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ON PROJECTIVE *Z*-FRAMES

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ABSTRACT. This paper deals with the projective objects in the category of all Z-frames, where the latter is a common generalization of different types of frames. The main result obtained here is that a Z-frame is **E**-projective if and only if it is stably Z-continuous, for a naturally arising collection **E** of morphisms.

In mathematics, characterizing the injective objects and their duals, the projective objects, in certain categories has quite a long history. The initial work occurred in algebra, concerning the characterization of projective modules. Later, people also investigated injective and projective objects in the category of distributive lattices, the category of sup complete lattices, and many other categories. A recent important result on injective topological spaces was obtained by Scott [13]. He discovered that the injective T_0-spaces are exactly the continuous lattices with their so-called Scott topology. Subsequently, Banaschewski generalized Scott's result to the category of all frames ([5], see also [3]).

The category **ZFrm** of all Z-frames is a generalization of the categories of various different types of frames, such as the category **Frm** of all frames [10], the category σ**Frm** of all σ-frames [2], the category **Dlat** of all distributive lattices, the category **Slat** of all meet-semilattices and the category **PreFrm** of all preframes [4] [11]. The basic properties of Z-frames have been discussed in [16]. In [15] we also investigated nuclei on Z-frames. The chief aim of this paper is to deal with some aspects of projective Z-frames. The main result obtained here is that a Z-frame is **E**-projective if and only if it is stably Z-continuous, where **E** is the collection of all Z-frame homomorphisms which have a right inverse as meet semilattice homomorphism. This establishes a natural relation between projectivity and continuity of Z-frames. For a recent discussion of projective frames, see [12].

1. *Z*-frames.

A set system on the category **Post** of all posets and order-preserving mappings was introduced by Bandelt and Ernè in defining Z-continuous posets [6]. Actually, as mentioned in [6], the notion of subset systems was originally introduced by Wright, Wagner, and Thatcher in [14]. In order to define Z-frames we define set systems on the category **Slat** of all meet semilattices.

In the following, by a semilattice we shall mean a finite-meet semilattice. Thus in particular, each semilattice has a top element. A semilattice homomorphism $f: S \longrightarrow T$ is a mapping from a semilattice S to the semilattice T which preserves finite meets (hence

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it must preserve the top element). Let **Slat** denote the category of all semilattices and semilattice homomorphisms.

A subset D of a poset P is called a *down-set*, if $D = \downarrow D = \{x \in P \mid \exists d \in D, x \leq d\}$. Denote by $\mathcal{D}(S)$ the set of all down-sets of S. For any semilattice S, $\mathcal{D}(S)$ is a semilattice (actually a complete lattice) with respect to the inclusion of sets.

DEFINITION 1.1. A set system $\mathbf{Z}$ on **Slat** is a function which assigns to each semilattice S a collection $\mathbf{Z}(S)$ of subsets of S, such that the following conditions are satisfied:

(Z1) $\mathbf{Z}(S)$ is a subsemilattice of $\mathcal{D}(S)$ containing all $\downarrow x$ for $x \in S$.

(Z2) For any $\mathcal{A} \in \mathbf{Z}\big(\mathbf{Z}(S)\big)$, $\cup\mathcal{A} \in \mathbf{Z}(S)$.

(Z3) For any semilattice homomorphism $f\colon S \longrightarrow T$ and any $D \in \mathbf{Z}(S)$, $\downarrow f(D) \in \mathbf{Z}(T)$.

REMARKS 1.2. (1) Elements of $\mathbf{Z}(S)$ will be called *Z-ideals*. A subset A of S is called a *Z-set* if $\downarrow A \in \mathbf{Z}(S)$.

(2) For each semilattice homomorphism $f\colon S \longrightarrow T$, the induced mapping $\mathbf{Z}(f)\colon \mathbf{Z}(S) \longrightarrow \mathbf{Z}(T)$ is a semilattice homomorphism, where $\mathbf{Z}(f)(D) = \downarrow f(D)$ for each $D \in \mathbf{Z}(S)$.

