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Hyperspace Dynamics of Generic Maps of the Cantor Space

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Abstract. We study the hyperspace dynamics induced from generic continuous maps and from generic homeomorphisms of the Cantor space, with emphasis on the notions of Li–Yorke chaos, distributional chaos, topological entropy, chain continuity, shadowing, and recurrence.

1 Introduction

The study of generic dynamics is a classical topic in the area of dynamical systems. In the context of topological dynamics, such a study has been developed during the last forty years by several authors. For the generic dynamics of continuous maps of the closed unit interval, see [4, 34], for instance. For the case of continuous maps and homeomorphisms of compact manifolds, we refer the reader to [7, 29, 32, 35], where further references can be found. Properties of generic continuous maps and of generic homeomorphisms of compact topological manifolds that hold almost everywhere with respect to a given Borel probability measure on the manifold were studied in [9–14]. A similar point of view was considered in [1]. Finally, for the generic dynamics of maps of the Cantor space, see [6, 15, 21, 22, 24, 26], for instance.

On the other hand, the study of collective dynamics is also an important topic in the area of dynamical systems. While the action of a system on points of the phase space can be thought of as individual dynamics, the action of the system on subsets of the phase space is a kind of collective dynamics, and it is natural to compare individual with collective dynamics. The most usual context for collective dynamics is that of the induced map on the hyperspace of all nonempty compact subsets endowed with the Hausdorff metric. We refer the reader to [2, 8, 25], where further references can be found.

It is natural to combine both topics and study the collective dynamics of generic maps. In order to state this in a more precise way, let us now fix some notations.

Given a compact metric space M with metric d, we denote by $\mathcal{C}(M)$ (resp. $\mathcal{H}(M)$) the space of all continuous maps from M into M (resp. of all homeomorphisms from M onto M) endowed with the metric

$$\tilde{d}(f,g) := \max_{x \in M} d\big(f(x), g(x)\big).$$

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Moreover, we denote by $\mathcal{K}(M)$ the hyperspace of all nonempty closed subsets of M endowed with the Hausdorff metric

$$d_H(X,Y) := \max\Big\{\max_{x\in X} d(x,Y), \max_{y\in Y} d(y,X)\Big\}.$$

It is well known that $\mathcal{K}(M)$ is also a compact metric space. Given $f \in \mathcal{C}(M)$, the induced map $\overline{f}\colon \mathcal{K}(M) \to \mathcal{K}(M)$ is defined by

$$\overline{f}(X) := f(X) \ \big(= \{f(x) : x \in X\}\big).$$

Note that $\overline{f} \in \mathcal{C}(\mathcal{K}(M))$. Moreover, if f is a homeomorphism, then so is $\overline{f}$. The book [27] contains a detailed treatment of hyperspaces.

Given a Baire space Z, to say that "the generic element of Z has a certain property P" means that the set of all elements of Z that satisfy property P is comeager in Z, that is, is the complement of a meager (*i.e.*, first category) set in Z. The word "typical" is sometimes used instead of the word "generic".

As mentioned before, it is natural to study the dynamics of the induced map $\overline{f}$ for the generic map $f \in \mathcal{C}(M)$ (resp. $f \in \mathcal{H}(M)$). In this paper we develop such a study in the case where M is the Cantor space. Recall that a Cantor space is a zero-dimensional compact metrizable space without isolated points. A classical result due to Brouwer [20] asserts that any Cantor space is homeomorphic to Cantor's ternary set. In particular, any two Cantor spaces are homeomorphic to each other. This explains why we often use the definite article "the" before the expression "Cantor space". In this paper we will work with the product space $\{0,1\}^{\mathbb{N}}$, where $\{0,1\}$ is endowed with the discrete topology. We consider the Cantor space $\{0,1\}^{\mathbb{N}}$ endowed with the compatible metric d given by $d(\sigma,\sigma) := 0$ and $d(\sigma,\tau) := \frac{1}{n}$, where n is the least positive integer such that $\sigma(n) \neq \tau(n)$ ($\sigma,\tau \in \{0,1\}^{\mathbb{N}}$, $\sigma \neq \tau$). If M is a Cantor space, it is well known that $\mathcal{K}(M)$ is a Cantor space as well. The following nice proof of this fact was suggested by the anonymous referee. Let $\{B_1, B_2, \dots\}$ be the countable set of all clopen subsets of M and define $c\colon \mathcal{K}(M) \to \{0,1\}^{\mathbb{N}}$ by $c(X)_n := 1$ if $X \subset B_n$, and $c(X)_n := 0$ otherwise. Since c is continuous and injective, it is an embedding. Moreover, it is easy to see that $\mathcal{K}(M)$ has no isolated point. Thus, $\mathcal{K}(M)$ is also a Cantor space.

Due to its importance, the dynamics of maps of the Cantor space has been extensively studied by several authors from different points of view. In the case of the generic dynamics of homeomorphisms, one of the most impressive results is the existence of a comeager conjugacy class for the group $\mathcal{H}(\{0,1\}^{\mathbb{N}})$. So, the dynamics of the generic element of $\mathcal{H}(\{0,1\}^{\mathbb{N}})$ reduces to the dynamics of a single element of this class. This result was first proved by Kechris and Rosendal [28] using model theoretic techniques, and a specific element of this class was described later by Akin, Glasner, and Weiss [6]. Bernardes and Darji [15] obtained a new proof of the existence of this class by giving a graph theoretic description of its elements, and applied this description to obtain many old and new dynamical properties of the elements of this class. Moreover, by using their graph theoretic techniques, they were also able to prove the somewhat surprising fact that there is a comeager subset of $\mathcal{C}(\{0,1\}^{\mathbb{N}})$ such that any two of its elements are conjugate by an element of $\mathcal{H}(\{0,1\}^{\mathbb{N}})$.

This work complements the previous works on the generic dynamics of maps of the Cantor space by establishing several new dynamical properties of these maps, from the point of view of collective dynamics.

2 Preliminaries

The main tools used in this paper are the graph theoretic descriptions of the generic continuous map and of the generic homeomorphism of the Cantor space obtained in [15]. In order to state these descriptions we need to recall some terminology from [15].

By a *partition* of $\{0,1\}^{\mathbb{N}}$, we mean a finite collection of pairwise disjoint nonempty clopen sets whose union is $\{0,1\}^{\mathbb{N}}$. The *mesh* of a partition $\mathcal{P}$ of $\{0,1\}^{\mathbb{N}}$ is defined by

$$\operatorname{mesh}(\mathcal{P}) := \max_{a \in \mathcal{P}} \operatorname{diam}(a).$$

For each $f \in \mathcal{C}(\{0,1\}^{\mathbb{N}})$ and each partition $\mathcal{P}$ of $\{0,1\}^{\mathbb{N}}$, we consider the digraph $\operatorname{Gr}(f,\mathcal{P})$ whose vertex set is $\mathcal{P}$ and whose edge set is

$$\left\{ \overrightarrow{ab} : a, b \in \mathcal{P} \text{ and } f(a) \cap b \neq \varnothing \right\}.$$

By a *component* of a digraph G we mean a largest (in vertices and edges) subgraph H of G such that given any two vertices a, b in H, there are vertices $a_1, \dots, a_n$ in H such that $a_1 = a$, $a_n = b$ and, for each $1 \le i < n$, $\overrightarrow{a_i a_{i+1}}$ or $\overrightarrow{a_{i+1} a_i}$ is an edge of H.

A digraph ℓ is a *loop* of length n if the vertex set of ℓ is a set $\{v_1, \dots, v_n\}$ with n elements and the edges of ℓ are $\overrightarrow{v_n v_1}$ and $\overrightarrow{v_i v_{i+1}}$ for $1 \le i < n$.

A digraph B is a *balloon* of type (s,t) if the vertex set of B is the union of two disjoint sets $p = \{v_1, \dots, v_s\}$, and $\ell = \{w_1, \dots, w_t\}$, and the edges of B are

- the edges of the path p, *i.e.*, $\overrightarrow{v_i v_{i+1}}$ for $1 \le i < s$,
- the edges of the loop formed by ℓ, and
- $\overrightarrow{v_s w_1}$.

We call v_1 the *initial vertex* of B. Unless otherwise specified, whenever we write a balloon B simply as

$$B = \{v_1, \dots, v_s\} \cup \{w_1, \dots, w_t\},$$

we implicitly assume that it is the balloon described above.

A digraph D is a *dumbbell* of type (r,s,t) if the vertex set of D is the union of three disjoint sets $\ell_1 = \{u_1, \dots, u_r\}$, $p = \{v_1, \dots, v_s\}$ and $\ell_2 = \{w_1, \dots, w_t\}$, and the edges of D are

- the edges of the loops formed by ℓ_1 and ℓ_2,
- the edges of the path p, and
- $\overrightarrow{u_1 v_1}$, $\overrightarrow{v_s w_1}$.

We say that s is the *length of the bar* of the dumbbell. If $r = t$, then we say that the dumbbell is *balanced* with *plate weight* r. Unless otherwise specified, whenever we write a dumbbell D simply as

$$D = \{u_1, \dots, u_r\} \cup \{v_1, \dots, v_s\} \cup \{w_1, \dots, w_t\},$$

we implicitly assume that it is the dumbbell described above.

Suppose that $f \in \mathcal{C}(\{0,1\}^{\mathbb{N}})$, $\mathcal{P}$ is a partition of $\{0,1\}^{\mathbb{N}}$, and B is a component of $\mathrm{Gr}(f,\mathcal{P})$ that is a balloon. Write

$$B = \{v_1,\dots,v_s\} \cup \{w_1,\dots,w_t\}$$

with usual labeling. We say that the balloon B is strict relative to f if $f(v_i) \subsetneq v_{i+1}$ for every $1 \le i < s$, $f(w_j) \subsetneq w_{j+1}$ for every $1 \le j < t$, and $f(v_s) \cup f(w_t) \subsetneq w_1$.

