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ON THE p-THIN PROBLEM FOR HYPERSURFACES OF R^n WITH ZERO GAUSSIAN CURVATURE

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ABSTRACT. A subset M of R^n is said to be p-thin if $T \in FL^p(R^n)$ and $\mathrm{supp}(T) \subset M$ imply $T = 0$. For a class of smooth $(n-1)$-dimensional submanifolds of R^n, we obtain the optimal result for the p-thin problem, which is applied to give the complete solution to a uniqueness problem of wave equations.

1. **Introduction.** Let $S(R^n)$ be the space of Schwartz class functions and $S'(R^n)$ be the dual space of $S(R^n)$. For $1 \leq p \leq \infty$, let $FL^p(R^n) = \{T \in S'(R^n), \hat{T} \in L^p(R^n)\}$. A subset M of R^n is said to be *p-thin* if $T \in FL^p(R^n)$ and $\mathrm{supp}(T) \subset M$ imply $T = 0$. F. Lust first studied this property ([9]) and showed that, for example, the unit sphere of R^n $(n \geq 2)$ has the p-thin property if and only if $p \leq \frac{2n}{n-1}$. Domar's method in [1] implies that Lust's result holds true for every smooth $(n-1)$-dimensional submanifold of R^n with nonzero Gaussian curvature. For a general smooth $(n-1)$-dimensional submanifold of R^n, without any curvature assumption, Hörmander showed ([5], Corollary 3.3), in our context, that M is p-thin if $p \leq \frac{2n}{n-1}$. A natural question is: Can we improve the index $\frac{2n}{n-1}$ if M has zero Gaussian curvature?

The purpose of this article is to answer the above question for a class of submanifolds of R^n with the so called constant relative nullity.

DEFINITION. Let U be open in R^{n-1} and let $F = \{(x, P(x))\ ;\ x \in U\}$ be a smooth hypersurface of R^n. If the Hessian matrix of P, $(\frac{\partial^2 P}{\partial x_i \partial x_j})$, has constant rank $n-1-\nu$ on U, $0 \leq \nu \leq n-1$, then we say that F has *constant relative nullity* ν. A smooth $(n-1)$-dimensional submanifold M of R^n is said to have *constant relative nullity* ν, if every localization F of M has constant relative nullity ν.

Since $\nu = n-1$ implies that M is a hyperplane of R^n, which is not interesting for our problem, we restrict ourselves on $0 \leq \nu \leq n-2$. Our main result is

THEOREM 1. *Let M be a smooth $(n-1)$-dimensional submanifold of R^n $(n \geq 2)$ with constant relative nullity ν such that $0 \leq \nu \leq n-2$, then M is p-thin if and only if $p \leq \frac{2(n-\nu)}{n-1-\nu}$.*

REMARK 1.1. A manifold with constant relative nullity 0 is just a manifold with nonzero Gaussian curvature. So from the inequality $\frac{2n}{n-1} \leq \frac{2(n-\nu)}{n-1-\nu}$, we see that Theorem 1 is a natural generalization of the known result for the manifold with nonzero Gaussian

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curvature. A typical manifold with nonzero nullity is the cone surface $x_n^2 = x_1^2 + \cdots + x_{n-1}^2$ in R^n (except the vertex, where the cone is not smooth), whose constant relative nullity is 1.

Noticing that the geometric property of the manifold M plays no role in the proof given in [5], we need a different approach to prove Theorem 1.

It turns out the the methods developed in the study of the so called spectral synthesis are the desired tools. The interested reader should consult [1], [2], [3] and [11] for information about the subject. Also in differential geometry there is a nice characterization of the manifold with constant relative nullity, which makes our approach possible.

We organize this article as follows. In Section 2, we state and prove Theorem 2, which tells that some FL^p information of a general distribution supported on a manifold can be transferred to a nicer distribution supported on the manifold. Theorem 3 will be stated and proved in Section 3. This theorem says that if μ is a smooth mass density on a manifold, then some FL^p information of μ will force $\mu = 0$. We prove Theorem 1 in Section 4 by combining Theorem 2 and Theorem 3. Then we give an example to show that the constant relative nullity hypothesis in Theorem 2 cannot be removed. At the end of this section, we apply Theorem 1 to obtain the complete solution to a uniqueness problem of wave equations (see Theorem 4). In [7], page 331, there is a discussion of the global unique continuation theorems, to which our uniqueness result is related. If the potential function $V(x)$ is identically zero, our result is better than the one described at the bottom of page 331 there. It would be very interesting if one can combine the spectral synthesis approach here with the approach in [7] to improve the global unique continuation theorems for a potential $V(x)$, being not identically zero.

Finally we point out that all the results of this article remains true if we only assume the manifold M to have differentiability up to a certain order.

