong digraph D such that $G = G(D)$.*

Figure 1 shows the smallest minimally strong digraph for which the converse of Corollary 4c fails to hold.

We will next give a procedure for constructing a large class of minimally strong blocks. We will see later that a closely related procedure in fact serves to construct all minimally strong blocks. Let D_1 and D_2 be digraphs with arcs $x_1 = u_1v_1 \in D_1$ and $x_2 = u_2v_2 \in D_2$, and let $D_1 \uparrow D_2$ be the digraph formed from $D_1 - x_1$ and $D_2 - x_2$ by identifying u_1 with u_2 to get point u and v_1 with v_2 to get point v and then adding arc $x = uv$.

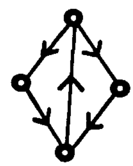

FIG. 1

Theorem 5. *If D_1 and D_2 are minimally strong blocks, neither of which contains a symmetric pair, then for any choice of $x_i \in D_i$, $D_1 \uparrow D_2$ is a minimally strong block.*

Proof. It is obvious that $D_1 \uparrow D_2$ is a block. To see that $D_1 \uparrow D_2$ is strong we must show that for any choice of points w_1 and w_2 there are $w_1 - w_2$ and $w_2 - w_1$ walks. There is certainly no difficulty if w_1 and w_2 are both points of D_1 or both points of D_2. Thus, without loss of generality, choose $w_1 \in D_1$ and $w_2 \in D_2$. But then since there is a $w_1 - u_1$ walk W_1 in D_1 and a $v_2 - w_2$ walk W_2 in D_2, $W_1 \times W_2$ is a $w_1 - w_2$ walk in $D_1 \uparrow D_2$. By symmetry, $D_1 \uparrow D_2$ is strong.

To see that $D_1 \uparrow D_2$ is minimally strong it suffices to show that each arc is basic. If x is not basic then there is a $u - v$ walk, and hence a $u - v$ path, in $D_1 \uparrow D_2$ which avoids uv. Since any walk from a point strictly in D_1 to one strictly in D_2 must contain either u or v, a $u - v$ path which avoids uv must lie completely in either D_1 or D_2, say, D_1. But then u_1v_1 is not basic in D_1, a contradiction. Suppose then that $y = w_1w_2$ is any other line of $D_1 \uparrow D_2$ and suppose that y is not basic. We may suppose that y is a line of D_1. Since y is basic in D_1, each $w_1 - w_2$ path P which avoids y must contain points of D_2. But then P must also contain both u and v. If u precedes v on P then replacing the subpath from u to v by uv we have a y-avoiding $w_1 - w_2$ path in D_1, which is impossible. Thus v precedes u on P. But even so, since D_1 has a $v - u$ path we also arrive at a contradiction unless every $v - u$ path (and hence every $v - u$ walk) contains w_1w_2. If so let $P' = vP_1w_1w_2P_2$ be such a $v - u$ path and note that the path formed by following vP_1w_1 by P contains a $v - u$ walk which does not contain w_1w_2. Thus $D_1 \uparrow D_2$ is minimally strong. Notice that if, say, D_1 had contained the symmetric pair (u_1v_1) then since there is a $u_2 - v_2$ path P in D_2, P together with vu forms a pseudocycle in $D_1 \uparrow D_2$.

If D is a digraph and $x = uv$ is an arc of D then the procedure of replacing arc x by a new point w and arcs uw and wv is called *insertion of a point of degree*

2. If D' is obtained from D by repeated insertion of points of degree 2 we say that D' is a *subdivision* of D.

Lemma 6. *If D is minimally strong then so is every subdivision of D.*

In particular the following procedure will thus construct only minimally strong blocks.

Step 1. Let D_1 be a triangle and form $D_2 = D_1 \uparrow D_1$.

Step 2. Let D_2' be any subdivision of D_2 and form $D_3 = D_2' \uparrow D_1$.

Step 3. Let D_n' be any subdivision of D_n and form $D_{n+1} = D_n' \uparrow D_1$.

Although Theorem 5 excluded symmetric pairs from the digraphs D_1 and D_2, we see from the proof that it was only necessary to insure that the points at which D_1 and D_2 were " merged " were not symmetrically adjacent; otherwise, as long as D_1 and D_2 were minimally strong $D_1 \uparrow D_2$ would be. We will now show that the only minimally strong block which contains a symmetric pair has exactly two points.

Theorem 7. A minimally strong block D with at least three points contains no symmetric pair.

Proof. Let (uv) be a symmetric pair in D and consider any choice apart from u and v themselves of $u_1 \in \Gamma^{-1}u$, $u_2 \in \Gamma u$, $v_1 \in \Gamma v$, $v_2 \in \Gamma^{-1}v$. We claim first that if each path between the u_i and the v_i (in either direction) contains (uv) then u is a cutpoint.