Suppose that $h \in \mathcal{H}(\{0,1\}^{\mathbb{N}})$, $\mathcal{P}$ is a partition of $\{0,1\}^{\mathbb{N}}$, and D is a component of $\mathrm{Gr}(h,\mathcal{P})$ that is a dumbbell. Write

$$D = \{u_1,\dots,u_r\} \cup \{v_1,\dots,v_s\} \cup \{w_1,\dots,w_t\}$$

with the usual labeling. We say that the dumbbell D contains a *left loop* of h (resp. a *right loop* of h) if there is a nonempty clopen subset a of u_1 (resp. of w_1) such that $h^r(a) = a$ (resp. $h^t(a) = a$).

We are now in position to state the above-mentioned results from [15].

Theorem A *The generic $f \in \mathcal{C}(\{0,1\}^{\mathbb{N}})$ has the following property:*

(Q) *For every $m \in \mathbb{N}$, there are a partition $\mathcal{P}$ of $\{0,1\}^{\mathbb{N}}$ of mesh $< 1/m$ and a multiple $q \in \mathbb{N}$ of m such that every component of $\mathrm{Gr}(f,\mathcal{P})$ is a balloon of type $(q!, q!)$ that is strict relative to f.*

Theorem B *The generic $h \in \mathcal{H}(\{0,1\}^{\mathbb{N}})$ has the following property:*

(P) *For every $m \in \mathbb{N}$, there are a partition $\mathcal{P}$ of $\{0,1\}^{\mathbb{N}}$ of mesh $< 1/m$ and a multiple $q \in \mathbb{N}$ of m such that every component of $\mathrm{Gr}(h,\mathcal{P})$ is a balanced dumbbell with plate weight $q!$ that contains both a left and a right loop of h.*

Moreover, it was proved in [15] that any two maps $f, g \in \mathcal{C}(\{0,1\}^{\mathbb{N}})$ (resp. $f, g \in \mathcal{H}(\{0,1\}^{\mathbb{N}})$) with property (Q) (resp. property (P)) are topologically conjugate to each other; that is, $f = h^{-1}gh$ for some $h \in \mathcal{H}(\{0,1\}^{\mathbb{N}})$.

Let us now introduce some further terminology and state a few simple facts. Given a partition $\mathcal{P}$ of $\{0,1\}^{\mathbb{N}}$, we define

$$\delta(\mathcal{P}) := \min\{d(a,b) : a, b \in \mathcal{P}, a \neq b\},$$
$$I_{\mathcal{P}}(X) := \{a \in \mathcal{P} : a \cap X \neq \varnothing\} \quad (X \subset \{0,1\}^{\mathbb{N}}).$$

Note that $\delta(\mathcal{P}) > 0$. The next two results follow immediately from the definitions.

Lemma 2.1 *For every $X, Y \in \mathcal{K}(\{0,1\}^{\mathbb{N}})$,*

$$d_H(X,Y) < \delta(\mathcal{P}) \Longrightarrow I_{\mathcal{P}}(X) = I_{\mathcal{P}}(Y).$$

Lemma 2.2 *For every $X, Y \in \mathcal{K}(\{0,1\}^{\mathbb{N}})$,*

$$I_{\mathcal{P}}(X) = I_{\mathcal{P}}(Y) \Longrightarrow d_H(X,Y) \le \mathrm{mesh}(\mathcal{P}).$$

3 Main Results

If $f\colon M \to M$ is a continuous map of a metric space M, recall that a pair $(x,y) \in M \times M$ is called a *Li–Yorke pair* for f if

$$\liminf_{n\to\infty} d\big(f^n(x), f^n(y)\big) = 0 \quad \text{and} \quad \limsup_{n\to\infty} d\big(f^n(x), f^n(y)\big) > 0.$$

A *scrambled set* for f is a subset S of M such that (x,y) is a Li–Yorke pair for f whenever x and y are distinct points in S. The map f is said to be *Li–Yorke chaotic* if there exists an uncountable scrambled set for f. This notion of chaos was introduced by Li and Yorke in [30].

It was proved in [15] that the generic $f \in \mathcal{C}(\{0,1\}^{\mathbb{N}})$ has no Li–Yorke pair (in particular, f is not Li–Yorke chaotic). We shall now extend this result by proving that the same property is satisfied by the induced map $\overline{f}$.

Theorem 3.1 *For the generic $f \in \mathcal{C}(\{0,1\}^{\mathbb{N}})$, $\overline{f}$ has no Li–Yorke pair.*

Proof Let $f \in \mathcal{C}(\{0,1\}^{\mathbb{N}})$ satisfy property (Q) of Theorem A. Suppose that $X, Y \in \mathcal{K}(\{0,1\}^{\mathbb{N}})$ satisfy

$$\liminf_{n\to\infty} d_H\big(\overline{f}^{\,n}(X), \overline{f}^{\,n}(Y)\big) = 0. \tag{3.1}$$

Given $\epsilon > 0$, there is a partition $\mathcal{P}$ of $\{0,1\}^{\mathbb{N}}$ of mesh $< \varepsilon$ such that every component of $\mathrm{Gr}(f,\mathcal{P})$ is a balloon. Moreover, by (3.1), there exists $n_0 \in \mathbb{N}$ such that

$$d_H\big(f^{n_0}(X), f^{n_0}(Y)\big) < \delta(\mathcal{P}).$$

By Lemma 2.1, $I_{\mathcal{P}}(f^{n_0}(X)) = I_{\mathcal{P}}(f^{n_0}(Y))$. Since each component B of $\mathrm{Gr}(f,\mathcal{P})$ is a balloon, f maps each vertex of B into a vertex of B, and so $I_{\mathcal{P}}(f^n(X)) = I_{\mathcal{P}}(f^n(Y))$ for every $n \geq n_0$. Hence, by Lemma 2.2,

$$d_H\big(f^n(X), f^n(Y)\big) < \epsilon \quad \text{for every } n \geq n_0.$$

This proves that

$$\lim_{n\to\infty} d_H\big(\overline{f}^{\,n}(X), \overline{f}^{\,n}(Y)\big) = 0,$$

and so (X,Y) is not a Li–Yorke pair for $\overline{f}$. ∎

It was proved in [15] that the generic $h \in \mathcal{H}(\{0,1\}^{\mathbb{N}})$ has no Li–Yorke pair (in particular, h is not Li–Yorke chaotic). In strong contrast to this fact and to the previous theorem, we will see that for the generic $h \in \mathcal{H}(\{0,1\}^{\mathbb{N}})$, the induced map $\overline{h}$ *is* Li–Yorke chaotic. In fact, we will see that $\overline{h}$ is even uniformly distributionally chaotic.

Let us recall the definition of uniform distributional chaos. Given $A \subset \mathbb{N}$, the upper density of A is defined by

$$\overline{\mathrm{dens}}(A) := \limsup_{n\to\infty} \frac{\mathrm{card}([1,n]\cap A)}{n}.$$

If $f\colon M \to M$ is a continuous map of a metric space M, a pair $(x,y) \in M \times M$ is called a *distributionally ε-chaotic pair* for f $(\varepsilon > 0)$ if

$$\overline{\mathrm{dens}}\{n \in \mathbb{N} : d(f^n(x), f^n(y)) \geq \varepsilon\} = 1$$

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and

$$\overline{\text{dens}}\{n \in \mathbb{N} : d(f^n(x), f^n(y)) < \delta\} = 1,$$

for all $\delta > 0$. A *distributionally ε-scrambled set* for f is a subset S of M such that (x, y) is a distributionally ε-chaotic pair for f whenever x and y are distinct points in S. The map f is said to be *uniformly distributionally chaotic* if there exists an uncountable distributionally ε-scrambled set for f, for some $\varepsilon > 0$. The notion of distributional chaos was introduced by Schweizer and Smítal in [33] (see also Oprocha [31]).

Theorem 3.2 *There is a dense open subset $\mathcal{O}$ of $\mathcal{H}(\{0,1\}^{\mathbb{N}})$ such that, for all $h \in \mathcal{O}$, $\overline{h}$ is uniformly distributionally chaotic. In particular, for the generic $h \in \mathcal{H}(\{0,1\}^{\mathbb{N}})$, $\overline{h}$ is uniformly distributionally chaotic.*

Proof Let $\mathcal{O}$ be the set of all $h \in \mathcal{H}(\{0,1\}^{\mathbb{N}})$ for which there is a partition $\mathcal{P}$ of $\{0,1\}^{\mathbb{N}}$ such that some component of $\mathrm{Gr}(h, \mathcal{P})$ is a balanced dumbbell with plate weight ≥ 2. Clearly, the set $\mathcal{O}$ is open in $\mathcal{H}(\{0,1\}^{\mathbb{N}})$. By Theorem B, $\mathcal{O}$ is also dense in $\mathcal{H}(\{0,1\}^{\mathbb{N}})$. Fix $h \in \mathcal{O}$ and let $\mathcal{P}$ be a partition of $\{0,1\}^{\mathbb{N}}$ such that there is a component D of $\mathrm{Gr}(h, \mathcal{P})$ that is a balanced dumbbell with plate weight $q \geq 2$. Write

$$D = \{u_1, \dots, u_q\} \cup \{v_1, \dots, v_s\} \cup \{w_1, \dots, w_q\},$$

with usual labeling. Since

$$u_1 \supset h^{-q}(u_1) \supset h^{-2q}(u_1) \supset \cdots \quad \text{and} \quad w_1 \supset h^{q}(w_1) \supset h^{2q}(w_1) \supset \cdots,$$

it follows that the intersections

$$F := \bigcap_{n=0}^{\infty} h^{-nq}(u_1) \quad \text{and} \quad G := \bigcap_{n=0}^{\infty} h^{nq}(w_1)$$

are nonempty. Moreover, $h^q(F) = F$ and $h^q(G) = G$. We define

$$X := F \cup h(F) \cup \cdots \cup h^{q-1}(F) \quad \text{and} \quad Y := G \cup h(G) \cup \cdots \cup h^{q-1}(G).$$

It is not difficult to verify that the following properties hold:

(a) X is a nonempty closed subset of $u_1 \cup \cdots \cup u_q$ with $h(X) = X$;
(b) $(u_1 \cup \cdots \cup u_q) \backslash X$ is exactly the set of all $\sigma \in u_1 \cup \cdots \cup u_q$ whose forward trajectory eventually goes to the bar of the dumbbell D (*i.e.*, there exists $r \in \mathbb{N}$ such that $h^r(\sigma) \in v_1$);
(a′) Y is a nonempty closed subset of $w_1 \cup \cdots \cup w_q$ with $h(Y) = Y$;
(b′) $(w_1 \cup \cdots \cup w_q) \backslash Y$ is exactly the set of all $\sigma \in w_1 \cup \cdots \cup w_q$ whose backward trajectory eventually goes to the bar of the dumbbell D (*i.e.*, there exists $r \in \mathbb{N}$ such that $h^{-r}(\sigma) \in v_s$).