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2. **Proof of Theorem 2.** Let $E = \{(x, \psi(x)) \; ; x \in U\}$, where U is a bounded open set in R^{n-1}, $\psi(x)$ is a real-valued function defined on U such that ψ is smooth and the rank of $(\frac{\partial^2 \psi}{\partial x_i \partial x_j}) = n - 1 - \nu$ on U, $0 \leq \nu \leq n-2$. Define the smooth mapping S on $U \times R$ by $S(x, y) = (x, y + \psi(x))$.

Given $T \in S'(R^n)$ with supp $T \subset E$, we can find $\gamma(x) \in C_0^\infty(U)$ such that $T = \gamma^2 T$. Also it is easy to see that supp$(T \circ S^{-1}) \subset U \times 0$. So from a result of Schwartz (*cf.* [1], page 34), we have for some q a representation

$$T \circ S^{-1} = \sum_{j=0}^{q} T_j \otimes \delta^{(j)},$$

where T_j are distributions on R^{n-1} such that $\operatorname{supp} T_j \subset U$, and where $\delta^{(j)}$ are the derivatives of order j of the Dirac measure on R. We can find $\gamma(x) \in C_0^\infty(U)$ such that $T_j = \gamma^2 T_j$ for $0 \le j \le q$. Let B^{n-1} be the open unit ball of R^{n-1} and let $\phi(x) \in C_0^\infty(B^{n-1})$. For $h > 0$, denote $\phi_h(x) = \frac{1}{h^{n-1}}\phi(\frac{x}{h})$ and $\check{\phi}_h(x) = \phi_h(-x)$. Then following Domar [1], we define $T_h \in S'(R^n)$ by

$$\langle T_h, f\rangle = \langle (T \circ S^{-1}) * \check{\phi}_h, f \circ S^{-1}\rangle, \text{ for } f \in S(R^n)$$

We choose h so small that for each $x \in \operatorname{supp}(\gamma)$, as a function of σ, $\phi_h(x-\sigma)$ is supported in U. It follows that $\big((T_j \circ S^{-1}) * \check{\phi}_h\big) \circ S$ for $0 \le j \le q$ are smooth mass densities on E, vanishing near the boundary of E. This implies that $\operatorname{supp}(T_h) \subset E$. We fix such a small h_0.

THEOREM 2. *Let E, T and T_{h_0} be as above. Then we have*

$$\|\hat{T}_{h_0}\|_p \le C(p)\|\hat{T}\|_p, \quad 1 \le p \le \infty.$$

To prove Theorem 2, we need several lemmas. In the rest of this article, different uniform constants may appear and will be denoted by the same letter C.

LEMMA 2.1 ([11], COROLLARY 3.2). *If $f \in C_0^\infty(R^{n-1})$, then*

$$\|\hat{f}\|_1 \le C|\operatorname{supp} f|^{\frac{1}{2}}\|f\|_{C^{n-1}(R^{n-1})},$$

where $|\operatorname{supp} f|$ denotes the Lebesgue measure of the support of f.

LEMMA 2.2 ([8]). *Let U, $\psi(x)$ be as in Theorem 2 and let $a(x) \in C_0^\infty(U)$. For $(\eta,\xi) \in R^{n-1} \times R$, set*

$$I_{\eta,\xi} = \int_{R^{n-1}} e^{-i(\eta\cdot x + \xi\psi(x))} a(x)\, dx.$$

Then

$$|I_{\eta,\xi}| \le C(1 + |\eta| + |\xi|)^{-\frac{n-1-\nu}{2}},$$

for all $(\eta,\xi) \in R^n$, where C is independent of (η,ξ).

LEMMA 2.3 ([11], THEOREM 4.1). *Let U, $\psi(x)$ be as in Theorem 2 and let $b(x) \in C_0^\infty(U)$. Then*

$$\|be^{-i\xi\psi}\|_{FL^1(R^{n-1})} \le C(1 + |\xi|)^{\frac{n-1-\nu}{2}},$$

for all real ξ, where C is independent of ξ.

LEMMA 2.4. *Let U, $\psi(x)$ be in Theorem 2 and let $\gamma(x)$ and $\phi_{h_0}(x)$ be the functions used in the construction of T_{h_0}. Set*

$$g_{\eta,\xi}(x) = \gamma^2(x) e^{-i\xi\psi(x)} \int_{R^{n-1}} e^{-i(\eta\cdot\sigma + \xi\psi(\sigma))}\phi_{h_0}(x-\sigma)\, d\sigma.$$

Then we have

(1) $$\int_{R^{n-1}} |\hat{g}_{\eta,\xi}(w)|\, dw \leq C$$

(2) $$\int_{R^{n-1}} |\hat{g}_{\eta,\xi}(w)|\, dn \leq C$$

where C is independent of η, ξ in (1) and independent of w, ξ in (2).

