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# MONOID GRADINGS ON ALGEBRAS AND THE CARTAN DETERMINANT CONJECTURE*

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In this work we tackle the Cartan determinant conjecture for finite-dimensional algebras through monoid gradings. Given an adequate $\Sigma$-grading on the left Artinian ring $A$, where $\Sigma$ is a monoid, we construct a generalized Cartan matrix with entries in $\mathbb{Z}\Sigma$, which is right invertible whenever $gl.dim\, A < \infty$. That gives a positive answer to the conjecture when $A$ admits a strongly adequate grading by an aperiodic commutative monoid. We then show that, even though this does not give a definite answer to the conjecture, it strictly widens the class of known graded algebras for which it is true.

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If $\{P_1, \ldots, P_n\}$ is a family of representatives, up to isomorphism, of the projective indecomposable modules over a left Artinian ring $A$ and $\{S_1 = P_1/JP_1, \ldots, S_n = P_n/JP_n\}$ is the corresponding family of simple modules, then the Cartan matrix $C_A$ of $A$ is the $n \times n$ integer matrix whose $(i,j)$-entry is the multiplicity of $S_i$ as a composition factor of $P_j$, for every $(i,j)$. It is well-known (cf. [**6**]) that if $gl.dim\, A < \infty$ then $\det C_A = \mp 1$. Based on the lack of examples with value $-1$, there has grown the so-called *Cartan determinant conjecture*, which asserts that if $gl.dim\, A < \infty$ then $\det C_A = 1$.

When looking at the different situations in which the Cartan determinant conjecture has been settled, one clearly sees two types of arguments. The first is by direct calculations, only possible in very particular cases. The second involves a certain grading on the algebra. The reader is referred to [**8**] for a good survey. Our objective is to give a new approach to the conjecture via monoid gradings on the algebra, generalizing known results on the relationship between gradability of the algebra and verification of the conjecture. In the first section we introduce the notion of (strongly) adequate grading on a left Artinian ring $A$ by a monoid $\Sigma$ and, associated with it, the construction of a generalized Cartan matrix $\hat{C}_A$ in $M_{n\times n}(\mathbb{Z}\Sigma)$, where $\mathbb{Z}\Sigma$ is the monoid ring. The first main result (Theorem 1.7) states that if $gl.dim\, A < \infty$ then $\hat{C}_A$ is right invertible in $M_{n\times n}(\mathbb{Z}\Sigma)$ and, when passing to the abelianization $\bar{\Sigma}$ of $\Sigma$, $\det \hat{C}_A$ is a unit of $\mathbb{Z}\bar{\Sigma}$. The second main result (Theorem 1.8) states the verification of the conjecture when $A$ admits a strongly adequate grading by an aperiodic commutative monoid.

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Once this result is at hand, one naturally asks if every f.d. algebra $A$ with $gl.dim\, A < \infty$ admits such a grading or, on the contrary, if any such algebra admitting the grading is already graded in the Wilson sense ([**14**]). In the second section we show that the answer to both questions is "no" (Propositions 2.6 & 2.8) and, on our way towards proving the latter proposition, we give a method of constructing explicitly a presentation by quiver and relations of the Auslander algebra $\mathrm{End}_B(B \oplus B/J^{\ell-1} \oplus \ldots \oplus B/J)$, where $B$ is a diagram algebra and $\ell = \ell(B)$ is the Loewy length of $B$.

Most terminology in our work is standard and should be familiar for both people working in Ring Theory and Representation of Algebras. By [**10**] we know that every f.d. split basic algebra over a field $K$ is isomorphic to $K\Delta/I$, where $\Delta$ is a uniquely determined quiver and I is an ideal of $K\Delta$ containing all paths of length $m$, for some $m \geq 2$, and consisting of $K$-linear combinations of paths of lengths $\geq 2$. Every such $I$ is called an *adequate ideal for $A$ in $K\Delta$* and any finite subset $\rho$ generating $I$ is called an *adequate set of relations for $A$ in $K\Delta$*. In the second section we will borrow from [**12**] the concept of *change of variable*, which is an algebra endomorphism $f: K\Delta \to K\Delta$ fixing the vertices and being bijective modulo the ideal $(I^+)^2$ of $K\Delta$ generated by the paths of length $\geq 2$. Finally, on what concerns $\mathbb{Z}$-gradability of an f.d. algebra $A$, we will follow the terminology already used in [**13**]. So $A$ will be called *gradable by the radical* when $A$ is isomorphic to its associated graded algebra $G(A)$ with respect to the powers of the radical. More generally, $A$ will be said to be *gradable in a semisimple way* when it admits a grading as in [**14**], i.e., a positive grading $A = \oplus_{n \geq 0} A_n$ such that $J(A) = \oplus_{n>0} A_n$.