To see this we first note that any path from either u_1 or u_2 to either of v_1 or v_2 containing (uv) must contain uv, and that any of the converse paths from some v_i to some u_i which contain (uv) contain vu. For example, if a $u_2 - v_1$ path contained only vu then the path formed from P be prefixing uu_2 would contain a $u-v$ walk which avoided uv, in which case uv would not be basic. Similar arguments will serve for the other cases.

It now follows that D contains two cycles Z_1 and Z_2 such that $Z_1 \cap Z_2 = \phi$, $Z_1 \cap (uv) = v$, $Z_2 \cap (uv) = u$. Since D is a block we can find a semipath which connects some point of Z_1 with some point of Z_2 and which avoids (uv). Choose a shortest such semipath and label its points $w_1, w_2, \ldots, w_n$, with $w_1 \in Z_1$, $w_n \in Z_2$ (see Figure 2 for an illustration of this situation).

Clearly there is $w_1 - v$ path which does not contain u. Then each $w_1 - v$ path must avoid u or else vu is not basic. Similarly, since uv is basic, every $v - w_1$ path avoids u. Suppose then that each $v - w_i$ path and each $w_i - v$ path avoids u. We show that the same must hold for w_{i+1}.

Case I: *Arc $w_{i+1}w_i$ in D.* If some $w_{i+1} - v$ path contains u there is a $w_{i+1} - w_i$ walk which does not contain $w_{i+1}w_i$, so that this arc is not basic. If some $v - w_{i+1}$ path contains u then the existence of a $w_i - v$ path which avoids u implies that of a $u - w_{i+1} - w_i - v$ walk not containing uv, which thus cannot be basic.

Case II: *Arc* $w_i w_{i+1}$ *in* D. If a $w_{i+1}-v$ path contains u then since each $v-w_i$ path misses u it follows that vu is not basic. Similarly, if a $v-w_{i+1}$ path contains u then since there is a w_i-v path which avoids u, $w_i w_{i+1}$ is not basic.

It then follows by induction that there is a $v-w_n$ path which avoids u, and thus a $v-u$ path which avoids (uv), so that vu is not basic. Hence u is a cutpoint. Therefore, since D is a block, some path from one of the u_i to one of the v_i (or vice versa) misses (uv). If the path is u_2-v_2, v_1-u_2, u_1-v_2, or v_1-u_1, we get a pseudocycle and thus some line of D cannot be basic. If it is a u_1-v_1 path (or equivalently a v_2-u_2 path) note that D also contains a v_1-u_1 path P. If P uses vu then uv is not basic. But since it cannot use uv it must then avoid (uv), in which case the path defined by vPu shows that vu is not basic.

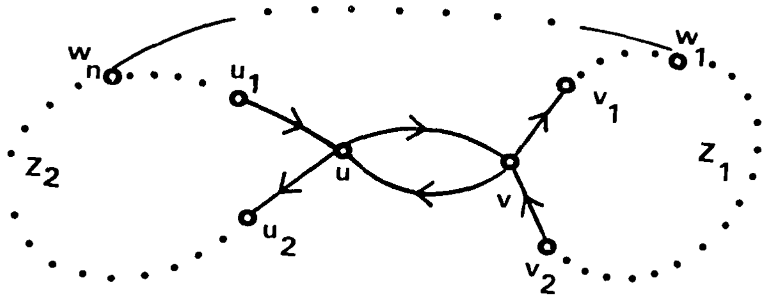

FIG. 2

It follows from this theorem that although some subdivision of a strong digraph D may be minimally strong, it is not necessarily true that D is. However, there are often many points of degree 2 which can be suppressed. To see this it is first necessary to show that a minimally strong block has points of degree 2.

Lemma 8. *Let u be any point of a minimally strong digraph with degree at least* 3. *Then $D-u$ is not unilateral.*

Proof. It is clearly impossible for either the indegree or the outdegree of u to be 0. Suppose first that $\text{id}(u) \geqq 2$ and let $u_1, u_2 \in \Gamma^{-1}u$. Suppose that $D-u$ is unilateral. Then there is a path P from, say, u_1 to u_2 which avoids u. But then P together with arcs u_1u and u_2u forms a pseudocycle in D, which is impossible. If $\text{od}(u) \geqq 2$ let $u_1, u_2 \in \Gamma u$ and note in a similar fashion that if there is, say, a u_1-u_2 path which avoids u then arc uu_2 is not basic.

Theorem 9. *Every minimally strong block with at least four points has at least two points of degree* 2.

Proof. There are certainly no points of degree 1. By the lemma, for each point v with degree greater than 2, $D-v$ is not unilateral. But Theorem 3 assures us of the existence of at least two points whose removal from D leaves a unilateral digraph, so that these points must have degree 2.