Moreover, we claim that:

(c) $\lim_{m\to\infty} \max_{\sigma \in v_1} d(h^{-m}(\sigma), X) = 0$;
(c′) $\lim_{m\to\infty} \max_{\sigma \in v_1} d(h^{m}(\sigma), Y) = 0$.

Indeed, suppose that (c) is false. Then there exist $\epsilon > 0$ and an increasing sequence $(m_j)_{j\in\mathbb{N}}$ of positive integers such that $\max_{\sigma \in v_1} d(h^{-m_j}(\sigma), X) > \epsilon$ for every $j \in \mathbb{N}$. Hence, for each $j \in \mathbb{N}$, there exists $\sigma_j \in v_1$ such that

$$d(h^{-m_j}(\sigma_j), X) > \epsilon. \tag{3.2}$$

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Note that $h^{-m_j}(\sigma_j) \in u_1 \cup \cdots \cup u_q$ for every $j \in \mathbb{N}$. By passing to a subsequence, if necessary, we may assume that there exists

$$\tau := \lim_{j\to\infty} h^{-m_j}(\sigma_j) \in u_1 \cup \cdots \cup u_q.$$

By (3.2), $\tau \notin X$. Thus, it follows from (b) that there exists $r \in \mathbb{N}$ such that $h^r(\tau) \in v_1$. By continuity,

$$\lim_{j\to\infty} h^{r-m_j}(\sigma_j) = h^r(\tau) \in v_1.$$

Therefore, $h^{r-m_j}(\sigma_j) \in v_1$ for every sufficiently large j. But this is impossible, since $h^n(v_1) \cap v_1 = \varnothing$ for every negative n. This proves (c). The proof of (c′) is analogous.

Now we fix a point $\sigma_0 \in v_1$. By (c) and (c′),

$$\text{(3.3)} \qquad \lim_{m\to\infty} d(h^{-m}(\sigma_0), X) = 0 \quad \text{and} \quad \lim_{m\to\infty} d(h^{m}(\sigma_0), Y) = 0.$$

Therefore, we can construct an increasing sequence

$$a_1 < b_1 < c_1 < d_1 < a_2 < b_2 < c_2 < d_2 < \cdots$$

of positive integers so that the following properties hold:

(d) $d(h^{n-t}(\sigma_0), X) < \frac{1}{j}$ whenever $n \le a_j$ and $t \ge b_j$;

(e) $\lim_{j\to\infty} \frac{c_j - b_j}{c_j} = 1$;

(d′) $d(h^{n-t}(\sigma_0), Y) < \frac{1}{j}$ whenever $n \ge d_j$ and $t \le c_j$;

(e′) $\lim_{j\to\infty} \frac{a_{j+1} - d_j}{a_{j+1}} = 1$.

Let H be a set of subsequences of the sequence (b_j) such that H has the cardinality of the continuum and any two distinct elements of H differ at infinitely many coordinates. Each element θ of H is a sequence of the form

$$\theta = (b_{j_1}, b_{j_2}, b_{j_3}, \dots),$$

with $j_1 < j_2 < j_3 < \cdots$. We associate with such a sequence θ, the sequence $\widetilde{\theta}$ given by

$$\widetilde{\theta} := (b_{j_1}, b_{j_1}+1, \dots, c_{j_1}, b_{j_2}, b_{j_2}+1, \dots, c_{j_2}, \dots).$$

For each $\theta \in H$, let

$$C_\theta := X \cup Y \cup \{h^{-\widetilde{\theta}(k)}(\sigma_0) : k \in \mathbb{N}\}.$$

It follows from (3.3) that each C_θ is a closed set; that is,

$$C_\theta \in \mathcal{K}(\{0,1\}^{\mathbb{N}}) \quad \text{for every } \theta \in H.$$

The set $S := \{C_\theta : \theta \in H\}$ has the cardinality of the continuum, and we shall prove that it is a distributionally $\delta(\mathcal{P})$-scrambled set for $\overline{h}$.

Let $\phi, \theta \in H$ with $\phi \ne \theta$. By the definition of H, we may assume that there are infinitely many b_j's that are terms of ϕ but not terms of θ. For each such b_j, we have that

$$h^n(C_\phi) \cap v_1 = \{\sigma_0\} \quad \text{and} \quad h^n(C_\theta) \cap v_1 = \varnothing \quad \text{for all } n \in \{b_j, b_j+1, \dots, c_j\},$$

which implies that

$$d_H\big(\overline{h}^n(C_\phi), \overline{h}^n(C_\theta)\big) \ge \delta(\mathcal{P}) \quad \text{for all } n \in \{b_j, b_j+1, \dots, c_j\}.$$

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By (e), we conclude that

$$\overline{\operatorname{dens}}\left\{ n \in \mathbb{N} : d_H\big(\overline{h}^n(C_\phi), \overline{h}^n(C_\theta)\big) \geq \delta(\mathcal{P})\right\} = 1. \tag{3.4}$$

Given $\theta \in H$, $j \in \mathbb{N}$, and $n \in \{d_j, d_j + 1, \dots, a_{j+1}\}$, we can write

$$h^n(C_\theta) = X \cup Y \cup \{h^{n-\widetilde{\theta}(k)}(\sigma_0) : \widetilde{\theta}(k) \leq c_j\} \cup \{h^{n-\widetilde{\theta}(k)}(\sigma_0) : \widetilde{\theta}(k) \geq b_{j+1}\}.$$

By (d),

$$d(h^{n-\widetilde{\theta}(k)}(\sigma_0), X) < \frac{1}{j+1} \quad \text{whenever } \widetilde{\theta}(k) \geq b_{j+1}.$$

By (d′),

$$d(h^{n-\widetilde{\theta}(k)}(\sigma_0), Y) < \frac{1}{j} \quad \text{whenever } \widetilde{\theta}(k) \leq c_j.$$

Therefore,

$$d_H\big(\overline{h}^n(C_\theta), X \cup Y\big) < \frac{1}{j}.$$

Finally, let $\phi, \theta \in H$. By what we have just seen,

$$d_H\big(\overline{h}^n(C_\phi), \overline{h}^n(C_\theta)\big) < \frac{2}{j}$$

whenever $j \in \mathbb{N}$ and $n \in \{d_j, d_j + 1, \dots, a_{j+1}\}$. Thus, (e′) implies that

$$\overline{\operatorname{dens}}\left\{ n \in \mathbb{N} : d_H\big(\overline{h}^n(C_\phi), \overline{h}^n(C_\theta)\big) < \delta\right\} = 1 \tag{3.5}$$

for every $\delta > 0$. By (3.4) and (3.5), S is a distributionally $\delta(\mathcal{P})$-scrambled set for $\overline{h}$. ■

Let us now recall the notion of topological entropy. Let $f\colon M \to M$ be a continuous map of a compact metric space M. For each $n \in \mathbb{N}$, we define an equivalent metric d_n on M by

$$d_n(x, y) := \max_{0 \leq k < n} d\big(f^k(x), f^k(y)\big).$$

A subset A of M is said to be (n, ϵ, f)-separated if $d_n(x, y) \geq \epsilon$ for every $x, y \in A$ with $x \neq y$. Let $N(n, \epsilon, f)$ be the maximum cardinality of an (n, ϵ, f)-separated set. The topological entropy of f is defined by

$$\operatorname{ent}(f) := \lim_{\epsilon \to 0^+} \Big(\limsup_{n \to \infty} \frac{1}{n} \log N(n, \epsilon, f)\Big).$$

The map f is said to be topologically chaotic if $\operatorname{ent}(f) > 0$. The notion of topological entropy was introduced by Adler, Konheim, and McAndrew [3]. Here, we are adopting the equivalent definition formulated by Bowen [17] and Dinaburg [23].

Theorem 3.3 *For the generic* $f \in \mathcal{C}(\{0,1\}^{\mathbb{N}})$, $\operatorname{ent}(\overline{f}) = 0$.

Proof Suppose that $f \in \mathcal{C}(\{0,1\}^{\mathbb{N}})$ satisfies property (Q), and fix $\epsilon > 0$. Then there exists a partition $\mathcal{P}$ of $\{0,1\}^{\mathbb{N}}$ of mesh $< \epsilon$ such that every component of $\operatorname{Gr}(f, \mathcal{P})$ is a balloon. If $X, Y \in \mathcal{K}(\{0,1\}^{\mathbb{N}})$ and $I_{\mathcal{P}}(X) = I_{\mathcal{P}}(Y)$, then $I_{\mathcal{P}}(f^n(X)) = I_{\mathcal{P}}(f^n(Y))$ for every $n \in \mathbb{N}$, and so

$$d_H\big(f^n(X), f^n(Y)\big) \leq \operatorname{mesh}(\mathcal{P}) < \epsilon \quad \text{for every } n \in \mathbb{N}.$$

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This implies that $N(n, \epsilon, \overline{f}) \leq 2^{\mathrm{card}(\mathcal{P})}$ for every $n \in \mathbb{N}$. Thus, $\mathrm{ent}(\overline{f}) = 0$. ■

Related to the previous theorem, let us mention that Blanchard, Glasner, Kolyada and Maass [16] solved a long-standing open question by proving that topological chaos implies Li–Yorke chaos. With this result, we see that Theorem 3.1 actually implies Theorem 3.3. However, we think it is instructive to establish Theorem 3.3 directly from Theorem A.