All modules in this paper are left unitary modules and, unless otherwise stated, they are always assumed to be finitely generated.

## 1. Adequate gradings by monoids: the generalized Cartan matrix

Throughout this section $A$ will be a left Artinian ring, $\{e_1, \ldots, e_n\}$ will stand for a basic family of primitive orthogonal idempotents, so that $\{P_1 = Ae_1, \ldots, P_n = Ae_n\}$ and $\{S_1 = Ae_1/Je_1, \ldots, S_n = Ae_n/Je_n\}$ are families of representatives, up to isomorphism, of the indecomposable projective and the simple $A$-modules, respectively.

**Definition.** Let $A$ be a left Artinian ring and $\Sigma$ a monoid (always with multiplicative notation). A $\Sigma$-grading on $A$, $A = \oplus_{\sigma \in \Sigma} A_\sigma$, is said to be *adequate* when $J(A) = \oplus_{\sigma \neq 1} A_\sigma$, where $J(A)$ denotes the Jacobson radical of $A$.

**Remark 1.1.** Notice that if $A = \oplus_{\sigma \in \Sigma} A_\sigma$ is an adequate $\Sigma$-grading of $A$, then $A_\sigma A_\tau = 0$ whenever $\sigma\tau = 1$ and $\sigma, \tau \neq 1$. Furthermore $A_1 \cong A/J(A)$ is a semisimple subring of $A$. When, moreover, $A_\sigma = 0$ for every unit $\sigma \neq 1$ of $\Sigma$, we shall say that the grading is *strongly adequate*.

Given an adequate grading $A = \oplus_{\sigma \in \Sigma} A_\sigma$, an important object of our study will be the skeletally small abelian category $(A, \Sigma) - gr$ whose objects are the $\Sigma$-graded (always finitely generated) $A$-modules. The above mentioned idempotents $\{e_1, \ldots, e_n\}$ can be chosen homogeneous of degree 1. That is equivalent to saying that each indecomposable projective $P_i$ has a canonical $\Sigma$-grading for which it is a direct summand of ${}_AA$ in the category $(A, \Sigma) - gr$. The simple $S_i$ with the trivial grading is also an object of $(A, \Sigma) - gr$, that we still denote by $S_i$. Finally, given $M \in (A, \Sigma) - gr$ and $\sigma \in \Sigma$, we define a new graded $A$-module, called the *σ-shifting* $M[\sigma]$, which coincides with $M$ as an $A$-module but has the following grading: $M[\sigma]_\tau = \Sigma\{M_\nu/\nu \in \Sigma \text{ and } \nu\sigma = \tau\}$. When dealing with the skeletally small abelian category $\mathscr{A}$, we can always consider the Grothendieck group $K_0(\mathscr{A})$. The next proposition is standard and its proof is left as an exercise:

**Proposition 1.2.** *Let $\mathscr{A}$ be a skeletally small abelian category all of whose objects have finite composition length. Then $K_0(\mathscr{A})$ is the free abelian group on the set of isomorphism classes of simple objects of $\mathscr{A}$.*

It is hence important to identify the simple objects of $(A, \Sigma) - gr$ in order to deal with $K_0((A, \Sigma) - gr)$, a group that we shall denote by $K_0(A, \Sigma)$ in the sequel.