PROOF. Lemma 2.1 and Lemma 2.2 yield

(3) $$\left\| \gamma(\cdot) \int_{R^{n-1}} e^{-i(\eta\cdot\sigma+\xi\psi(\sigma))} \phi_{h_0}(\cdot - \sigma)\, d\sigma \right\|_{FL^1(R^{n-1})} \leq C(1+|\xi|)^{-\frac{n-1-\nu}{2}}.$$

So (1) follows from (3) and Lemma 2.3. To verify (2), let

$$H_{\xi,w}(\sigma) = e^{-i\xi\psi(\sigma)} \int_{R^{n-1}} e^{-i(w\cdot x+\xi\psi(x))} \gamma^2(x) \phi_{h_0}(x-\sigma)\, dx.$$

Then it is easy to check that $H_{\xi,w}(\sigma) \in C_0^\infty(U)$ and for η, $w \in R^{n-1}$, $\xi \in R$, $\hat{H}_{\xi,w}(\eta) = \hat{g}_{\eta,\xi}(w)$. Now (2) follows from the proof of (1). The proof of Lemma 2.4 is complete.

PROOF OF THEOREM 2. Let $(\eta,\xi) \in R^{n-1} \times R$ and $(x,y) \in R^{n-1} \times R$. Let $X(x,y) = e^{i(\eta\cdot x+\xi y)}$. Then from the construction of T_{h_0}, we have

$$\begin{aligned}
\hat{T}_{h_0}(\eta,\xi) &= \langle T_{h_0}, X\rangle \\
&= \left\langle \sum_{j=0}^{q} (\gamma^2 T_j * \check{\phi}_{h_0}) \otimes \delta^j, X \circ S^{-1} \right\rangle \\
&= \sum_{j=0}^{q} \langle \gamma^2 T_j * \check{\phi}_{h_0}, e^{i\eta x - \xi\psi(x)} \rangle (i\xi)^j \\
&= \sum_{j=0}^{q} \left\langle T_j, \gamma^2(x) \int_{R^{n-1}} e^{i(n\cdot\sigma - \xi\psi(\sigma))} \phi_{h_0}(x-\sigma)\, d\sigma \right\rangle (i\xi)^j \\
&= \sum_{j=0}^{q} \left\langle T_j, \gamma^2(x) e^{-i\xi\psi(x)} e^{i\xi\psi(x)} \int_{R^{n-1}} e^{i(\eta\cdot\sigma - \xi\psi(\sigma))} \phi_{h_0}(x-\sigma)\, d\sigma \right\rangle (i\xi)^j \\
&= \sum_{j=0}^{q} \langle T_j, e^{-i\xi\psi(x)} g_{\eta,\xi}(x) \rangle (i\xi)^j \\
&= C \sum_{j=0}^{q} \left\langle T_j, e^{-i\xi\psi(x)} \int_{R^{n-1}} \hat{g}_{\eta,\xi}(w) e^{ix\cdot w}\, dw \right\rangle (i\xi)^j \\
&= C \int_{R^{n-1}} \left(\sum_{j=0}^{q} \langle T_j, e^{i(w\cdot x - \xi\psi(x))} \rangle (i\xi)^j \right) \hat{g}_{\eta,\xi}(w)\, dw \\
&= C \int_{R^{n-1}} \langle T, e^{i(w\cdot x+\xi y)} \rangle \hat{g}_{\eta,\xi}(w)\, dw \\
&= C \int_{R^{n-1}} \hat{T}(w,\xi) \hat{g}_{\eta,\xi}(w)\, dw
\end{aligned}$$

Thus (1) implies $\|\hat{T}_{h_0}\|_\infty \leq C\|\hat{T}\|_\infty$.

For $1 \leq p < \infty$, from (1) and Jensen's convex inequality, we have

$$|\hat{T}_{h_0}(\eta,\xi)|^p \leq C(p) \int_{R^{n-1}} |\hat{T}(w,\xi)|^p\, |\hat{g}_{\eta,\xi}(w)|\, dw$$

and hence (2) gives

$$\int_R \int_{R^{n-1}} |\hat{T}_{h_0}(\eta,\xi)|^p \, d\eta \, d\xi \le C(p) \int_R \int_{R^{n-1}} \int_{R^{n-1}} |\hat{T}(w,\xi)|^p \, |\hat{g}_{\eta,\xi}(w)| \, dw \, d\eta \, d\xi$$
$$\le C(p) \int_R \int_{R^{n-1}} |\hat{T}(w,\xi)|^p \int_{R^{n-1}} |\hat{g}_{\eta,\xi}(w)| \, d\eta \, dw \, d\xi$$
$$\le C(p) \int_R \int_{R^{n-1}} |\hat{T}(w,\xi)|^p \, dw \, d\xi$$

The proof of Theorem 2 is complete.