(3) For any $a \in S$ and $D \in \mathbf{Z}(S)$, we have

$$\downarrow\{a \wedge x \mid x \in D\} = (\downarrow a) \cap D \in \mathbf{Z}(S).$$

(4) For each semilattice S, the functions $\mathcal{D}(S)$ and $\mathcal{P}(S) = \{\downarrow x \mid x \in S\}$ define the largest and the smallest set systems, respectively.

It is also noticed that for any family $\{\mathbf{Z}_\alpha \mid \alpha \in I\}$ of set systems, the function $\mathbf{Z}$ defined by $\mathbf{Z}(S) = \bigcap_\alpha \mathbf{Z}_\alpha(S)$ is a set system.

A semilattice S is called *Z-complete* if $\vee D = \sup D$ exists for each $D \in Z(S)$, and hence also for each Z-set D of S.

Given two Z-complete semilattices S and T, a Z-complete homomorphism $f\colon S \longrightarrow T$ is a semilattice homomorphism such that $f(\vee D) = \vee f(D)$ for all Z-sets D. Let **ZComSlat** denote the category of all Z-complete semilattices and Z-complete homomorphisms.

DEFINITION 1.3. A Z-complete semilattice A is called a *Z-frame* if the following equation holds for any $a \in A$ and $D \in \mathbf{Z}(A)$:

$$a \wedge \vee D = \vee\{a \wedge x \mid x \in D\}.$$

Notice that by Remark 1.2(3) the set $\{a \wedge x \mid x \in D\}$ is a Z-set of A, so the right side of the above equation does exist.

It can be proved that a Z-complete semilattice A is a Z-frame iff the mapping $\vee\colon Z(A) \longrightarrow A$ is a Z-complete homomorphism.

A Z-complete homomorphism between two Z-frames is also called a *Z-frame* homomorphism. We use **ZFrm** to denote the category of all Z-frames and Z-frame homomorphisms. **ZFrm** is a full subcategory of **ZComSlat**.

LEMMA 1.4. *For any semilattice S, $Z(S)$ is a Z-frame, and the correspondence $S \longmapsto Z(S)$ defines a functor $Z\colon$ **Slat** $\longrightarrow$ **ZFrm**, left adjoint to the inclusion functor **ZFrm** $\longrightarrow$ **Slat**, with adjuncion maps $\eta_S\colon S \longrightarrow Z(S)$ taking $x \in S$ to $\downarrow x \in Z(S)$.*

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PROOF. The proof of that $Z(S)$ is a Z-frame is straightforward. For any semilattice homomorphism $f\colon S \longrightarrow T$, the semilattice homomorphism $Z(f)\colon Z(S) \longrightarrow Z(T)$ given in Remark 1.2 preserves all joins of Z-sets since these are actually unions. Further, for any semilattice homomorphism $f\colon S \longrightarrow A$, where A is a Z-frame, $\overline{f}\colon Z(S) \longrightarrow A$ such that $\overline{f}(D) = \vee f(D)$ is easily seen to be a Z-frame homomorphism for which $\overline{f} \circ \eta_S = f$.

2. Z-continuity. Parallel to the notion of Z-frame is the concept of Z-continuous semilattice. A semilattice A is said to be Z-continuous if it is Z-complete and satisfies the condition:

(ZC) For each $a \in A$, $\{x \in A \mid x \ll_Z a\} \in \mathbf{Z}(A)$ and $a = \vee\{x \in A \mid x \ll_Z a\}$,

where the binary relation $\ll_Z$ is defined by $x \ll_Z a$ iff for each Z-set D, $\vee D \geq a$ implies the existence of a $d \in D$ such that $x \leq d$.

REMARKS 2.1. (1) A more general structure, namely Z-continuous posets, has been studied in [6] by Bandelt and Erné, using set systems on ***Pos***. Although we are using set systems on **Slat** to define Z-continuous semilattices, many results on Z-continuous posets apply to Z-continuous semilattices.

(2) For any x and any a in S, $x \ll_Z a$ iff $a \leq \vee D$ implies $x \in D$, for each $D \in Z(S)$.

(3) Similarly as for continuous posets, it can be proved that a Z-complete semilattice A is Z-continuous iff the map

$$\vee\colon \mathbf{Z}(A) \longrightarrow A, \quad D \longmapsto \vee D$$

has a left adjoint.