**Lemma 1.3.** *Let $A$ and $\Sigma$ be as above, $A = \oplus_{\sigma\in\Sigma} A_\sigma$ being an adequate $\Sigma$-grading on $A$. $\{S_i[\sigma]/i = 1, \ldots, n \text{ and } \sigma \in \Sigma\}$ is a set of representatives, up to isomorphism, of the simple objects of $(A, \Sigma) - gr$.*

**Proof.** Since $J = J(A)$ is a graded ideal of $A$, for every object $T$ of $(A, \Sigma) - gr$ $J(A)T$ is a graded submodule of $T$. By Nakayama's Lemma, $J(A)T \neq T$. When $T$ is a simple object, we get $J(A)T = 0$. From that it follows easily that each homogeneous component $T_\sigma$ of $T$ is actually a graded submodule of $T$. But the simplicity of $T$ implies that only one $\sigma \in \Sigma$ exists for which $T_\sigma \neq 0$, and in that case $T_\sigma$ is a simple $A$-module. It is clear now that $T \cong S_i[\sigma]$, where $S_i$ is the simple $A$-module which is isomorphic to $T_\sigma$. On the other hand, if $S_i[\sigma] \cong S_j[\tau]$ a look at the non-zero homogeneous component of both sides of the isomorphism shows that $\sigma = \tau$. But $S_i[\sigma] \cong S_j[\sigma]$ clearly implies $S_i \cong S_j$ and so $i = j$. □

The above proof shows that $J(A) \subseteq J^{gr}(A)$, where $J^{gr}(A)$ is the graded Jacobson radical. But, since every simple $A$-module is canonically graded, the converse inclusion holds as well. Therefore $J(A) = J^{gr}(A)$.

We are now ready to prove a fundamental fact.

**Proposition 1.4.** *Let $A$, $\Sigma$ and $A = \oplus_{\sigma\in\Sigma} A_\sigma$ as above. Then $K_0(A, \Sigma)$ is the free abelian group on the set $\{S_i[\sigma]/i = 1, \ldots, n \text{ and } \sigma \in \Sigma\}$. Moreover, $M \cdot \sigma = M[\sigma]$ yields a structure of a free right $\mathbb{Z}\Sigma$-module on $K_0(A, \Sigma)$, with basis $\{S_1, \ldots, S_n\}$.*

**Proof.** The only thing that remains to be proved is the statement about the $\mathbb{Z}\Sigma$-module structure on $K_0(A, \Sigma)$. For that, bearing in mind the first part of the proposition, we only need to check that $(S_i[\sigma]\cdot\tau)\cdot v = S_i[\sigma]\cdot(\tau v)$, which is a direct consequence of the associativity in $\Sigma$ and the fact that $T[\sigma][\tau] = T[\sigma\tau]$, for every simple object $T$ of $(A,\Sigma)-gr$ and all $\sigma,\tau\in\Sigma$. Finally, $\{S_i[\sigma]/i = 1,\ldots,n \ \& \ \sigma\in\Sigma\}$ being a basis of $K_0(A,\Sigma)$ as abelian group easily implies that $\{S_1,\ldots,S_n\}$ is a basis as $\mathbb{Z}\Sigma$-module. □

In the proof of our first main result we will need the following property of gradability of the projective covers.

**Definition.** Let $M\in(A,\Sigma)-gr$ and $\varepsilon: P\to M$ be a homomorphism, where $P$ is a projective $A$-module. We shall say that $\varepsilon$ is *gradable* when it is possible to give $P$ a $\Sigma$-grading so that $P$ becomes a graded $A$-module isomorphic in $(A,\Sigma)-gr$ to a finite direct sum $\oplus\{P_i[\sigma]^{u(i,\sigma)}/i = 1,\ldots,n,\sigma\in\Sigma\}$ and $\varepsilon(P_\tau)\subseteq M_\tau$, for every $\tau\in\Sigma$.

The crucial fact now is the following.

**Proposition 1.5.** *Given an adequate $\Sigma$-grading on the left Artinian ring $A$ and a $\Sigma$-graded module $M$, there is a minimal projective resolution of $M$ which is gradable, i.e., all its morphisms are gradable.*

