THE MAXIMUM IDEMPOTENT-SEPARATING CONGRUENCE ON A REGULAR SEMIGROUP

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(Received 4th December 1971)

1. Introduction

It has been established by G. Lallement (**3**) that the set of idempotent-separating congruences on a regular semigroup S coincides with the set $\Sigma(\mathscr{H})$ of congruences on S which are contained in Green's equivalence $\mathscr{H}$ on S. In view of this and Lemma 10.3 of A. H. Clifford and G. B. Preston (**1**) it is obvious that the maximum idempotent-separating congruence on a regular semigroup S is given by

$$\mu = \{(a, b) \in S \times S\colon (sat, sbt) \in \mathscr{H}, \forall s, t \in S^1\}. \tag{1}$$

The expression (1) for μ suffers from two maladies: it provides us with no information about μ which is not immediately deducible from Lallement's theorem and it is clearly not the sort of expression which may be readily used to decide if two given elements a and b of S are related under μ.

In (**2**), J. M. Howie gave and alternative expression for μ in the case where S is an inverse semigroup. He determined that the maximum idempotent-separating congruence on an inverse semigroup S is given by

$$\mu = \{(a, b) \in S \times S \mid aea^{-1} = beb^{-1} \text{ for all idempotents } e \text{ of } S\}. \tag{2}$$

A slightly different expression for μ in the case where S is an inverse semigroup was obtained by Lallement in (**3**). In (**4**), the author has developed a characterization of the maximum idempotent-separating congruence on an orthodox semigroup (a regular semigroup whose idempotents form a subsemigroup). This characterization is similar to Howie's characterization (2): in fact, if S is an orthodox semigroup, then μ is given by

$$\mu = \{(a, b) \in S \times S \mid \text{there are inverses } a' \text{ of } a \text{ and } b' \text{ of } b \text{ for which } aea' = beb' \text{ and } a'ea = b'eb \text{ for all idempotents } e \text{ of } S\}. \tag{3}$$

In this note we develop an analogous characterization of the maximum idempotent-separating congruence on a regular semigroup.

2. Preliminaries

We use the notation and terminology of A. H. Clifford and G. B. Preston (**1**). In addition we denote the set of idempotents of a semigroup S by E_S and the set of inverses of an element a in a regular semigroup S by $V(a)$. Recall that we can define a partial ordering on the $\mathscr{L}$-classes of a semigroup S by

$$L_a \leqq L_b \text{ iff } S^1a \subseteq S^1b \tag{4}$$

and we can define a partial ordering on the $\mathscr{R}$-classes of S by

$$R_a \leqq R_b \text{ iff } aS^1 \subseteq bS^1. \tag{5}$$

We also note that if e and f are idempotents of S then

$$L_e \leqq L_f \text{ iff } ef = e \text{ and } R_e \leqq R_f \text{ iff } fe = e. \tag{6}$$

We now introduce the following notation: if a is an element of the semigroup S then we define

$$EL(a) = \{e \in E_S \mid L_e \leqq L_a\} \tag{7}$$

and

$$ER(a) = \{e \in E_S \mid R_e \leqq R_a\}. \tag{8}$$

We remark that if S is regular then for any $a \in S$, $EL(a) \neq \square$ and $ER(a) \neq \square$.

3. The characterization of μ

We now prove the following theorem.

Theorem 2.1. *The maximum idempotent-separating congruence on a regular semigroup S is given by:*

$$\mu = \{(a, b) \in S \times S \mid \text{there are inverses } a' \text{ of } a \text{ and } b' \text{ of } b \text{ such that } aea' = beb' \ \forall e \in EL(a) \cup EL(b), \text{ and } a'fa = b'fb \ \forall f \in ER(a) \cup ER(b)\}. \tag{9}$$

Proof. It is obvious that μ is reflexive and symmetric. To show that μ is transitive, we first show that μ is contained in Green's equivalence $\mathscr{H}$.

Let $(a, b) \in \mu$ and let a', b' be the corresponding inverses of a, b respectively, as in the definition (9) of μ.