3. **Proof of Theorem 3.**

THEOREM 3. *Let E be as in Theorem 2. Let α be a smooth mass density on E, vanishing near the boundary of E. Assume that α is not identically zero on E. Then $\alpha \in FL^p(R^n)$ if and only if $p > \frac{2(n-\nu)}{n-1-\nu}$.*

The proof of this theorem depends on the following Hartman's result ([4]).

LEMMA 3.1. (*cf.* [11], LEMMA 5.2). *Let M be a smooth $(n-1)$-dimensional submanifold of R^n with constant relative nullity ν. Then for each $m_0 \in M$, there exists a bijective affine-linear transform τ_{m_0} of R^n such that at $\tau_{m_0}(m_0)$ the manifold $M^\tau = \tau_{m_0}(M)$ has a chart (X,Ω) with the following properties.*

(i) $\Omega = B_{2\delta}^{n-1-\nu} \times B_\delta^\nu$, where δ is a small positive number, B_δ^k is the open ball in R^k with radius δ and the center at the origin.

(ii) For $v = (v', v'') \in \Omega$, $X(v) \in R^n = R^{n-1-\nu} \times R^\nu \times R$ such that

$$X(v',v'') = \big(a(v')\cdot v'' + b(v'), v'', c(v')\cdot v'' + d(v')\big),$$

where a is a smooth matrix-valued function, b, c are smooth vector-valued functions and d is a smooth scalar function.

(iii) For each $v_0' \in B_{2\delta}^{n-1-\nu}$, the space of vectors normal to M^τ at a point $m' \in M' = \{X(v_0', v'') : v'' \in B_\delta^\nu\}$ is independent of m'.

(iv) Let $\Gamma(v',v'') = \big(a(v')\cdot v'' + b(v'), v''\big)$. Then Γ is a C^∞ diffeomorphism from $\Omega = B_{2\delta}^{n-1-\nu} \times B_\delta^\nu$ onto $\Gamma(\Omega)$ with $\Gamma(0,v'') = (0,v'')$. Let $x = (x',x'') \in \Gamma(\Omega)$. Then $x'' = v''$.

(v) If we define $\psi(x) \in C^\infty\big(\Gamma(\Omega)\big)$ by $\psi \circ \Gamma(v',v'') = c(v')\cdot v'' + d(v')$, then $\nabla\psi(0,x'') = 0$ for all $x'' \in B_\delta^\nu$, and $D^2\psi(0,0)$ is a diagonal matrix with real entries such that $D^2\psi(0,0)_{i,i} = k_i \neq 0$ for $i = 1,\ldots,n-1-\nu$, $D^2\psi(0,0)_{i,i} = 0$ for $i = n-\nu,\ldots,n-1$. And there exists $\delta_1 > 0$ such that $\det[(\frac{\partial^2\psi}{\partial x_i \partial x_j})]|_{i,j=1}^{i,j=n-1-\nu}] \ge \delta_1 > 0$ for all $(x',x'') \in \Omega$.

PROOF OF THEOREM 3. By a smooth partition of unity and by a bijective affine-linear transform of R^n, we may assume that E has the form (X,Ω) with the properties stated in Lemma 3.1. Here we used the fact that FL^p spaces are invariant under a bijective affine-linear transform. Also we may assume that the density function $A(v',v'')$ of α has the property that $A(v',v'') \in C_0^\infty(B_\delta^{n-1-\nu} \times B_\delta^\nu)$ and $A(v',v'')$ is not identically zero on Ω. For $(\eta_1,\eta_2,\xi) \in R^{n-1-\nu} \times R \times R^\nu = R^n$, we have

$$\hat{\alpha}(\eta_1,\eta_2,\xi) = \int_{B_\delta^\nu} \int_{B_{2\delta}^{n-1-\nu}} e^{-i[\eta_1\cdot(a(v')\cdot v''+b(v'))+\eta_2\cdot(c(v')\cdot v''+d(v'))+\xi\cdot v'']} A(v',v'') \, dv' \, dv''. \tag{4}$$

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For $\eta = (\eta_1, \eta_2) \in R^{n-1-\nu} \times R = R^{n-\nu}$ and $\eta \neq 0$, let $\eta^0 = \frac{1}{|\eta|}\eta$ so that $\eta^0 = (\eta_1^0, \eta_2^0) \in S^{n-1-\nu}$ and $\eta = |\eta|\eta^0$. For each $\eta^0 \in S^{n-1-\nu}$, denote

$$g_{\eta^0,\nu''}(\nu') = \eta^0 \cdot \big(a(\nu') \cdot \nu'' + b(\nu'), c(\nu') \cdot \nu'' + d(\nu')\big).$$