**Proof.** Let $M\in(A,\Sigma)-gr$. Since $J = J(A)$ is a homogeneous ideal $JM$ is a graded submodule of $M$, so that $M/JM$ is also a graded module. Semisimplicity of $M/JM$ as an $A$-module immediately implies the same property as an object of $(A,\Sigma)-gr$. But then it admits a decomposition $M/JM = \oplus_{i=1}^{n}\oplus_{\sigma\in\Sigma}(\oplus_{j=1}^{m(i,\sigma)}X_{i,\sigma,j})$, where $X_{i,\sigma,j}$ is a graded submodule of $M/JM$ isomorphic to $S_i[\sigma]$, for every triple $(i,\sigma,j)$ (notice that $m(i,\sigma)$ is then the multiplicity of $S_i[\sigma]$ as a direct summand of $M/JM$ and so $m(i,\sigma)=0$ for all but finitely many pairs $(i,\sigma)$). Since, clearly, $M_\sigma/(JM)_\sigma \cong (M/JM)_\sigma = \oplus_{i=1}^{n}(\oplus_{j=1}^{m(i,\sigma)}X_{i,\sigma,j})$, we can choose a family of homogeneous elements in $M$, $\{x_{i,\sigma,j}/(i,\sigma)\in\mathbb{N}_n\times\Sigma$, $j = 1,\ldots,m(i,\sigma)\}$ (almost all zero!) such that the degree of $x_{i,\sigma,j}$ is $\sigma$ and $A\bar{x}_{i,\sigma,j} = X_{i,\sigma,j}$, where $\bar{x}_{i,\sigma,j}$ is the class of $x_{i,\sigma,j}$ modulo $JM$. Nakayama's Lemma tells as that $M = \Sigma Ax_{i,\sigma,j}$. Now, for each nonzero $x_{i,\sigma,j}$, we choose a projective cover $\varepsilon_{i,\sigma,j}: P_i = Ae_i \to Ax_{i,\sigma,j}$ that maps $e_i$ onto $x_{i,\sigma,j}$. By giving $e_i$ degree $\sigma$, we can view $\varepsilon_{i,\sigma,j}$ as a morphism in $(A,\Sigma)-gr$ $P_i[\sigma]\to Ax_{i,\sigma,j}$. It is now obvious that we get a graded homomorphism from a suitable direct sum of the $P_i[\sigma]$'s onto $M = \Sigma Ax_{i,\sigma,j}$, which is actually a projective cover of $A$-modules. □

Given an adequate $\Sigma$-grading $A = \oplus_{\sigma\in\Sigma}A_\sigma$ as above, we can define a $n\times n$ matrix ($n =$ the number of nonisomorphic simples) with entries in the monoid ring $\mathbb{Z}\Sigma$, that we shall denote by $\hat{C}_A$ and call the *generalized Cartan matrix* (*associated with the given $\Sigma$-grading*). Its $(i,j)$-th entry will be, by definition, $\hat{c}_{ij} = \sum_{\sigma\in\Sigma}\hat{c}_{ij}(\sigma)\sigma$, where $\hat{c}_{ij}(\sigma)$ denotes the multiplicity of $S_i[\sigma]$ as a composition factor (in the category $(A,\Sigma)-gr$) of $P_j$ with the canonical grading mentioned at the beginning.

**Remarks 1.6.** (a) As an immediate consequence of the definition, we have the following equality in the $\mathbb{Z}\Sigma$-module $K_0(A,\Sigma)$: $P_j = \sum_{i=1}^{n} S_i \cdot \hat{c}_{ij}$. In other words, the $j$-th column of $\hat{C}_A$ consists of the coordinates of $P_j$ with respect to the basis of the right $\mathbb{Z}\Sigma$-module $K_0(A,\Sigma)$ given by the simple $A$-modules.

(b) $\hat{c}_{ii} = 1 + \sum_{\sigma\neq 1} \hat{c}_{ii}(\sigma)\sigma$, for all $i$, while $\hat{c}_{ij} = \sum_{\sigma\neq 1} \hat{c}_{ij}(\sigma)\sigma$. Indeed from Remark 1.1 follows that $JP_j$ is a graded submodule of $P_j$ contained in $\oplus_{\sigma\neq 1}(P_j)_\sigma$ and so $JP_j = \oplus_{\sigma\neq 1}(P_j)_\sigma$. But then $S_i = S_i[1]$ is a composition factor of $P_j = P_j[1]$ in $(A,\Sigma) - gr$ if, and only if, $i = j$ in which case its multiplicity is exactly 1.