In contrast to the previous theorem, we have the following result.

Theorem 3.4 *For the generic* $h \in \mathcal{H}(\{0,1\}^{\mathbb{N}})$, $\mathrm{ent}(\overline{h}) = \infty$.

Proof Let $h \in \mathcal{H}(\{0,1\}^{\mathbb{N}})$ satisfy property (P). For each $n \in \mathbb{N}$, define

$$d_n(X, Y) := \max_{0 \leq k < n} d_H\big(\overline{h}^k(X), \overline{h}^k(Y)\big) \quad (X, Y \in \mathcal{K}(\{0,1\}^{\mathbb{N}})).$$

Given $m \in \mathbb{N}$, there is a partition $\mathcal{P}$ of $\{0,1\}^{\mathbb{N}}$ of mesh $< 1/m$ such that every component of $\mathrm{Gr}(h, \mathcal{P})$ is a balanced dumbbell with plate weight $q \geq 2$. Put $\delta := \delta(\mathcal{P}) > 0$. Let

$$D_i = \{u_{i,1}, \dots, u_{i,q}\} \cup \{v_{i,1}, \dots, v_{i,s_i}\} \cup \{w_{i,1}, \dots, w_{i,q}\} \quad (1 \leq i \leq N)$$

be the components (dumbbells) of $\mathrm{Gr}(h, \mathcal{P})$. In order to simplify the indexing in the sequel, we do not use the usual labeling for these dumbbells. The only difference is that we consider $\overrightarrow{u_{i,q}v_{i,1}}$ as an edge of $\mathrm{Gr}(h, \mathcal{P})$ instead of $\overrightarrow{u_{i,1}v_{i,1}}$. For each $t \in \mathbb{N}$, let $\mathcal{P}_t$ be the partition of $\{0,1\}^{\mathbb{N}}$ obtained from $\mathcal{P}$ by replacing each $u_{i,j}$ by its partition given by the sets

$$h^{-kq+j-1}(v_{i,1}) \quad (1 \leq k \leq t) \quad \text{and} \quad u_{i,j} \backslash \Big(\bigcup_{k=1}^{t} h^{-kq+j-1}(v_{i,1}) \Big).$$

Note that $\mathrm{card}(\mathcal{P}_t) = \mathrm{card}(\mathcal{P}) + tqN$. Now, we claim that the following property holds:

$(*)$ If $\mathcal{C}_1$ and $\mathcal{C}_2$ are distinct nonempty subsets of $\mathcal{P}_t$, then $d_{tq+1}(\bigcup \mathcal{C}_1, \bigcup \mathcal{C}_2) \geq \delta$.

In fact, let $X := \bigcup \mathcal{C}_1$ and $Y := \bigcup \mathcal{C}_2$. Without loss of generality, let us assume that there exists $a \in \mathcal{C}_1$ such that $a \notin \mathcal{C}_2$. If a is some $v_{i,j}$ or some $w_{i,j}$, then

$$d_H(X, Y) \geq \delta.$$

If $a = h^{-kq+j-1}(v_{i,1})$ for some i, j, k, then $v_{i,1} \subset h^{kq-j+1}(X)$ and $v_{i,1} \cap h^{kq-j+1}(Y) = \varnothing$, implying that

$$d_H\big(\overline{h}^{kq-j+1}(X), \overline{h}^{kq-j+1}(Y)\big) \geq \delta.$$

Finally, if $a = u_{i,j} \backslash \big(\bigcup_{k=1}^{t} h^{-kq+j-1}(v_{i,1}) \big)$ for some i, j, then $h^{tq-j+1}(a) = u_{i,1}$. Hence, $u_{i,1} \subset h^{tq-j+1}(X)$ and $u_{i,1} \cap h^{tq-j+1}(Y) = \varnothing$, which gives

$$d_H\big(\overline{h}^{tq-j+1}(X), \overline{h}^{tq-j+1}(Y)\big) \geq \delta.$$

In any case, we see that $d_{tq+1}(X, Y) \geq \delta$.

Now, for every $t \in \mathbb{N}$, $(*)$ tells us that the set $\{\bigcup \mathcal{C}; \mathcal{C} \subset \mathcal{P}_t, \mathcal{C} \neq \varnothing\}$ is $(tq+1, \delta, \overline{h})$-separated, and so

$$N(tq+1, \delta, \overline{h}) \geq 2^{\mathrm{card}(\mathcal{P}_t)} - 1 = 2^{\mathrm{card}(\mathcal{P}) + tqN} - 1 \geq 2^{\mathrm{card}(\mathcal{P}) + tqN - 1}.$$

Thus,

$$\frac{1}{tq+1}\log N(tq+1,\delta,\overline{h}) \geq \frac{\operatorname{card}(\mathcal{P})+tqN-1}{tq+1}\log 2 \quad (t\in\mathbb{N}),$$

which implies that

$$\limsup_{n\to\infty}\frac{1}{n}\log N(n,\delta,\overline{h}) \geq N\log 2.$$

Hence, $\operatorname{ent}(\overline{h}) \geq N\log 2$. Since $N\to\infty$ as $m\to\infty$, we conclude that $\operatorname{ent}(\overline{h})=\infty$. ∎

Given a map f from a metric space M into itself, recall that f is said to be equicontinuous at a point $x\in M$ if for every $\varepsilon>0$ there exists $\delta>0$ such that

$$d(y,x)<\delta \Longrightarrow d\big(f^n(y),f^n(x)\big)<\varepsilon \text{ for all } n\geq 0.$$

Moreover, f is said to be chain continuous at x [5, 9] if for every $\varepsilon>0$ there exists $\delta>0$ such that for any choice of points

$$x_0\in B(x;\delta),\ x_1\in B(f(x_0);\delta), x_2\in B(f(x_1);\delta),\dots,$$

we have that

$$d(x_n,f^n(x))<\varepsilon \quad \text{for all } n\geq 0.$$

Of course, chain continuity is a much stronger property than equicontinuity. It was proved in [15] that the generic $f\in\mathcal{C}(\{0,1\}^{\mathbb{N}})$ is chain continuous at every point. We shall now see that the induced map $\overline{f}$ has the same property.

Theorem 3.5 *For the generic $f\in\mathcal{C}(\{0,1\}^{\mathbb{N}})$, $\overline{f}$ is chain continuous at every point.*

Proof Let $f\in\mathcal{C}(\{0,1\}^{\mathbb{N}})$ satisfy property (Q) and fix $\varepsilon>0$. There is a partition $\mathcal{P}$ of $\{0,1\}^{\mathbb{N}}$ of mesh $<\varepsilon$ such that every component of $\operatorname{Gr}(f,\mathcal{P})$ is a balloon. Put $\delta:=\delta(\mathcal{P})>0$. Fix $X\in\mathcal{K}(\{0,1\}^{\mathbb{N}})$ and let

$$X_0\in B(X;\delta), X_1\in B(\overline{f}(X_0);\delta), X_2\in B(\overline{f}(X_1);\delta),\dots.$$

We have to prove that $d_H\big(X_n,\overline{f}^{\,n}(X)\big)<\varepsilon$ for all $n\geq 0$. In view of Lemma 2.2, it is enough to prove that

$$I_{\mathcal{P}}(X_n)=I_{\mathcal{P}}\big(\overline{f}^{\,n}(X)\big) \tag{3.6}$$

for all $n\geq 0$. Since $d_H(X_0,X)<\delta$, Lemma 2.1 shows that (3.6) holds for $n=0$. Assume that (3.6) holds for a certain $n\geq 0$. Since every component of $\operatorname{Gr}(f,\mathcal{P})$ is a balloon, it follows that

$$I_{\mathcal{P}}\big(\overline{f}(X_n)\big)=I_{\mathcal{P}}\big(\overline{f}^{\,n+1}(X)\big). \tag{3.7}$$

On the other hand,

$$I_{\mathcal{P}}(X_{n+1})=I_{\mathcal{P}}\big(\overline{f}(X_n)\big), \tag{3.8}$$

because $d_H\big(X_{n+1},\overline{f}(X_n)\big)<\delta$. Equalities (3.7) and (3.8) show that (3.6) also holds with $n+1$ in place of n. By induction we have the desired result. ∎

It was proved in [15] that the generic $h \in \mathcal{H}(\{0,1\}^{\mathbb{N}})$ is not equicontinuous at each point of an uncountable set, and so the same property holds for the induced map $\overline{h}$. However, we have the following result, where $R(h)$ denotes the set of all recurrent points of h.