Then (iii) of Lemma 3.1 implies that the set $\{\nu' \in B_{2\delta}^{n-1-\nu}, \nabla g_{\eta^0,\nu''}(\nu') = 0\}$ is independent of ν''. Also from (v) of Lemma 3.1, we see that for each $\nu'' \in B_\delta^\nu$, the $(n-1-\nu)$-dimensional submanifold $F_{\nu''} = \big\{\big(a(\nu') \cdot \nu'' + b(\nu'), c(\nu') \cdot \nu'' + d(\nu')\big), \nu' \in B_{2\delta}^{n-1-\nu}\big\}$ is smooth and the Gaussian curvature $k_{\nu''}(\nu')$ of $F_{\nu''}$ is away from zero, uniformly for $(\nu', \nu'') \in \Omega$. It follows that for δ small, we may assume that there exists $\epsilon > 0$ such that for each $\eta^0 \in S^{n-1-\nu}$, either

$$|\nabla g_{\eta^0,\nu''}(\nu')| \geq \epsilon, \text{ uniformly for } (\nu', \nu'') \in B_\delta^{n-1-\nu} \times B_\delta^\nu. \tag{5}$$

or there exists one and only one $\nu_0' \in B_{2\delta}^{n-1-\nu}$ such that

$$\nabla g_{\eta^0,\nu''}(\nu_0') = 0, \tag{6}$$

where ν_0' is independent of $\nu'' \in B_\delta^\nu$.

For the case (5), integration by parts first for ν', then for ν'' yields

$$|\hat{\alpha}(\eta, \xi)| \leq C(N)(1 + |\eta|)^{-N}\Big[1 + \big(|\xi| - s(\eta^0)|\eta|\big)^2\Big]^{-\frac{N}{2}} \tag{7}$$

where N is any positive integer and $s(\eta^0)$ is some measurable function of η^0.

For the case (6), we apply the stationary phase method (*cf.* [8], [10], [12]). Let

$$I(\eta_1, \eta_2, \nu'') = \int_{B_{2\delta}^{n-1-\nu}} e^{-i[\eta_1 \cdot (a(\nu') \cdot \nu'' + b(\nu')) + \eta_2 \cdot (c(\nu') \cdot \nu'' + d(\nu'))]} A(\nu', \nu'')\, d\nu'.$$

Then as $|\eta| \longrightarrow \infty$, $I(\eta_1, \eta_2, \nu'') = P(\eta_1, \eta_2, \nu'') + E(\eta_1, \eta_2, \nu'')$, where P is the principal term of I and E is the error term. Let

$$P(\eta, \xi) = \int_{B_\delta^\nu} P(\eta_1, \eta_2, \nu'') e^{-i\xi \cdot \nu''}\, d\nu'',$$
$$E(\eta, \xi) = \int_{B_\delta^\nu} E(\eta_1, \eta_2, \nu'') e^{-i\xi \cdot \nu''}\, d\nu''.$$

The formula for $P(\eta_1, \eta_2, \nu'')$ is well known (*cf.* [10], page 331). As $|\eta| \longrightarrow \infty$, $P(\eta_1, \eta_2, \nu'')$ is

$$C(\nu_0', \nu'') A(\nu_0', \nu'') |k_{\nu''}(\nu_0')|^{-\frac{1}{2}} e^{-i[\eta_1 \cdot (a(\nu_0') \cdot \nu'' + b(\nu_0')) + \eta_2 \cdot (c(\nu_0') \cdot \nu'' + d(\nu_0'))]} |\eta|^{-\frac{n-1-\nu}{2}},$$

where $C(\nu_0', \nu'')$ is a constant, uniformly bounded for all $\nu_0' \in B_{2\delta}^{n-1-\nu}$ and $\nu'' \in B_\delta^\nu$. Thus

$$P(\eta, \xi) = C|\eta|^{-\frac{n-1-\nu}{2}} \int_{B_\delta^\nu} e^{-i[\eta_1 \cdot (a(\nu_0') \cdot \nu'' + b(\nu_0')) + \eta_2 \cdot (c(\nu_0') \cdot \nu'' + d(\nu_0')) + \xi \cdot \nu'']} B(\nu_0', \nu'')\, d\nu'', \tag{8}$$

where ν_0' is a smooth function of η^0, $B(\nu_0', \nu'')$ is a smooth function of (ν_0', ν'') and is not identically zero.