(c) The augmentation map $\mathbb{Z}\Sigma \to \mathbb{Z}$, or its induced homomorphism $M_{n\times n}(\mathbb{Z}\Sigma) \to M_{n\times n}(\mathbb{Z})$, takes $\hat{C}_A$ onto the usual Cartan matrix $C_A$.

We shall denote by $\bar{\Sigma}$ the abelianization of $\Sigma$, i.e., the factor of $\Sigma$ by the congruence relation generated by the relations $\sigma\tau \equiv \tau\sigma$, with $\sigma, \tau \in \Sigma$. There is a canonical ring homomorphism $\mathbb{Z}\Sigma \to \mathbb{Z}\bar{\Sigma}$ which induces another one $M_{n\times n}(\mathbb{Z}\Sigma) \to M_{n\times n}(\mathbb{Z}\bar{\Sigma})$. The image of $\hat{C}_A$ by this latter homomorphism will be denoted by $C'_A$ and called the *abelianized version* of $\hat{C}_A$. The first main result is now available.

**Theorem 1.7.** *Let $A = \oplus_{\sigma\in\Sigma} A_\sigma$ be an adequate grading on the left Artinian ring $A$, and let $\hat{C}_A$ and $C'_A$ be the generalized Cartan matrix and its abelianized version, respectively. If the global dimension of $A$ is finite, then $\hat{C}_A$ has a right inverse in $M_{n\times n}(\mathbb{Z}\Sigma)$ and* $\det C'_A$ *is an invertible element in $\mathbb{Z}\bar{\Sigma}$.*

**Proof.** Let $S_i$ be any of the simple $A$-modules considered with its trivial grading. We can take a minimal projective resolution of $S_i$ which is gradable, where the grading of each term is a suitable direct sum of shiftings $P_j[\sigma]$. Suppose $O \to Q_{i,r} \to \dots \to Q_{i,1} \to Q_{i,o} \to S_i \to O$ is that resolution. Then, in $K_0(A,\Sigma)$, we get $S_i = \sum_0^r (-1)^k Q_{i,k}$. But, on the other hand, $Q_{i,k} = \sum_{l=1}^{n} \sum_{\sigma\in\Sigma} P_l[\sigma] \cdot u(i,k,l,\sigma)$, where $u(i,k,l,\sigma)$ denotes the multiplicity of $P_l[\sigma]$ as a direct summand of $Q_{i,k}$. Moreover $P_l[\sigma] = P_l \cdot \sigma$, by definition of the $\mathbb{Z}\Sigma$-module structure on $K_0(A,\Sigma)$. By suitable substitutions, we get the following equality in the latter $\mathbb{Z}\Sigma$-module: $S_i = \sum_{l=1} P_l \cdot \{\sum_{k=0} \sum_{\sigma\in\Sigma} (-1)^k u(i,k,l,\sigma)\sigma\}$. Since, by the first remark above, $P_l = \sum_{j=1} S_j \cdot \hat{c}_{jl}$ we also obtain $S_i = \sum_{j=1} S_j [\sum_{l=1} \hat{c}_{jl} \cdot (\sum_{\sigma\in\Sigma} \sum_{k=0} (-1)^k u(i,k,l,\sigma)\sigma)]$. By comparing with $S_i = \sum_{j=1} S_j \delta_{ji}$ and bearing in mind that $\{S_i, \dots, S_n\}$ is a basis of the $\mathbb{Z}\Sigma$-module $K_0(A,\Sigma)$, we conclude that, for all $i, j = 1, \dots, n$, $\delta_{ji} = \sum_{l=1} \hat{c}_{jl} \cdot (\sum_{\sigma\in\Sigma} \sum_{k=0} (-1)^k u(i,k,l,\sigma)\sigma)$. It is now clear that $d_{li} = \sum_{\sigma\in\Sigma} \sum_{k=0} (-1)^k u(i,k,l,\sigma)\sigma$ yields the $(1,i)$-th entry of a matrix in $M_{n\times n}(\mathbb{Z}\Sigma)$ which is right inverse to $\hat{C}_A$.