Theorem 3.6 *For the generic $h \in \mathcal{H}(\{0,1\}^{\mathbb{N}})$, $\overline{h}$ is chain continuous at every point $X \in \mathcal{K}(\{0,1\}^{\mathbb{N}})$ with $X \cap R(h) = \varnothing$; in particular, $\overline{h}$ is chain continuous at every point of a dense open set.*

Proof Let $h \in \mathcal{H}(\{0,1\}^{\mathbb{N}})$ satisfy property (P). Fix $X \in \mathcal{K}(\{0,1\}^{\mathbb{N}})$ with $X \cap R(h) = \varnothing$. For each $\sigma \in X$, there exists $t'_\sigma > 0$ such that

$$B(\sigma; t'_\sigma) \cap \{h(\sigma), h^2(\sigma), \dots\} = \varnothing.$$

Moreover, since h is equicontinuous at every nonrecurrent point [15, Theorem 4.6], there exists $t''_\sigma > 0$ such that

$$d(\tau,\sigma) < t''_\sigma \implies d\big(h^n(\tau), h^n(\sigma)\big) < \frac{t'_\sigma}{2} \text{ for all } n \in \mathbb{N}.$$

Define $t_\sigma := \min\left\{\frac{t'_\sigma}{2}, t''_\sigma\right\}$ for each $\sigma \in X$. Then

$$B(\sigma; t_\sigma) \cap \{h(\tau), h^2(\tau), \dots\} = \varnothing \text{ whenever } \sigma \in X \text{ and } \tau \in B(\sigma; t_\sigma).$$

Now a simple compactness argument shows that there exists $t > 0$ such that

$$B(\sigma; t) \cap \{h(\sigma), h^2(\sigma), \dots\} = \varnothing \text{ for all } \sigma \in X. \tag{3.9}$$

Fix $\varepsilon > 0$ and let $\mathcal{P}$ be a partition of $\{0,1\}^{\mathbb{N}}$ of mesh $< \min\{\varepsilon, t\}$ such that every component of $\mathrm{Gr}(h,\mathcal{P})$ is a balanced dumbbell with plate weight $q \geq 2$. Let

$$D_i = \{u_{i,1},\dots,u_{i,q}\} \cup \{v_{i,1},\dots,v_{i,s_i}\} \cup \{w_{i,1},\dots,w_{i,q}\} \quad (1 \leq i \leq N)$$

be the components (dumbbells) of $\mathrm{Gr}(h,\mathcal{P})$. As in the proof of Theorem 3.4, we consider $\overrightarrow{u_{i,q}v_{i,1}}$ as an edge of $\mathrm{Gr}(h,\mathcal{P})$ instead of $\overrightarrow{u_{i,1}v_{i,1}}$. By (3.9), we must have

$$h^{q-j+1}(X \cap u_{i,j}) \subset v_{i,1} \text{ whenever } i \in \{1,\dots,N\} \text{ and } j \in \{1,\dots,q\}. \tag{3.10}$$

Let $\mathcal{P}'$ be the partition of $\{0,1\}^{\mathbb{N}}$ obtained from $\mathcal{P}$ by replacing each $u_{i,j}$ by

$$\left\{ h^{-(q-j+1)}(v_{i,1}), u_{i,j} \backslash h^{-(q-j+1)}(v_{i,1}) \right\}.$$

Define $\delta := \delta(\mathcal{P}') > 0$. It follows from (3.10) that the relations

$$X_0 \in B(X;\delta),\ X_1 \in B\big(\overline{h}(X_0);\delta\big), X_2 \in B\big(\overline{h}(X_1);\delta\big), \dots$$

imply

$$d\big(X_n, \overline{h}^n(X)\big) < \varepsilon \quad \text{for all } n \geq 0.$$

This proves that $\overline{h}$ is chain continuous at X.

Finally, since $R(h)$ is closed and has empty interior in $\{0,1\}^{\mathbb{N}}$ [15, Theorem 4.5], the set of all $X \in \mathcal{K}(\{0,1\}^{\mathbb{N}})$ with $X \cap R(h) = \varnothing$ is open and dense in $\mathcal{K}(\{0,1\}^{\mathbb{N}})$. ■

Given a homeomorphism $h\colon M \to M$ of a metric space M, recall that a sequence $(x_n)_{n\in\mathbb{Z}}$ is called a δ-pseudotrajectory ($\delta > 0$) of h if

$$d(h(x_n), x_{n+1}) \le \delta \quad \text{for all } n \in \mathbb{Z}.$$

The homeomorphism h is said to have the shadowing property [18, 19] (also called pseudo-orbit tracing property) if for every $\varepsilon > 0$ there exists $\delta > 0$ such that every δ-pseudotrajectory $(x_n)_{n\in\mathbb{Z}}$ of h is ε-shadowed by a real trajectory of h; *i.e.,* there exists $x \in X$ such that

$$d(x_n, h^n(x)) < \varepsilon \quad \text{for all } n \in \mathbb{Z}.$$

It was proved in [15] that the generic $h \in \mathcal{H}(\{0,1\}^{\mathbb{N}})$ has the shadowing property. In this case, we will see that this property of the generic h is also satisfied by $\overline{h}$.

Theorem 3.7 *For the generic $h \in \mathcal{H}(\{0,1\}^{\mathbb{N}})$, $\overline{h}$ has the shadowing property.*

Proof Suppose $h \in \mathcal{H}(\{0,1\}^{\mathbb{N}})$ satisfies property (P) and fix $\varepsilon > 0$. Then there exists a partition $\mathcal{P}$ of $\{0,1\}^{\mathbb{N}}$ of mesh $< \varepsilon$ such that every component of $\mathrm{Gr}(h,\mathcal{P})$ is a balanced dumbbell with plate weight $q \ge 2$. Choose

$$0 < \delta < \delta(\mathcal{P}).$$

Let $(X_n)_{n\in\mathbb{Z}}$ be a δ-pseudotrajectory of $\overline{h}$. We have to find $X \in \mathcal{K}(\{0,1\}^{\mathbb{N}})$ such that

$$d_H\big(X_n, \overline{h}^n(X)\big) < \varepsilon \quad \text{for all } n \in \mathbb{Z}.$$

In view of Lemma 2.2, it is enough to find $X \in \mathcal{K}(\{0,1\}^{\mathbb{N}})$ such that

$$I_{\mathcal{P}}(X_n) = I_{\mathcal{P}}\big(\overline{h}^n(X)\big) \quad \text{for all } n \in \mathbb{Z}.$$

Let us fix a component

$$D = \{u_1,\dots,u_q\} \cup \{v_1,\dots,v_s\} \cup \{w_1,\dots,w_q\}$$

of $\mathrm{Gr}(h,\mathcal{P})$. In order to simplify the indexing in the sequel, we do not use the usual labeling for this dumbbell. As in the proofs of Theorems 3.4 and 3.6, the only difference is that we consider $\overrightarrow{u_q v_1}$ as an edge of $\mathrm{Gr}(h,\mathcal{P})$ instead of $\overrightarrow{u_1 v_1}$. Note that it is enough to prove the existence of a closed subset Y of $\bigcup D$ such that

$$I_{\mathcal{P}}\Big(X_n \cap \big(\textstyle\bigcup D\big)\Big) = I_{\mathcal{P}}\big(\overline{h}^n(Y)\big) \quad \text{for all } n \in \mathbb{Z}. \tag{3.11}$$

For this purpose, we fix points

$$\sigma_j \in \bigcap_{n=0}^{\infty} h^{-nq}(u_j) \subset u_j, \quad \sigma_{j,k} \in h^{-kq+j-1}(v_1) \subset u_j,$$

$$\tau_j \in \bigcap_{n=0}^{\infty} h^{nq}(w_j) \subset w_j, \quad \tau_{j,k} \in h^{(k-1)q+j}(v_s) \subset w_j,$$

for each $j \in \{1,\dots,q\}$ and each $k \in \mathbb{N}$. We also fix a point $\theta_i \in v_i$ for each $i \in \{1,\dots,s\}$. Note that both the forward and the backward trajectories of σ_j go around the "loop" $\{u_1,\dots,u_q\}$ forever. The same is true for the backward trajectory of $\sigma_{j,k}$, but the forward trajectory of $\sigma_{j,k}$ goes around the "loop" $\{u_1,\dots,u_q\}$ exactly $k-1$ times, then passes through the bar of the dumbbell, and finally keeps going around the "loop" $\{w_1,\dots,w_q\}$ forever. We have analogous geometric descriptions for the trajectories of τ_j and $\tau_{j,k}$.

Now, we define a set A consisting of σ_j's, $\sigma_{j,k}$'s, τ_j's, $\tau_{j,k}$'s, and θ_i's in the following way:

- $\sigma_j \in A \iff X_{nq} \cap u_j \neq \varnothing$ for all $n \geq 0$,
- $\sigma_{j,k} \in A \iff X_{kq-j+1} \cap v_1 \neq \varnothing$,
- $\tau_j \in A \iff X_{-nq} \cap w_j \neq \varnothing$ for all $n \geq 0$,
- $\tau_{j,k} \in A \iff X_{-(k-1)q-j} \cap v_s \neq \varnothing$,
- $\theta_i \in A \iff X_0 \cap v_i \neq \varnothing$.

We claim that

$$I_{\mathcal{P}}\big(X_n \cap \big(\bigcup D\big)\big) = I_{\mathcal{P}}\big(\overline{h}^n(A)\big) \tag{3.12}$$

for all $n \in \mathbb{Z}$. In order to prove this assertion, note that

$$I_{\mathcal{P}}(X_{n+1}) = I_{\mathcal{P}}\big(\overline{h}(X_n)\big) \quad \text{for all } n \in \mathbb{Z}, \tag{3.13}$$

because of our choice of δ. By (3.13) and the way the set A was defined, it is clear that (3.12) holds for $n = 0$. Assume that (3.12) holds for a certain $n \geq 0$. We shall prove that (3.12) also holds with $n + 1$ in place of n. By (3.13), it is enough to show that

$$I_{\mathcal{P}}\big(\overline{h}\big(X_n \cap \big(\bigcup D\big)\big)\big) = I_{\mathcal{P}}\big(\overline{h}^{n+1}(A)\big). \tag{3.14}$$

In view of the dumbbell structure, it follows that

$$I_{\mathcal{P}}\big(\overline{h}\big(X_n \cap \big(\bigcup D\big)\big)\big) \setminus \{u_1, v_1\} = I_{\mathcal{P}}\big(\overline{h}^{n+1}(A)\big) \setminus \{u_1, v_1\}.$$

So, we have only to worry about the vertices u_1 and v_1. Assume that at least one of these vertices belongs to some of the sets in (3.14). Then we must have

$$u_q \in I_{\mathcal{P}}\big(X_n \cap \big(\bigcup D\big)\big) = I_{\mathcal{P}}\big(\overline{h}^n(A)\big).$$