For small $|\eta|$, from the definition of $P(\eta_1, \eta_2, v'')$ and $P(\eta, \xi)$, it is easy to see

$$|P(\eta, \xi)| \leq C(N)(1 + |\xi|)^{-N} \text{ for any integer } N > 0. \tag{9}$$

Hence from (8) and (9), we have

$$|P(\eta, \xi)| \leq C(N)(1 + |\eta|)^{-\frac{n-1-\nu}{2}} \left[1 + \left(|\xi| - t(\eta^0)|\eta|\right)^2\right]^{-\frac{N}{2}}, \tag{10}$$

where $t(\eta^0)$ is some measurable function of η^0. Also there is a solid cone S in $R^{n-\nu}$ with the vertex at the origin such that for $\eta \in S$, $\xi \in R^\nu$

$$|P(\eta, \xi)| \approx C(N)(1 + |\eta|)^{-\frac{n-1-\nu}{2}} \left[1 + \left(|\xi| - t(\eta^0)|\eta|\right)^2\right]^{-\frac{N}{2}}. \tag{11}$$

Moreover a detail calculation in the stationary phase method yields

$$|E(\eta, \xi)| \leq C(N)(1 + |\eta|)^{-(\frac{n-1-\nu}{2}+1)} \left[1 + \left(|\xi| - u(\eta^0)|\eta|\right)^2\right]^{-\frac{N}{2}}, \tag{12}$$

where $u(\eta^0)$ is some measurable function of η^0.

From (7), (10) and (12), we have

$$|\hat{\alpha}(\eta, \xi)| \leq C(N)(1 + |\eta|)^{-\frac{n-1-\nu}{2}} \left[1 + \left(|\xi| - v(\eta^0)|\eta|\right)^2\right]^{-\frac{N}{2}}, \tag{13}$$

where $(\eta, \xi) \in R^n$ and $v(\eta^0)$ is some measurable function of η^0.

And (7), (11) and (12) yield for $(\eta, \xi) \in S \times R^\nu$,

$$|\hat{\alpha}(\eta, \xi)| \approx C(N)(1 + |\eta|)^{-\frac{n-1-\nu}{2}} \left[1 + \left(|\xi| - v(\eta^0)|\eta|\right)^2\right]^{-\frac{N}{2}}. \tag{14}$$

It follows from (13) and (14) that $\hat{\alpha} \in L^p(R^n)$ if and only if $p > \frac{2(n-\nu)}{n-1-\nu}$.

This gives the proof of Theorem 3.

COROLLARY OF THEOREM 3. *Let E and T_{h_0} be as in Theorem 2. If $T_{h_0} \neq 0$, then $T_{h_0} \in FL^p(R^n)$ only if $p > \frac{2(n-\nu)}{n-1-\nu}$.*

PROOF. Suppose that $T_{h_0} \in FL^p(R^n)$ for some p, $1 \leq p \leq \frac{2(n-\nu)}{n-1-\nu}$, we need to show that $T_{h_0} = 0$.

Since $\text{supp}(T_{h_0}) \subset E$, which is bounded in R^n, we can find $\beta(x) \in C_0^\infty(R^n)$ such that $T_{h_0} = \beta T_{h_0}$. Noticing that $\hat{\beta}$ is a nice function, we see that $T_{h_0} \in FL^\infty$ by Hölder's inequality. This again implies that $T_{h_0} \in FL^p(R^n)$ for $p > \frac{2(n-\nu)}{n-1-\nu}$.

Let $(\eta_1, \eta_2, \xi) \in R^{n-1-\nu} \times R \times R^\nu$ as in Theorem 3. From the construction of T_{h_0} and Theorem 3, we have

$$\hat{T}_{h_0}(\eta_1, \eta_2, \xi) = \sum_{j=0}^{q} F_j(\eta_1, \eta_2, \xi)(i\eta_2)^j,$$

where $F_j \in FL^p(R^n)$ if and only if $p > \frac{2(n-\nu)}{n-1-\nu}$.

When $q = 0$, Theorem 3 can be applied directly. When $q = 1$, $\hat{T}_{h_0}(\eta_1, \eta_2, \xi) = F_0(\eta_1, \eta_2, \xi) + F_1(\eta_1, \eta_2, \xi)(i\eta_2)$, so $F_1(\eta_1, \eta_2, \xi)(i\eta_2) \in FL^p(R^n)$ for all $p > \frac{2(n-\nu)}{n-1-\nu}$ since

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both $\hat{T}_{h_0}(\eta_1, \eta_2, \xi)$ and $F_0(\eta_1, \eta_2, \xi)$ do. But from the proof of Theorem 3, we see that if F_1 is not identically zero, then $F_1(\eta_1, \eta_2, \xi)(i\eta_2) \notin FL^{p_0}(R^n)$ for some $p_0 > \frac{2(n-\nu)}{n-1-\nu}$. Hence we must have $F_1 = 0$.