Note that $aa' \in R_a$, so $a'a = a'(aa')a = b'(aa')b$ and similarly

$$b'b = b'(bb')b = a'(bb')a, \quad aa' = b(a'a)b' \text{ and } bb' = a(b'b)a'.$$

Hence

$$\begin{aligned} aa' &= a(a'a)a' = a(b'aa'b)a' \\ &= b(b'aa'b)b' \qquad (b'aa'b \in E_S \text{ because } b'aa'b = a'a), \\ &= (bb')(aa')(bb'). \end{aligned}$$

Hence

$$(bb')(aa') = (bb')(aa')(bb') = aa'$$

and

$$(aa')(bb') = (bb')(aa')(bb') = aa'.$$

By symmetry, $bb' = (bb')(aa') = (aa')(bb')$, and so $aa' = bb'$. Similarly, $a'a = b'b$, and so since $aa' = bb'$ and $a'a = b'b$, it follows that $(a, b) \in \mathscr{H}$, and so $\mu \subseteq \mathscr{H}$.

In the sequel, if $a' \in V(a)$ and $(a, b) \in \mathscr{H}$, we let b' denote the inverse of b which is $\mathscr{H}$-related to a'.

We now prove that the relation μ defined by (7) is transitive. Let $(a, b) \in \mu$ and $(b, c) \in \mu$. Then $a\mathscr{H}b\mathscr{H}c$ and there are inverses a' of a, b' and b^* of b and c^* of c such that $aea' = beb'$, $beb^* = cec^*$, $\forall e \in EL(a) = EL(b) = EL(c)$, and $a'fa = b'fb$, $b^*fb = c^*fc$, $\forall f \in ER(a) = ER(b) = ER(c)$. Then $aa' = bb'$, $a'a = b'b$, $bb^* = cc^*$, $b^*b = c^*c$, and there exists $a^* \in V(a)$ and $c' \in V(c)$ such that $aa' = cc'$, $c'c = a'a$, $aa^* = cc^*$ and $a^*a = c^*c$. Then for each

$$e \in EL(a) = EL(a'a) = EL(b) = EL(b'b) = EL(c),$$

we have

$$\begin{aligned} a(e)a^* &= a(ea'a)a^* = (aea')(aa^*) = (aea')(bb^*) = (beb')(bb^*) \\ &= b(eb'b)b^* = beb^* = cec^*, \end{aligned}$$

and for each $f \in ER(a) = ER(b) = ER(c) = ER(aa') = ER(bb')$, we have

$$\begin{aligned} a^*fa &= a^*(aa'f)a = (a^*a)(a'fa) = (b^*b)(a'fa) = (b^*b)(b'fb) \\ &= b^*(bb'f)b = b^*fb = c^*fc. \end{aligned}$$

Hence $(a, c) \in \mu$, and so μ is transitive.

To prove that μ is left compatible, we first show that if $(a, b) \in \mu$ then $(ca, cb) \in \mathscr{H}$ for each $c \in S$. We let $(a, b) \in \mu$ and $c \in S$ and choose $a' \in V(a)$, $b' \in V(b)$ in accordance with the definition of μ. Note first that

$$(a, b) \in \mu \subseteq \mathscr{H} \subseteq \mathscr{R},$$

and $\mathscr{R}$ is a left congruence on S, so $(ca, cb) \in \mathscr{R}$.

$$\begin{aligned} \text{Now} \quad ca &= (ca)(ca)'(ca) \text{ (where } (ca)' \text{ is some inverse of } ca) \\ &= (ca)(a'a)(ca)'ca(a'a) = (ca)(b'b)(ca)'ca(b'b) \\ &= cab'[b((ca)'(ca))b']b. \end{aligned}$$

But $(ca)'(ca) = ((ca)'c)a \in S^1a$, so $(ca)'(ca) \in EL(b)$, and it follows that $b((ca)'(ca))b' = a((ca)'(ca))a'$.