Write n in the form $n = kq - j$ with $j \in \{1, \dots, q\}$. Then,

$$\begin{aligned} v_1 \in I_{\mathcal{P}}\big(\overline{h}\big(X_n \cap \big(\bigcup D\big)\big)\big) &\iff v_1 \in I_{\mathcal{P}}(X_{n+1}) \\ &\iff X_{kq-j+1} \cap v_1 \neq \varnothing \\ &\iff \sigma_{j,k} \in A \\ &\iff h^{kq-j+1}(A) \cap v_1 \neq \varnothing \\ &\iff v_1 \in I_{\mathcal{P}}\big(\overline{h}^{n+1}(A)\big). \end{aligned}$$

Moreover, $u_1 \in I_{\mathcal{P}}(\overline{h}(X_n \cap (\bigcup D)))$ if and only if $u_1 \in I_{\mathcal{P}}(X_{n+1}) = I_{\mathcal{P}}(X_{kq-j+1})$, and this happens if and only if

$$X_{rq} \cap u_j \neq \varnothing \quad \text{for every } r \geq 0$$

or

$$X_{k'q-j+1} \cap v_1 \neq \varnothing \quad \text{for some } k' > k,$$

which is equivalent to saying that $u_1 \in I_{\mathcal{P}}\big(\overline{h}^{n+1}(A)\big)$. This completes the proof of (3.14). By induction we have that (3.12) holds for every $n \geq 0$. A similar induction argument shows that (3.12) also holds for $n \leq 0$.

Finally, let $Y := \overline{A}$, which is a closed subset of $\cup D$. We have to prove that (3.11) holds. By (3.12), it is enough to show that

$$I_{\mathcal{P}}\big(\overline{h}^n(Y)\big) = I_{\mathcal{P}}\big(\overline{h}^n(A)\big) \tag{3.15}$$

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for all $n \in \mathbb{Z}$. Since $A \subset Y$, the inclusion "$\supset$" is clear. In order to prove the reverse inclusion, suppose $y \in Y \backslash A$. Then either y belongs to some u_j of y belongs to some w_j. We will consider only the first case, since the second one is analogous. So assume that $y \in u_j$ for a certain j. There must exists a subsequence $(\sigma_{j,k})_{k\in N}$ of $(\sigma_{j,k})_{k\in\mathbb{N}}$ contained in A such that y is a limit point of this subsequence. Since

$$h^{kq-j+1}(\sigma_{j,k}) \in v_1 \quad \text{for all } k \in \mathbb{N},$$

the whole trajectory of y must be contained in the "loop" $\{u_1, \dots, u_q\}$. This information together with the fact that the subsequence $(\sigma_{j,k})_{k\in N}$ lies in A imply that

$$I_{\mathcal{P}}\big(\overline{h}^n(\{y\})\big) \subset I_{\mathcal{P}}\big(\overline{h}^n(A)\big) \quad \text{for all } n \in \mathbb{Z}.$$

Since this holds for each $y \in Y \backslash A$, we conclude that (3.15) is true. ■

Given a map f from a metric space M into itself, we denote by $P(f)$ (resp. $R(f)$, $\Omega(f), CR(f)$) the set of all periodic (resp. recurrent, nonwandering, chain recurrent) points of f.

It was proved in [21] (resp. [7]) that the generic $f \in \mathcal{C}(\{0,1\}^{\mathbb{N}})$ (resp. $h \in \mathcal{H}(\{0,1\}^{\mathbb{N}})$) has no periodic point. We will see that the situation is completely different for the induced map $\overline{f}$ (resp. $\overline{h}$).

Theorem 3.8 *For the generic $f \in \mathcal{C}(\{0,1\}^{\mathbb{N}})$, the following properties hold:*

(i) *$\overline{f}$ has uncountably many periodic points of each period $p \geq 1$;*
(ii) *$R(\overline{f}) = \Omega(\overline{f}) = CR(\overline{f})$;*
(iii) *$CR(\overline{f})$ has empty interior in $\overline{f}\big(\mathcal{K}(\{0,1\}^{\mathbb{N}})\big)$;*
(iv) *$P(\overline{f})$ is dense in $CR(\overline{f})$.*

Proof Suppose $f \in \mathcal{C}(\{0,1\}^{\mathbb{N}})$ satisfies property (Q).
(i) Fix $p \in \mathbb{N}$ and assume that $\overline{f}$ has only countably many periodic points of period p. Let $X_1, X_2, X_3, \dots$ be a list of all these periodic points. There are a partition $\mathcal{P}_1$ of $\{0,1\}^{\mathbb{N}}$ and an integer $q_1 \geq p$ such that every component of $\mathrm{Gr}(f, \mathcal{P}_1)$ is a balloon of type $(q_1!, q_1!)$. Moreover, by choosing $\mathcal{P}_1$ with mesh$(\mathcal{P}_1)$ small enough, we can also guarantee that $\mathrm{Gr}(f, \mathcal{P}_1)$ has at least two components. So, we can choose a component B_1 of $\mathrm{Gr}(f, \mathcal{P}_1)$ with $X_1 \backslash (\bigcup B_1) \neq \varnothing$. Now, there are a partition $\mathcal{P}_2$ of $\{0,1\}^{\mathbb{N}}$ of mesh $< \delta(\mathcal{P}_1)$ and a positive integer q_2 such that every component of $\mathrm{Gr}(f, \mathcal{P}_2)$ is a balloon of type $(q_2!, q_2!)$. The condition mesh$(\mathcal{P}_2) < \delta(\mathcal{P}_1)$ implies that $\mathcal{P}_2$ is a refinement of $\mathcal{P}_1$. Hence, every component B of $\mathrm{Gr}(f, \mathcal{P}_2)$ is *contained* in a component B' of $\mathrm{Gr}(f, \mathcal{P}_1)$ in the sense that $\bigcup B \subset \bigcup B'$. Moreover, by choosing $\mathcal{P}_2$ with mesh$(\mathcal{P}_2)$ small enough, we can also guarantee that $\mathrm{Gr}(f, \mathcal{P}_2)$ has at least two components whose initial vertices are contained in the initial vertex of the component B_1 of $\mathrm{Gr}(f, \mathcal{P}_1)$. So, we can choose such a component B_2 of $\mathrm{Gr}(f, \mathcal{P}_2)$ with $X_2 \backslash (\bigcup B_2) \neq \varnothing$. We continue this process to obtain the sequences $(\mathcal{P}_j)$, (q_j), and (B_j).

Since B_j is a component of $\mathrm{Gr}(f, \mathcal{P}_j)$, it is a balloon of the form

$$B_j = \{v_{j,1}, \dots, v_{j,q_j!}\} \cup \{w_{j,1}, \dots, w_{j,q_j!}\}.$$

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By construction, $\mathcal{P}_{j+1}$ refines $\mathcal{P}_j$ and $v_{j+1,1} \subset v_{j,1}$, which implies that

$$q_{j+1} \geq q_j \quad \text{and} \quad w_{j+1,1} \subset w_{j,1} \quad (j \in \mathbb{N}).$$

For each $j \in \mathbb{N}$, let

$$F_j := \bigcap_{n=0}^{\infty} f^{nq_j!}(w_{j,1}).$$

Clearly, $F_j \in \mathcal{K}(\{0,1\}^{\mathbb{N}})$ and $f^{q_j!}(F_j) = F_j$. Write $q_j! = k_j p$. Then

$$Y_j := F_j \cup f^p(F_j) \cup f^{2p}(F_j) \cup \cdots \cup f^{(k_j-1)p}(F_j)$$

is a periodic point of $\overline{f}$ of period p. Since $Y_1 \supset Y_2 \supset Y_3 \supset \cdots$, the set

$$Y := \bigcap_{j=1}^{\infty} Y_j$$

is also a periodic point of $\overline{f}$ of period p. Finally, since $X_j \backslash (\bigcup B_j) \neq \varnothing$ and $Y \subset Y_j \subset \bigcup B_j$, we have that $Y \neq X_j$ ($j \in \mathbb{N}$). This contradiction completes the proof of item (i).

(ii) Follows from the fact that $\overline{f}$ is chain continuous at every point (Theorem 3.5).

(iii) Fix $X \in R(\overline{f})$ and $\varepsilon > 0$. There are a partition $\mathcal{P}$ of $\{0,1\}^{\mathbb{N}}$ of mesh $< \varepsilon$ and a positive integer q such that every component of $\mathrm{Gr}(f, \mathcal{P})$ is a balloon of type (q,q). Let

$$B_i = \{v_{i,1}, \ldots, v_{i,q}\} \cup \{w_{i,1}, \ldots, w_{i,q}\} \quad (1 \leq i \leq N)$$

be the components (balloons) of $\mathrm{Gr}(f, \mathcal{P})$. Since X is a recurrent point of $\overline{f}$, we must have

$$X \subset \bigcup_{i=1}^{N} \bigcup_{j=1}^{q} w_{i,j}.$$

Hence, we may define Y as the unique union of some of the sets

$$f(v_{i,q}), \ldots, f^q(v_{i,q}),$$

with i varying in $\{1, \ldots, N\}$, that satisfies $I_{\mathcal{P}}(Y) = I_{\mathcal{P}}(X)$. Then

$$Y \in \overline{f}\big(\mathcal{K}(\{0,1\}^{\mathbb{N}})\big) \quad \text{and} \quad d_H(Y, X) < \varepsilon.$$

Moreover, it follows from the balloon structure that Y is not a recurrent point of $\overline{f}$.