(4) The relation $\ll_Z$ in a Z-continuous semilattice S is interpolating, that is if $x \ll_Z y$ then there is a $z \in S$ with $x \ll_Z z \ll_Z y$.

An element a of A is call Z-compact if $a \ll_Z a$. A Z-algebraic semilattice A is a Z-complete semilattice such that for each $a \in A$,

$$\downarrow \{x \in A \mid x \text{ is } Z\text{-compact and } x \leq a\} \in \mathbf{Z}(A)$$

and

$$a = \vee\{x \in A \mid x \text{ is } Z\text{-compact and } x \leq a\}.$$

Obviously every Z-algebraic semilattice is Z-continuous.

A Z-continuous semilattice A is said to be *stable* if (i) $x \ll_Z a$ and $x \ll_Z b$ imply $x \ll_Z a \wedge b$, and (ii) $1_A \ll 1_A$, where 1_A is the top element of A. A stable and Z-continuous semilattice is briefly called a *stably Z-continuous semilattice*.

A Z-frame is said to be *coherent* if it is Z-algebraic and is stable as Z-continuous semilattice. It can be proved that a coherent Z-frame is exactly a Z-algebraic semilattice such that it is stable as Z-continuous semilattice, or equivalently, the Z-compact elements form a subsemilattice.

LEMMA 2.2. *For each semilattice S, $\mathbf{Z}(S)$ is a coherent Z-frame.*

PROOF. For each $x \in S$, $\downarrow x \in \mathbf{Z}(S)$ and $\downarrow x \ll_Z \downarrow x$, *i.e.* $\downarrow x$ is Z-compact. Given any $D \in \mathbf{Z}(S)$, $D = \cup\{\downarrow x \mid x \in D\}$. Observe that $\{\downarrow x \mid x \in D\}$ is the image of the Z-set D under the semilattice homomorphism

$$\eta_S = \downarrow : S \longrightarrow \mathbf{Z}(S),$$

and by (Z2) it follows that $\{\downarrow x \mid x \in D\}$ is a Z-set of $\mathbf{Z}(S)$. From this it is then easily seen that the set

$$\{B \in \mathbf{Z}(S) \mid B \ll_Z B, B \leq D\}$$

is a Z-set and

$$D = \vee\{B \in \mathbf{Z}(S) \mid B \ll_Z B, B \leq D\}.$$

Hence $\mathbf{Z}(S)$ is Z-algebraic. We now prove that $\mathbf{Z}(S)$ is stable. Suppose that $E \ll_Z D$ and $E \ll_Z F$ hold in $\mathbf{Z}(S)$. By the above discussion, there exist $x \in D$ with $E \subseteq \downarrow x$ and $y \in F$ with $E \subseteq \downarrow y$. Hence $E \subseteq \downarrow x \cap \downarrow y = \downarrow(x \wedge y)$, and since $x \wedge y \in D \wedge F$, we have $E \ll_Z D \wedge F$. Further $S = \downarrow 1_S$, where 1_S is the top element of S, is obviously Z-compact. Hence $\mathbf{Z}(S)$ is a coherent Z-frame.

In a category $\mathcal{C}$, an object A is called a *retract* of the object B if there are morphisms $f : A \longrightarrow B$ and $r : B \longrightarrow A$ in $\mathcal{C}$ such that $r \circ f = id_A$.

LEMMA 2.3. *In* **ZComSlat**, *the following notions are stable under retraction:*

(i) Being a Z-frame,

(ii) Z-continuity, and

(iii) stable Z-continuity.