As we saw above it is a corollary of Theorem 3 that in every minimally strong block there are some arcs whose removal leaves a unilateral digraph. It is interesting to note at this point that Harary, Norman, and Cartwright (**4**, p. 260) have a condition for every arc of a minimally strong block to have this property: If D is a minimally strong block then for each arc x, $D-x$ is unilateral if and only if for each pair of points u and v, whenever there is an arc x_1 which is in every $u-v$ path, there is an arc x_2 in every $v-u$ path. Notice that the minimally strong block D of Figure 1, which has an arc uv such that $D-uv$ is not unilateral also has the property that there is no arc which lies in every $v-u$ path.

Let D be a minimally strong block and let $w_1u_1u_2...u_nw_2$ be a path in D such that each u_i has degree 2. Let θ be a contractive map which identifies all of the u_i to a single point u of degree 2, and let the image of D under θ be denoted D/θ. Clearly D/θ is also a minimally strong block. In fact, for any minimally strong block D the *reduced digraph* $D/$ formed by contracting each path consisting of points of degree 2 to a single point of degree 2 is likewise a minimally strong block.

Corollary 9a. *Every minimally strong block has two points of degree* 2 *which are separated from each other by points of higher degree.*

If we consider a reduced minimally strong block E we can suppress any point u of degree 2 by replacing the arcs from $\Gamma^{-1}u$ to u and from u to Γu by a single arc from $\Gamma^{-1}u$ to Γu. Call the resulting digraph E/u. It is not always the case, as it is with the digraph of Figure 1, that such a digraph E/u cannot be minimally strong.

Theorem 10. *If E is a reduced minimally strong block and u is a point of degree* 2 *then E/u is minimally strong if and only if $E-u$ is strictly unilateral.*

Proof. Clearly if E/u is minimally strong then $E-u$ cannot be strong. To see that $E-u$ is unilateral it is only necessary to note that E has a closed spanning walk which can contain u only once so that $E-u$ must have a spanning path.

For the converse note first that E/u must be strong. Thus if E/u is not minimally strong then the arc x from $\Gamma^{-1}u$ to Γu to Γu is not basic and thus $E/u-x$ is strong. But $E/u-x \cong E-u$, which completes the proof.

If E is a reduced minimally strong block call a point u of degree 2 such that E/u is minimally strong an *essential point*. It is clear that the digraph $E/$ formed by suppressing all essential points is also a minimally strong block. It is not hard to see that $E/$ has no essential points. But by Theorem 3, $E/$ has two points u_i of degree 2 such that each $E/-u_i$ is unilateral. Thus each $E/-u_i$ is strong, and hence minimally strong. Further, note that if $E/-u_i$ is unilateral then so is $E-u_i$. But since it is not possible for u_i to be an essential point of E, it must be true that $E-u_i$ is strong.

Theorem 11. *If E is a reduced minimally strong block there are at least two points u_i of degree* 2 *such that $E-u_i$ is minimally strong.*

Suppose that u is such a point in E and let v_1u and uv_2 be arcs of E such that $v_2v_1 \notin E$. Then consider any v_1-v_2 path in $E-u$, and let v_1w be the first arc on the path. If some v_2-w path failed to contain v_1 then there would be, in E, a v_1-w walk which avoided v_1w, so that v_1w would not be basic. Thus each v_2-w path contains v_1: whenever v_1 and v_2 are joined by at least two paths $v_1w_iP_iv_2$ such that each v_2-w_i path contains $<v_1$, we say that v_1 and v satisfy condition (α).

If we remove points of degree 2 from minimally strong blocks in such a way as to leave minimally strong digraphs then we will eventually arrive at a digraph E_1 which is either a triangle, a minimally strong block which is not reduced, or a minimally strong digraph which is not a block. In the second case we continue with $E_1/$ while in the third case we continue on the strong blocks of E_1 separately. The process will terminate in one or more triangles. We have thus proved:

Theorem 12. *Any minimally strong block can be obtained by starting with triangles and at each step applying one of the following operations to any minimally strong blocks D_i produced thus far:*

(i) *Subdivision.*

(ii) *Choose u and v such that $uv \notin D_1$ but either $vu \in D_1$ or u and v satisfy (α) and add a new point w together with arcs uw and wv.*

(iii) *Let $D_1, \ldots, D_n$ be minimally strong blocks produced by the procedure and choose a pair of distinct points u_{i-1} and v_i in each D_i, making sure that u_0 and v_1 satisfy (α). To the minimally strong digraph formed by identifying each u_j with each v_j add a new point w and arcs u_0w and wv_n.*

Corollary 12*a* follows immediately from this procedure, and Corollary 12*b*, which also involves Corollary 4*c*, is a weak form of a result which appears in both (**2**) and (**5**).

Corollary 12a. *For each minimally strong block D the chromatic number* $\chi(G(D)) \leqq 3$.

Corollary 12b. *If G is a minimally 2-connected graph then* $\chi(G) \leqq 3$.

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