(iv) Let X, ε, $\mathcal{P}$, q, and B_i be as in the proof of (iii). For each $i \in \{1, \ldots, N\}$, let

$$F_i := \bigcap_{n=0}^{\infty} f^{nq}(w_{i,1}).$$

Then F_i is a periodic point of $\overline{f}$ of period q. Let Z be the unique union of some of the sets

$$F_i, f(F_i), \ldots, f^{q-1}(F_i),$$

with i varying in $\{1, \ldots, N\}$, that satisfies $I_{\mathcal{P}}(Z) = I_{\mathcal{P}}(X)$. Then

$$\overline{f}^q(Z) = Z \quad \text{and} \quad d_H(Z, X) < \varepsilon,$$

which completes the proof. ∎

Theorem 3.9 *For the generic $h \in \mathcal{H}(\{0,1\}^{\mathbb{N}})$, the following properties hold:*

(i) *$\overline{h}$ has uncountably many periodic points of each period $p \geq 1$;*

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(ii) $R(\overline{h}) \neq \Omega(\overline{h}) = CR(\overline{h})$;
(iii) $CR(\overline{h})$ *has empty interior in* $\mathcal{K}(\{0,1\}^{\mathbb{N}})$;
(iv) $P(\overline{h})$ *is dense in* $CR(\overline{h})$.

Proof Let $h \in \mathcal{H}(\{0,1\}^{\mathbb{N}})$ satisfy property (P).
(i) Let $\mathcal{P}$ be a partition of $\{0,1\}^{\mathbb{N}}$ such that every component of $\mathrm{Gr}(h,\mathcal{P})$ is a balanced dumbbell with plate weight $q \geq 2$. Fix such a component

$$D = \{u_1, \dots, u_q\} \cup \{v_1, \dots, v_s\} \cup \{w_1, \dots, w_q\}.$$

Let X and Y be defined as in the proof of Theorem 3.2. For each $\sigma \in v_1$, let

$$Z_\sigma := X \cup Y \cup \{h^{kp}(\sigma) : k \in \mathbb{Z}\}.$$

It follows from properties (c) and (c′) in the proof of Theorem 3.2 that each Z_σ is a closed set. Hence, $\{Z_\sigma : \sigma \in v_1\}$ is an uncountable set of periodic points of $\overline{h}$ with period p.
(ii) Let $\mathcal{P}$, q, D, X and Y be as in the proof of (i). Define

$$Z := X \cup Y \cup v_1 \in \mathcal{K}(\{0,1\}^{\mathbb{N}}).$$

Since

$$d_H\big(Z, \overline{h}^n(Z)\big) \geq \delta(\mathcal{P}) \quad \text{for all } n \in \mathbb{N},$$

we have that Z is not a recurrent point of $\overline{h}$. On the other hand, for each $k \in \mathbb{N}$, let

$$Z_k := Z \cup h^{-k}(v_1) \in \mathcal{K}(\{0,1\}^{\mathbb{N}}).$$

Then

$$h^k(Z_k) = Z \cup h^k(v_1),$$

and it follows from properties (c) and (c′) in the proof of Theorem 3.2 that

$$Z_k \to Z \quad \text{and} \quad h^k(Z_k) \to Z.$$

Thus, Z is a nonwandering point of $\overline{h}$.

The equality $\Omega(\overline{h}) = CR(\overline{h})$ follows from (iv).
(iii) Fix $X \in CR(\overline{h})$ and $\varepsilon > 0$. There are a partition $\mathcal{P}$ of $\{0,1\}^{\mathbb{N}}$ of mesh $< \varepsilon$ and a positive integer q such that every component of $\mathrm{Gr}(h,\mathcal{P})$ is a balanced dumbbell with plate weight q. Let

$$D_i = \{u_{i,1}, \dots, u_{i,q}\} \cup \{v_{i,1}, \dots, v_{i,s_i}\} \cup \{w_{i,1}, \dots, w_{i,q}\} \quad (1 \leq i \leq N)$$

be the components (dumbbells) of $\mathrm{Gr}(h,\mathcal{P})$. We define Y as the unique union of some of the sets

$$h^{-q}(v_{i,1}), \dots, h^{-1}(v_{i,1}), v_{i,1}, \dots, v_{i,s_i}, h(v_{i,s_i}), \dots, h^q(v_{i,s_i}),$$

with i varying in $\{1, \dots, N\}$, that satisfies $I_{\mathcal{P}}(Y) = I_{\mathcal{P}}(X)$. Then

$$d_H(Y, X) < \varepsilon.$$

Moreover, it follows from the dumbbell structure that Y is not a chain recurrent point of $\overline{h}$.

(iv) Let X, ε, $\mathcal{P}$, q and D_i be as in the proof of (iii). For each $i \in \{1,\dots,N\}$, we define

$$F_i := \bigcap_{n=0}^{\infty} h^{-nq}(u_{i,1}) \quad \text{and} \quad G_i := \bigcap_{n=0}^{\infty} h^{nq}(w_{i,1}).$$

Clearly,

$$h^q(F_i) = F_i \quad \text{and} \quad h^q(G_i) = G_i.$$

Since X is a chain recurrent point of $\overline{h}$, there is a finite sequence $X_0,\dots,X_k$ in $\mathcal{K}(\{0,1\}^{\mathbb{N}})$ such that

$$d_H(X,X_0) < \delta(\mathcal{P}), d_H\big(\overline{h}(X_0),X_1\big) < \delta(\mathcal{P}),\dots, d_H\big(\overline{h}(X_{k-1}),X_k\big) < \delta(\mathcal{P})$$

and $X_k = X$. Therefore,

$$\begin{aligned}(3.16)\qquad I_{\mathcal{P}}(X) = I_{\mathcal{P}}(X_0),\ I_{\mathcal{P}}\big(\overline{h}(X_0)\big) &= I_{\mathcal{P}}(X_1),\dots, I_{\mathcal{P}}\big(\overline{h}(X_{k-1})\big)\\ &= I_{\mathcal{P}}(X_k) = I_{\mathcal{P}}(X).\end{aligned}$$

Let Z_1 be the union of all sets of the form $h^j(F_i)$, with $i \in \{1,\dots,N\}$ and $j \in \{0,\dots,q-1\}$, such that $u_{i,j+1} \cap X \neq \varnothing$. By (3.16), if $u_{i,j+1} \cap X \neq \varnothing$ and $u_{i,\ell+1}$ is the vertex that contains $h^{-k}(u_{i,j+1})$, then we also have $u_{i,\ell+1} \cap X \neq \varnothing$, and so $h^{-k}(h^j(F_i)) = h^{\ell}(F_i) \subset Z_1$. This implies that

$$(3.17)\qquad h^k(Z_1) = Z_1.$$

Let Z_2 be the union of all sets of the form $h^j(G_i)$, with $i \in \{1,\dots,N\}$ and $j \in \{0,\dots,q-1\}$, such that $w_{i,j+1} \cap X \neq \varnothing$. By (3.16), if $w_{i,j+1} \cap X \neq \varnothing$ and $w_{i,\ell+1}$ is the vertex that contains $h^k(w_{i,j+1})$, then $w_{i,\ell+1} \cap X \neq \varnothing$, and so $h^k(h^j(G_i)) = h^{\ell}(G_i) \subset Z_2$. Thus,

$$(3.18)\qquad h^k(Z_2) = Z_2.$$

Let Z_3 be the union of all sets of the form $\bigcup_{n\in\mathbb{Z}} h^{nk}(v_{i,j})$, with $i \in \{1,\dots,N\}$ and $j \in \{1,\dots,s_i\}$, such that $v_{i,j} \cap X \neq \varnothing$. Obviously,

$$(3.19)\qquad h^k(Z_3) = Z_3.$$

Finally, we define $Z := Z_1 \cup Z_2 \cup Z_3$. It folows from (3.16) that

$$I_{\mathcal{P}}(Z) = I_{\mathcal{P}}(X).$$

By properties (c) and (c′) in the proof of Theorem 3.2, $Z \in \mathcal{K}(\{0,1\}^{\mathbb{N}})$. By (3.17), (3.18), and (3.19), $h^k(Z) = Z$. Thus, Z is a periodic point of $\overline{h}$ and $d_H(Z,X) < \varepsilon$. ■

4 Remarks on the Dynamics Induced on Products

Another very natural context for collective dynamics consists in looking at the action of a system on k-tuples of points of the phase space. In other words, given $f \in C(M)$ and $k \in \mathbb{N}$, it consists in studying the dynamics of the induced product map

$$f^{\times k}\colon (x_1,\dots,x_k) \in M^k \mapsto (f(x_1),\dots,f(x_k)) \in M^k.$$

In the case of the generic dynamics of maps of the Cantor space, it turns out that the dynamics of these induced product maps can be easily derived from the dynamics of the original maps. In this direction, we have the following results.