PROOF. We only give the proof of the second and the third assertions. Suppose that A is a Z-complete semilattice that is a retract of a Z-continuous semilattice L. Then there are Z-complete homomorphisms $r : L \longrightarrow A$ and $f : A \longrightarrow L$ such that $r \circ f = id_A$. Let $a \in A$ be an arbitrary element of A. Since L is Z-continuous, so

$$f(a) = \vee\{x \in L \mid x \ll_Z f(a)\} \quad \text{and} \quad \{x \in L \mid x \ll_Z f(a)\} \in \mathbf{Z}(L).$$

Next r is a Z-complete homomorphism, so

$$\begin{aligned} a &= r\big(f(a)\big) \\ &= r\big(\vee\{x \in L \mid x \ll_Z f(a)\}\big) \\ &= \vee\{r(x) \mid x \in L, x \ll_Z f(a)\}. \end{aligned}$$

Now $\{r(x) \mid x \ll_Z f(a)\}$ is a Z-set the join of which is a. We now only need to verify that for each $x \ll_Z f(a)$, $r(x) \ll_Z a$. Let D be an arbitrary Z-set of A with $\vee D \geq a$. Then $\{f(y) \mid y \in D\}$ is a Z-set of L and $\vee\{f(y) \mid y \in D\} = f(\vee D) \geq f(a)$. So there is $d \in D$ with $f(d) \geq x$ and hence $d = r\big(f(d)\big) \geq r(x)$. This shows that $r(x) \ll_Z a$. It then follows immediately that $a = \vee\{y \in A \mid y \ll_Z a\}$ and $\{y \in A \mid y \ll_Z a\}$ is a Z-set of A. Thus A is Z-continuous.

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Suppose now that L is also stable; we show that A is stable as well. First, $1_L \ll_Z 1_L = f(1_L)$. By the above discussion this implies that $1_A = r(1_L) \ll_Z 1_A$, *i.e.* 1_A is Z-compact. If $y, a, b \in A$ such that $y \ll_Z a$, $y \ll_Z b$, then there are elements $x_1 \ll_Z f(a)$ and $x_2 \ll_Z f(b)$ such that $y \leq r(x_1)$, $y \leq r(x_2)$. But L is stable, so $x_1 \wedge x_2 \ll_Z f(a) \wedge f(b) = f(a \wedge b)$. This then indicates that $r(x_1 \wedge x_2) \ll_Z a \wedge b$. As $y \leq r(x_1) \wedge r(x_2) = r(x_1 \wedge x_2)$, it follows that $y \ll_Z a \wedge b$. Hence A is stable.

3. **E-projective *Z*-frames.** Let $\mathcal{C}$ be a category and $\mathbf{E}$ be a collection of morphisms in $\mathcal{C}$. An object A of $\mathcal{C}$ is called $\mathbf{E}$-*projective* if for any morphism $r: B \longrightarrow C$ in $\mathbf{E}$ and any $f: A \longrightarrow C$ in $\mathcal{C}$ there exists a morphism $\tilde{f}: A \longrightarrow B$ such that $f = r \circ \tilde{f}$. If $\mathbf{E}$ is the collection of all epimorphisms then the $\mathbf{E}$-projective objects are exactly the projective objects of $\mathcal{C}$.

It is well known that retracts of $\mathbf{E}$-projective objects are $\mathbf{E}$-projective.

If $\mathbf{G}: \mathcal{C} \longrightarrow \mathcal{D}$ and $\mathbf{F}: \mathcal{D} \longrightarrow \mathcal{C}$ are functors such that $\mathbf{F}$ is left adjoint to $\mathbf{G}$, with back adjunction $\epsilon: \mathbf{F} \circ \mathbf{G} \longrightarrow \mathrm{Id}_{\mathcal{C}}$, then it is natural to consider the $\mathbf{E}$-projective objects of $\mathcal{C}$ for the collection $\mathbf{E}$ of all $f: A \longrightarrow B$ such that $\mathbf{G}f$ has a section, that is, a right inverse. The basic result concerning these is the following:

LEMMA 3.1. *For any $A \in \mathcal{C}$, the following are equivalent:*

(1) A is $\mathbf{E}$-projective.

(2) $\epsilon_A: \mathbf{F} \circ \mathbf{G}A \longrightarrow A$ has a right inverse.

(3) A is a retract of some $\mathbf{F}X$.

PROOF. (1) $\Rightarrow$ (2) since $\epsilon_A \in \mathbf{E}$ by the adjunction identities, (2) $\Rightarrow$ (3) is trivial, and (3) $\Rightarrow$ (1) follows from the fact that $\mathbf{F}X$ is $\mathbf{E}$-projective and an elementary calculation.