Hence

$$\begin{aligned} ca &= (ca)(b'a)(ca)'c(aa')b \\ &= (ca)(b'a)(ca)'(cb) \in S^1cb, \end{aligned}$$

and so $L_{ca} \leqq L_{cb}$. Similarly, $L_{cb} \leqq L_{ca}$, and thus $(ca, cb) \in \mathscr{L}$. It follows that $(ca, cb) \in \mathscr{H}$, as required. We remark that in view of this fact, if $(ca)' \in V(ca)$, then $(cb)'$ denotes the inverse of cb which is $\mathscr{H}$- equivalent to $(ca)'$. We now proceed to the proof of the left compatibility of μ. Let $(a, b) \in \mu$ and $c \in S$.

Let $e \in EL(ca) = EL(cb)$. We show that if $(ca)' \in V(ca)$, then

$$cae(ca)' = cbe(cb)'.$$

As usual, a' and b' denote the inverses of a and b respectively which appear in the definition of μ.

Now $e \in EL(ca) = EL((ca)'ca)$ and so $e(ca)'(ca) = e$. Also,

$$L_{(ca)'(ca)} \leqq L_{a'a} = L_a,$$

so $ea'a = e$ and $L_e \leqq L_a$.

Hence

$$\begin{aligned} cae(ca)' &= cae(a'a)(ca)' = c(aea')a(ca)' \\ &= c(beb')a(ca)' = (cb)eb'a(ca)'(ca)(ca)' \\ &= (cb)eb'a(ca)'(cb)(cb)' \\ &= (cb)eb'a(ca)'(cb)(b'b)(cb)' \\ &= (cb)eb'[a(ca)'(ca)a']b(cb)' \\ &= (cb)eb'[b(ca)'(ca)b']b(cb)', \text{ (because } L_{(ca)'ca} \leqq L_a = L_b), \\ &= (cb)(eb'b)(ca)'(ca)(a'a)(cb)' \\ &= (cb)e(cb)'(cb)(cb)' = (cb)e(cb)'. \end{aligned}$$

Now let $f \in ER(ca) = ER(cb) = ER((ca)(ca)')$. Then $(ca)(ca)'f = f$ and it follows that $(ca)'f(ca) \in E_S$ and that $(ca)'f(ca) \in EL(a) = EL(b)$.

Hence,

$$\begin{aligned} (ca)'f(ca) &= (ca)'(ca)(ca)'f(ca)(ca)'(ca) \\ &= (cb)'(cb)[(ca)'f(ca)](cb)'(cb) \\ &= (cb)'(cb)(b'b)[(ca)'f(ca)](a'a)(cb)'(cb) \\ &= (cb)'(cbb')b[(ca)'f(ca)](b'b)(cb)'(cb) \\ &= (cb)'(cbb')a[(ca)'f(ca)]a'b(cb)'(cb) \\ &= (cb)'(ca)(a'a)[(ca)'f(ca)]a'b(cb)'(cb) \\ &= (cb)'(ca)(ca)'fc(aa')b(cb)'(cb) \\ &= (cb)'(ca)(ca)'f(cb)(cb)'(cb) \\ &= (cb)'(cb)(cb)'f(cb)(cb)'(cb) = (cb)'f(cb). \end{aligned}$$

Thus μ is left compatible: a similar argument shows that μ is right compatible, and so μ is a congruence. Since $\mu \subseteq \mathscr{H}$, μ is an idempotent-separating congruence.

Finally, let ρ be any idempotent-separating congruence on S, and let $(a, b) \in \rho$. Then $(a, b) \in \mathscr{H}$, and so there are inverses a' of a and b' of b such that $aa' = bb'$ and $a'a = b'b$.

Let $e \in EL(a) = EL(a'a) = EL(b'b) = EL(b)$. Then $ea'a = e = eb'b$, and $(aea')(aea') = a(ea'a)ea' = aeea' = aea' \in E_S$, and similarly $beb' \in E_S$. Also, $b' = b'bb' = b'aa'\rho b'ba' = a'aa' = a'$, so $(a', b') \in \rho$, and hence $(aea', beb') \in \rho$. Since $aea', beb' \in E_S$, this implies that $aea' = beb'$. Similarly, $a'fa = b'fb$ for each $f \in ER(a) = ER(b)$, and so $(a, b) \in \mu$. Thus $\rho \subseteq \mu$ and this completes the proof that μ is the maximum idempotent-separating congruence on S.

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