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Theorem 4.1 *For the generic $f \in \mathcal{C}(\{0,1\}^{\mathbb{N}})$, the following properties hold for each $k \in \mathbb{N}$:*

(i) *$f^{\times k}$ has no Li–Yorke pair. In particular,* $\operatorname{ent}(f^{\times k}) = 0$*;*
(ii) *$f^{\times k}$ is chain continuous at every point;*
(iii) *$P(f^{\times k}) = \varnothing$;*
(iv) *$R(f^{\times k}) = \Omega(f^{\times k}) = CR(f^{\times k})$;*
(v) *$R(f^{\times k})$ is a Cantor set with empty interior in $f^{\times k}((\{0,1\}^{\mathbb{N}})^k)$.*

Theorem 4.2 *For the generic $h \in \mathcal{H}(\{0,1\}^{\mathbb{N}})$, the following properties hold for each $k \in \mathbb{N}$:*

(i) *$h^{\times k}$ has no Li–Yorke pair. In particular,* $\operatorname{ent}(h^{\times k}) = 0$*;*
(ii) *$h^{\times k}$ has the shadowing property;*
(iii) *$h^{\times k}$ is chain continuous at every point of a dense open set, but it is not equicontinuous at each point of an uncountable set;*
(iv) *$P(h^{\times k}) = \varnothing$;*
(v) *$R(h^{\times k}) = \Omega(h^{\times k}) = CR(h^{\times k})$;*
(vi) *$R(h^{\times k})$ is a Cantor set with empty interior in $(\{0,1\}^{\mathbb{N}})^k$.*

We will say a few words about the proof of Theorem 4.2 (the proof of Theorem 4.1 follows the same reasoning). Properties (i), (ii), (iii), and (iv) follow easily from the corresponding properties of the map h as presented in [15]. Properties (v) and (vi) also follow from the corresponding properties of h provided we prove that

$$R(h^{\times k}) = R(h)^k.$$

Since the inclusion "$\subset$" is obvious, let us take a point $(\sigma_1, \dots, \sigma_k) \in R(h)^k$. Given $\varepsilon > 0$, let $\mathcal{P}$ be a partition of $\{0,1\}^{\mathbb{N}}$ of mesh $< \varepsilon$ such that every component of $\operatorname{Gr}(h, \mathcal{P})$ is a balanced dumbbell with plate weight $q \geq 2$. Let

$$D_i = \{u_{i,1}, \dots, u_{i,q}\} \cup \{v_{i,1}, \dots, v_{i,s_i}\} \cup \{w_{i,1}, \dots, w_{i,q}\} \quad (1 \leq i \leq N)$$

be the components (dumbbells) of $\operatorname{Gr}(h, \mathcal{P})$. For each $1 \leq j \leq k$, let $1 \leq i_j \leq N$ be such that σ_j belongs to a vertex of the dumbbell D_{i_j}. Since σ_j is a recurrent point of h, either σ_j belongs to $u_{i_j,1} \cup \dots \cup u_{i_j,q}$ or σ_j belongs to $w_{i_j,1} \cup \dots \cup w_{i_j,q}$. Moreover, in the case $\sigma_j \in u_{i_j,1} \cup \dots \cup u_{i_j,q}$, we must have $h^n(\sigma_j) \in u_{i_j,1} \cup \dots \cup u_{i_j,q}$ for all $n \in \mathbb{N}$. In both cases, we see that

$$d(h^{nq}(\sigma_j), \sigma_j) < \varepsilon \quad \text{for all } n \in \mathbb{N}_0.$$

Since this holds for each $1 \leq j \leq k$, and since $\varepsilon > 0$ is arbitrary, we conclude that $(\sigma_1, \dots, \sigma_k) \in R(h^{\times k})$.

References

[1] F. Abdenur and M. Andersson, *Ergodic theory of generic continuous maps*. Commun. Math. Phys. **318**(2013), no. 3, 831–855.
http://dx.doi.org/10.1007/s00220-012-1622-9

[2] G. Acosta, A. Illanes, and H. Méndez-Lango, *The transitivity of induced maps*. Topology Appl. **156**(2009), no. 5, 1013–1033.
http://dx.doi.org/10.1016/j.topol.2008.12.025

[3] R. L. Adler, A. G. Konheim, and M. H. McAndrew, *Topological entropy*. Trans. Amer. Math. Soc. **114**(1965), 309–319.
http://dx.doi.org/10.1090/S0002-9947-1965-0175106-9
[4] S. J. Agronsky, A. M. Bruckner, and M. Laczkovich, *Dynamics of typical continuous functions*. J. London Math. Soc. (2) **40**(1989), no. 2, 227–243.
http://dx.doi.org/10.1112/jlms/s2-40.2.227
[5] E. Akin, *On chain continuity*. Discrete Contin. Dynam. Systems **2**(1996), no. 1, 111–120.
http://dx.doi.org/10.3934/dcds.1996.2.111
[6] E. Akin, E. Glasner, and B. Weiss, *Generically there is but one self homeomorphism of the Cantor set*. Trans. Amer. Math. Soc. **360**(2008), no. 7, 3613–3630.
http://dx.doi.org/10.1090/S0002-9947-08-04450-4
[7] E. Akin, M. Hurley, and J. A. Kennedy, *Dynamics of topologically generic homeomorphisms*. Mem. Amer. Math. Soc. **164**(2003), no. 783.
[8] W. Bauer and K. Sigmund, *Topological dynamics of transformations induced on the space of probability measures*. Monatsh. Math. **79**(1975), 81–92.
http://dx.doi.org/10.1007/BF01585664
[9] N. C. Bernardes Jr., *On the predictability of discrete dynamical systems*. Proc. Amer. Math. Soc. **130**(2002), no. 7, 1983–1992.
http://dx.doi.org/10.1090/S0002-9939-01-06247-5
[10] ———, *On the predictability of discrete dynamical systems. II*. Proc. Amer. Math. Soc. **133**(2005), no. 12, 3473–3483.
http://dx.doi.org/10.1090/S0002-9939-05-07924-4
[11] ———, *Limit sets of typical continuous functions*. J. Math. Anal. Appl. **319**(2006), no. 2, 651–659.
http://dx.doi.org/10.1016/j.jmaa.2005.06.056
[12] ———, *On the predictability of discrete dynamical systems. III*. J. Math. Anal. Appl. **339**(2008), no. 1, 58–69.
http://dx.doi.org/10.1016/j.jmaa.2007.06.029
[13] ———, *Limit sets of typical homeomorphisms*. Canad. Math. Bull. **55**(2012), no. 2, 225–232.
http://dx.doi.org/10.4153/CMB-2011-066-2
[14] ———, *Addendum to "Limit sets of typical homeomorphisms"*. Canad. Math. Bull., to appear.
http://dx.doi.org/10.4153/CMB-2012-033-1
[15] N. C. Bernardes Jr. and U. B. Darji, *Graph theoretic structure of maps of the Cantor space*. Adv. Math. **231**(2012), no. 3–4, 1655–1680.
http://dx.doi.org/10.1016/j.aim.2012.05.024
[16] F. Blanchard, E. Glasner, S. Kolyada, and A. Maass, *On Li–Yorke pairs*. J. Reine Angew. Math. **547**(2002), 51–68.
[17] R. Bowen, *Entropy for group endomorphisms and homogeneous spaces*. Trans. Amer. Math. Soc. **153**(1971), 401–414.
http://dx.doi.org/10.1090/S0002-9947-1971-0274707-X
[18] ———, *Equilibrium states and the ergodic theory of Anosov diffeomorphisms*. Lecture Notes in Mathematics, 470, Springer-Verlag, Berlin-New York, 1975.
[19] ———, *ω-limit sets for axiom A diffeomorphisms*. J. Differential Equations **18**(1975), no. 2, 333–339.
http://dx.doi.org/10.1016/0022-0396(75)90065-0
[20] L. E. J. Brouwer, *On the structure of perfect sets of points*. Proc. Akad. Amsterdam **12**(1910), 758–794.
[21] E. D'Aniello and U. B. Darji, *Chaos among self-maps of the Cantor space*. J. Math. Anal. Appl. **381**(2011), no. 2, 781–788.
http://dx.doi.org/10.1016/j.jmaa.2011.03.065
[22] E. D'Aniello, U. B. Darji, and T. H. Steele, *Ubiquity of odometers in topological dynamical systems*. Topology Appl. **156**(2008), no. 2, 240–245.
http://dx.doi.org/10.1016/j.topol.2008.07.003
[23] E. I. Dinaburg, *A connection between various entropy characterizations of dynamical systems*. (Russian) Izv. Akad. Nauk SSSR Ser. Mat. **35**(1971), 324–366.
[24] E. Glasner and B. Weiss, *The topological Rohlin property and topological entropy*. Amer. J. Math. **123**(2001), no. 6, 1055–1070.
http://dx.doi.org/10.1353/ajm.2001.0039
[25] J. L. G. Guirao, D. Kwietniak, M. Lampart, P. Oprocha, and A. Peris, *Chaos on hyperspaces*. Nonlinear Anal. **71**(2009), no. 1–2, 1–8.
http://dx.doi.org/10.1016/j.na.2008.10.055
[26] M. Hochman, *Genericity in topological dynamics*. Ergodic Theory Dynam. Systems **28**(2008), no. 1, 125–165.

[27] A. Illanes and S. B. Nadler Jr., *Hyperspaces. Fundamentals and recent advances*. Monographs and Textbooks in Pure and Applied Mathematics, 216, Marcel Dekker, Inc., New York, 1999.

[28] A. S. Kechris and C. Rosendal, *Turbulence, amalgamation, and generic automorphisms of homogeneous structures*. Proc. Lond. Math. Soc. (3) **94**(2007), no. 2, 302–350.
http://dx.doi.org/10.1112/plms/pdl007

[29] H. Lehning, *Dynamics of typical continuous functions*. Proc. Amer. Math. Soc. **123**(1995), no. 6, 1703–1707.
http://dx.doi.org/10.1090/S0002-9939-1995-1239798-X

[30] T. Y. Li and J. A. Yorke, *Period three implies chaos*. Amer. Math. Monthly **82**(1975), no. 10, 985–992.
http://dx.doi.org/10.2307/2318254

[31] P. Oprocha, *Distributional chaos revisited*. Trans. Amer. Math. Soc. **361**(2009), no. 9, 4901–4925.
http://dx.doi.org/10.1090/S0002-9947-09-04810-7

[32] S. Yu. Pilyugin, *The space of dynamical systems with the C^0-topology*. Lecture Notes in Mathematics, 1571, Springer-Verlag, Berlin, 1994.

[33] B. Schweizer and J. Smítal, *Measures of chaos and a spectral decomposition of dynamical systems on the interval*. Trans. Amer. Math. Soc. **344**(1994), no. 2, 737–754.
http://dx.doi.org/10.1090/S0002-9947-1994-1227094-X

[34] K. Simon, *On the periodic points of a typical continuous function*. Proc. Amer. Math. Soc. **105**(1989), no. 1, 244–249.
http://dx.doi.org/10.1090/S0002-9939-1989-0929418-0

[35] K. Yano, *A remark on the topological entropy of homeomorphisms*. Invent. Math. **59**(1980), no. 3, 215–220.
http://dx.doi.org/10.1007/BF01453235

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