For the inclusion functor $\underline{\mathbf{ZFrm}} \longrightarrow \underline{\mathbf{Slat}}$ and its left adjoint $Z: \underline{\mathbf{Slat}} \longrightarrow \underline{\mathbf{ZFrm}}$, $\mathbf{E}$ is the collection of all Z-frame homomorphisms which have a section in $\underline{\mathbf{Slat}}$, and the back adjunction is given by the maps $\vee: Z(A) \longrightarrow A$. For this, we now have the following main result of this paper.

THEOREM 3.2. *A Z-frame A is $\mathbf{E}$-projective if and only if it is stably Z-continuous.*

PROOF. By Lemma 3.1 it is enough to establish that $\vee: Z(A) \longrightarrow A$ has a right inverse in $\underline{\mathbf{ZFrm}}$ iff A is stably Z-continuous. Since ($\Rightarrow$) follows from Lemmas 2.2 and 2.3 it remains to prove ($\Leftarrow$). We claim that

$$\omega: A \longrightarrow Z(A), \ \omega(a) = \{x \in A \mid x \ll_Z a\}, \quad a \in A,$$

defines the desired right inverse of $\vee: Z(A) \longrightarrow A$. Since A is Z-continuous, so $\omega(a) \in Z(A)$ and $\vee\omega(a) = a$ for each $a \in A$. That ω is a meet-semilattice homomorphism follows from the stability of A. Finally, for any $D \in Z(A)$ if $x \ll_Z \vee D$, then $x \ll_Z y \ll_Z \vee D$ for some y by Remark 2.1(4), hence $x \in \omega(y)$ and $y \in D$, and therefore $x \in \cup\omega(D)$; it follows that $\omega(\vee D) \subseteq \cup\omega(D) = \vee\omega(D)$, the non-trivial part of the identity $\omega(\vee D) = \vee\omega(D)$. So ω preserves all joins of Z-sets. Hence ω is a Z-complete homomorphism, and is an right inverse of $\vee$.

REMARKS 3.3. (1) Obviously each morphism $f: A \longrightarrow B$ in $\mathbf{E}$ is a surjective map, so it is an epimorphism in $\underline{\mathbf{ZFrm}}$. Thus every projective Z-frame is $\mathbf{E}$-projective.

(2) If we take $\mathbf{E_1}$ to be the collection of all those surjective Z-frame homomorphisms $f: A \longrightarrow B$ which have a right adjoint $f_*: B \longrightarrow A$, then $\mathbf{E_1} \subseteq \mathbf{E}$. And it is not difficult to see that A is $\mathbf{E}$-projective if and only if A is $\mathbf{E_1}$-projective.

REMARK 3.4. As pointed out in Remark 7 of [3], the usual stably continuous frames can be characterized as the projectives relative to those surjective frame homomorphisms whose right adjoint preserves finitary joins. In terms of this paper: for the category $\mathbf{D}$ of bounded distributive lattices, the functor $\mathcal{I}: \mathbf{D} \longrightarrow \underline{\mathbf{Frm}}$ assigning to each $A \in \mathbf{D}$ its ideal lattice $\mathcal{I}A$ is a *lattice version* of the set systems here considered for semilattices, and all the arguments presented for the later have their exact counterparts for these more specialized set systems. In particular, the $\mathcal{I}$-frames are exactly the frames. Further $\mathcal{I}$ is left adjoint to the inclusion functor $\underline{\mathbf{Frm}} \longrightarrow \mathbf{D}$, and $\mathcal{I}$-continuous just means continuous. Hence the stably continuous frames are exactly the frames projective with respect to the homomorphisms which have a section in $\mathbf{D}$. Notice that [3] says slightly different thing: it refers to those homomorphisms which are surjective and have their right adjoints in $\mathbf{D}$.

We call a Z-frame homomorphism *proper* if it preserves $\ll_Z$. Let $\underline{\mathbf{SZCFrm}}$ be the category of all stably Z-continuous frames with proper homomorphisms. Then Z is a functor from $\underline{\mathbf{ZFrm}}$ to $\underline{\mathbf{SZCFrm}}$ because $Z(f): Z(A) \longrightarrow Z(B)$ is proper for any $f: A \longrightarrow B$ in $\underline{\mathbf{ZFrm}}$. Now following the same method as in [3], it can be proved that for any Z-frame A, $Z(A)$ is the coreflection to $\underline{\mathbf{SZCFrm}}$, with coreflection mapping $\vee: Z(A) \longrightarrow A$.