The induction argument deals with a general q. We have therefore finished the proof of this corollary.

REMARK 3.1. For the measure α in Theorem 3, the well-known asymptotic estimate given in [8] is $|\hat{\alpha}(\eta, \xi)| \leq C(1 + |\eta| + |\xi|)^{-\frac{n-1-\nu}{2}}$, which does not give the precise FL^p information of α as our Theorem 3 does.

4. **Proof of Theorem 1.**

PROOF OF THEOREM 1. Theorem 3 implies the only if part of the theorem, so it remains for us to show that if $T \in FL^p(R^n)$, $1 \leq p \leq \frac{2(n-\nu)}{n-1-\nu}$ and $\operatorname{supp} T \subset M$, then $T = 0$. For any $\beta \in C_0^\infty(R^n)$, we have

$$||\beta T||_{FL^p} \leq C||\beta||_{FL^1}||T||_{FL^p} \leq C||T||_{FL^p},$$

so we may assume that supp T is compact. Since supp T is compact, by a smooth partition of unity, we may assume that M is of the form E as in Theorem 2. Thus we can find T_{h_0} such that

$$||\hat{T}_{h_0}||_p \leq C(p)||\hat{T}||_p, \quad 1 \leq p \leq \frac{2(n-\nu)}{n-1-\nu}. \tag{15}$$

From the construction of T_{h_0}, it is easy to see that for $T \neq 0$, we can always find such $T_{h_0} \neq 0$. On the other hand, (15) and corollary of Theorem 3 force $T_{h_0} = 0$ for any such T_{h_0}, so we must have $T = 0$.

This finishes the proof of Theorem 1.

The constant relative nullity hypothesis in Theorem 1 seems to be the right one, as we can see from the following example, which shows that in Theorem 2 this hypothesis is necessary.

EXAMPLE. Let $x = (x_1, x_2) \in R^2$ and $x_0 \in R^2$ with $|x_0| = 3$. Let $U_1 = \{x\ ;\ |x| < 1\}$, $U_2 = \{x\ ;\ |x - x_0| < 1\}$, $U_1' = \{x\ ;\ |x| < \frac{1}{2}\}$ and $U_2' = \{x\ ;\ |x - x_0| < \frac{1}{2}\}$. Let $U = \{x\ ;\ |x| < 5\}$ and choose $\alpha(x) \in C_0^\infty(U)$ such that $\alpha(x) = 1$ on $U_1' \cup U_2'$ and $\alpha(x) = 0$ on $U \backslash (U_1 \cup U_2)$. Define $\psi(x) \in C^\infty(U)$ by letting $\psi(x) = (2 - |x|^2)^{\frac{1}{2}}\alpha(x)$ for $x \in U_1$, $\psi(x) = (x_1^2 + x_2^2)^{\frac{1}{2}}\alpha(x)$ for $x \in U_2$, $\psi(x) = 0$ for $x \in U\backslash(U_1 \cup U_2)$.

Let $E = \{(x, \psi(x)), x \in U\}$, then E is a smooth 2-dimensional manifold of R^3, which contains a sphere-piece $E_1 = \{(x, (2-|x|^2)^{\frac{1}{2}})\ ;\ x \in U_1'\}$ and a cone-piece $E_2 = \{(x, |x|)\ ;\ x \in U_2'\}$. Choose a nice measure T on E with the smooth density function contained in the piece of the sphere. Then from Theorem 3, $T \in FL^p(R^3)$ for $p > 3$. In particular, $T \in FL^4(R^3)$. If Theorem 2 would be true for E, then we could construct T_{h_0} as in Theorem 2 such that

$$||\hat{T}_{h_0}||_4 \leq C||\hat{T}||_4.$$

We can apply Theorem 2 several times to make T_{h_0} to be such a distribution on E that it does not vanish on the piece of the cone. Choose $\phi(x) \in C_0^\infty(R^3)$ such that ϕT_{h_0} is contained in the piece of the cone and is nonzero as a distribution. Since $T_{h_0} \in FL^4(R^3)$ implies $\phi T_{h_0} \in FL^4(R^3)$, and $4 = \frac{2(3-1)}{3-1-1}$, Theorem 3 forces $\phi T_{h_0} = 0$, which contradicts the fact that ϕT_{h_0} is a nonzero distribution. We have therefore shown that for the above manifold E, Theorem 2 is false.

Now we consider an application of Theorem 1.