4. **Some applications.** The above theorem applies easily to many special cases.

EXAMPLE 4.1. For each semilattice S, take $\mathbf{Z}(S) = \mathcal{D}(S)$, the set of all down-sets of S. Then A is a Z-frame if and only if it is a frame, *i.e.* iff A is a complete lattice such that for any $a \in A$ and any nonempty $X \subseteq A$, the following equation holds:

$$a \wedge \vee X = \vee\{a \wedge x \mid x \in X\}.$$

By Raney's characterization, for this Z, A is Z-continuous if and only if it is a completely distributive lattice. A frame homomorphism $f: A \longrightarrow B$ is a semilattice homomorphism that preserves joins of arbitrary sets. Hence every frame homomorphism $f: A \longrightarrow B$ has a right adjoint $f_*: B \longrightarrow A$, and f_* is a section of f iff f is surjective. By Remark 3.3, $\mathbf{E}$ is the collection of all surjective frame homomorphisms, which in turn are exactly the regular epimorphisms in $\underline{\mathbf{Frm}}$. So $\mathbf{E}$-projective frames are exactly the regular-projective frames. Theorem 3.2 then says that the regular-projective frames are exactly the stably completely distributive lattices. This is the main result obtained by Banaschewski and Niefield in [3] where completely distributive lattices are called *supercontinuous lattices*.

EXAMPLE 4.2. For each semilattice S let $\mathbf{Z}(S) = \mathrm{Idl}(S)$, the collection of all ideals of S. Here $D \in \mathrm{Idl}(S)$ iff it is a down-set and up-directed. Then $\mathbf{Z}$ is a set system on $\underline{\mathbf{Slat}}$. In this case, a Z-frame A is a semilattice in which every up-directed set has a join and the equation

$$a \wedge \vee D = \vee\{a \wedge x \mid x \in D\}$$

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holds for any $a \in A$ and $D \in \mathrm{Idl}(A)$. Thus Z-frames are exactly the meet continuous semilattices [8], or preframes that have been studied by Banaschewski [4], Johnstone and Vickers [11]. Now a Z-continuous semilattice is exactly a continuous semilattice in the sense of [8]. Hence Theorem 3.2 implies that a preframe is **E**-projective if and only if it is a stably continuous semilattice.

EXAMPLE 4.3. For each semilattice S let $\mathbf{Z}(S) = \{\downarrow E \mid E$ is a finite subset of $S\}$. **Z** defines a set system on **Slat**. A Z-complete semilattice is just a lattice. A Z-frame now is exactly a distributive lattice. A Z-continuous semilattice A is a lattice satisfies the property that for each element $a \in A$, there is a finite set $D = \{d_i \mid i = 1, 2, \ldots, n\}$ with $\vee D = a$ and for each d_i if $x \vee y \geq a$ then $x \geq d_i$ or $y \geq d_i$.

EXAMPLE 4.4. For each semilattice S, define $\mathbf{Z}(S) = \{\downarrow E \mid E \subseteq S$ is a countable set$\}$. Then a Z-frame is a so-called σ-frame which has been studied by many authors especially Banaschewski [2]. The Theorem 2.1 here characterizes the **E**-projective σ-frames.

EXAMPLE 4.5. For the smallest set system $Z(S) = P(S)$ given in Remarks 1.2, Z-frames are exactly the semilattices, and $\ll_Z = \leq$. So in this case every Z-frame is **E**-projective.

REMARKS 4.6. (1) In [5] it was shown that there is only one projective frame, namely the frame **2**, the two elements chain. In [1] it was shown that **2** is also the unique projective distributive lattice. Thus it is natural to consider more general types of projective objects such as the **E**-projective Z-frames.

(2) We still have not dealt with the determination of the injective Z-frames. In some special cases one has perfect characterizations of these. For instance, the injective semilattices are the frames [7] [9], whereas, at the other extreme, there are no non-trivial injective frames [4].

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