THEOREM 4. *Let u be a solution in the distributional sense of the equation:*

$$\frac{\partial^2 u}{\partial x_1^2} + \cdots + \frac{\partial^2 u}{\partial x_a^2} - \frac{\partial^2 u}{\partial x_{a+1}^2} - \cdots - \frac{\partial^2 u}{\partial x_n^2} = 0, \tag{16}$$

where $n \geq 3$ and a is an integer such that $1 \leq a \leq n-1$. Then

(i) $u \in L^p(R^n)$, $1 \leq p \leq \frac{2(n-1)}{n-2}$ implies $u = 0$.

(ii) For each $p > \frac{2(n-1)}{n-2}$, there exists a nonzero solution u of (16) such that $u \in L^p(R^n)$.

PROOF OF THEOREM 4. From (16) and by taking the Fourier transform of u, we see that $\text{supp}(\hat{u}) \subset M = \{(\xi_1, \ldots, \xi_n) \; ; \; \xi_1^2 + \cdots + \xi_a^2 = \xi_{a+1}^2 + \cdots + \xi_n^2\}$. To show the main idea, we only prove the theorem for the cases of $a = 1$, $n \geq 3$ and $a = 2$, $n = 4$.

For $a = 1$, it is easy to check that except at origin, the cone surface M is smooth and has the constant relative nullity 1. Denote $\hat{u}$ by T. Then we know that $\text{supp}\, T \subset M$ and $T \in FL^p(R^n)$, $1 \leq p \leq \frac{2(n-1)}{n-2}$. For any $\alpha \in C_0^\infty(R^n \setminus \{0\})$, let $T_\alpha = \alpha T$. Since $\|\hat{T}_\alpha\|_p \leq C(\alpha)\|\hat{T}\|_p$, we can apply Theorem 1 with $\nu = 1$ and $n \geq 3$ to T_α to yield that $T_\alpha = 0$. So it is only possible that $\text{supp}\, T \subset \{0\}$. It follows from an elementary result (*cf.* [6], page 46, Theorem 2.3.4) that there exists an integer $k \geq 0$ such that

$$T = \Sigma_{|\alpha| \leq k} a_\alpha \partial^\alpha(\delta).$$

In this case, it is easy to see that the assumption that $T \in FL^p(R^n)$, $1 \leq p \leq \frac{2(n-1)}{n-2}$ implies that $a_\alpha = 0$ for all α. Thus $T = 0$ and hence $u = 0$. This verifies (i) of the theorem. To verify (ii) of the theorem, we choose any smooth mass density μ on M such that its density function $\alpha \in C_0^\infty(M \setminus \{0\})$. Then one sees that $\hat{\mu}$ is a C^∞ solution of (16) by differentiation and that $\hat{\mu} \in L^p(R^n)$ for $p > \frac{2(n-1)}{n-2}$ from the if part of Theorem 3.

For $n = 4, a = 2$, we denote $M^+ = \{(\xi_1, \ldots, \xi_4) \; ; \; \xi_4 = (\xi_1^2 + \xi_2^2 - \xi_3^2)^{\frac{1}{2}} \; ; \; \xi_1^2 + \xi_2^2 > \xi_3^2\}$. Without loss of generality, we may assume that $\hat{\mu}$ is supported on M^+ and the boundary of M^+: $\partial M^+ = \{(\xi_1, \ldots, \xi_4) \; ; \; \xi_1^2 + \xi_2^2 = \xi_3^2, \xi_4 = 0\}$. A routine calculation shows that M^+ is a smooth 3-dimensional submanifold of R^4 with the constant relative nullity 1, so an application of Theorem 3 gives (ii) of the theorem. Furthermore Theorem 1 and the argument used above for the case $a = 1$ lead to that $\hat{\mu} = 0$ on M^+. Hence the task of verifying (i) is reduced from a 3-dimensional submanifold of R^4 to a 2-dimensional cone ∂M^+ in R^3. Noticing that $\frac{2(4-1)}{4-2} \leq \frac{2(3-1)}{3-2}$, we may apply Theorem 1 again to $\hat{\mu}$ on ∂M^+ to yield that $\hat{\mu} = 0$ since $a = 1$ for ∂M^+.

For the general case, the geometric meaning of the constant relative nullity helps one to figure it out that except some lower dimensional submanifold(s) of M, M is a smooth $(n-1)$-dimensional submanifold of R^n with the constant relative nullity 1. Then one uses the inequality $\frac{2(n-1)}{n-2} \leq \frac{2(n-2)}{n-3}$ for $n \geq 4$ and induction on the dimension n.

This is the end of the proof of Theorem 4.

REMARK 4.2. Hörmander's general result in [5] implies that if u is a solution of (16) and $u \in L^p(R^n)$ for $1 \leq p \leq \frac{2n}{n-1}$, then $u = 0$. Our Theorem 4 is not a general result, but it does give the complete solution of the L^p uniqueness problem for the wave equations.

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