Finally, via the canonical homomorphism $M_{n\times n}(\mathbb{Z}\Sigma) \to M_{n\times n}(\mathbb{Z}\bar{\Sigma})$, we get that $C'_A$ is an invertible matrix in $M_{n\times n}(\mathbb{Z}\bar{\Sigma})$, which is equivalent to saying that $\det C'_A$ is an invertible element in $\mathbb{Z}\bar{\Sigma}$. □

From the above theorem we shall derive a new partial affirmative answer to the Cartan determinant conjecture, not only for finite dimensional algebras over a field, but for left Artinian rings in general. First recall some terminology (see **[11]**).

**Definition.** Let $\Sigma$ be a commutative monoid. We shall say that $\Sigma$ is *torsionfree* in the case that $\sigma^n = \tau^n$ implies $\sigma = \tau$, for $\sigma$, $\tau \in \Sigma$ and $n \geq 1$. In the case that every nontrivial cyclic submonoid of $\Sigma$ is infinite, we shall say that $\Sigma$ is *aperiodic*.

**Theorem 1.8.** *Suppose that the left Artinian ring $A$ admits a strongly adequate grading $A = \oplus_{\sigma\in\Sigma} A_\sigma$, where $\Sigma$ of an aperiodic commutative monoid. If gl.dim $A < \infty$ then* $\det C_A = 1$.

**Proof.** Our previous theorem states that $\det C'_A$ is an invertible element in $\mathbb{Z}\Sigma$. On the other hand, by construction of $C'_A$, its entries can be written in the form $\sum_{\upsilon\in\Sigma} m_\upsilon \upsilon$, where $m_\upsilon = 0$ whenever $A_\upsilon = 0$. In particular, by Remark 1.1, the support of the entries of $C'_A$ always consists of nonunits of $\Sigma$ and 1. Bearing in mind Remark 1.6(b), the latter implies that $\det C'_A = 1 + \sum_{\upsilon\in\Sigma\setminus\{1\}} m_\upsilon \upsilon$, where $m_\upsilon = 0$ whenever $\upsilon$ is a unit of $\Sigma$. Now the set of all relations of the form $\upsilon \equiv \varphi$ whenever there is a $n \geq 1$ such that $\upsilon^n = \varphi^n$ in $\Sigma$ is already a congruence relation, which is compatible with the multiplication in $\Sigma$. When we factor by it, the new (commutative) monoid $\hat{\Sigma}$ is aperiodic and torsionfree. Moreover, if $q: \Sigma \to \hat{\Sigma}$ is the canonical projection and $\upsilon \in \operatorname{Ker} q$, then there exists $n \geq 1$ such that $\upsilon^n = 1$ in $\Sigma$. The aperiodicity of $\Sigma$ implies that $\upsilon = 1$. As a consequence, the induced homomorphism of monoid rings $\mathbb{Z}\Sigma \to \mathbb{Z}\hat{\Sigma}$ maps $\det C'_A$ onto an element of the form $1 + \sum_{s\in\hat{\Sigma}\setminus\{1\}} m_s s$, which must be invertible in $\mathbb{Z}\hat{\Sigma}$. By Theorem 11.15 in [**11**], that element is 1. By applying now the augmentation map $\mathbb{Z}\hat{\Sigma} \to \mathbb{Z}$ and taking into account Remark 1.6(c), one gets $\det C_A = 1$ as desired. □

We have now the following well-known extension of Wilson's result (see Section 6 in [**9**]) as a straightforward consequence:

**Corollary 1.9.** *Let $A = \oplus_{n\in\mathbb{N}} A_n$ be a positively graded left Artinian ring such that $J(A) = \oplus_{n>0} A_n$. If gl.dim $A < \infty$ then* $\det C'_A = 1 = \det C_A$. □

## 2. The monoid associated with a representation of an algebra by quiver and relations

As mentioned in the introduction, two natural questions arise for an f.d. algebra $A$ such that *gl.dim* $A < \infty$:

(1) Does there exist always a strongly adequate $\Sigma$-grading $A = \oplus_{\sigma\in\Sigma} A_\sigma$, where $\Sigma$ is an aperiodic commutative monoid?

(2) If $A$ actually admits such a grading, is $A$ gradable in a semisimple way?

A positive answer to Question 1 would imply the truth of the Cartan determinant conjecture. In turn, a positive answer to Question 2 would imply that our last theorem means really no advance in the conjecture with respect to Wilson's result ([**14**]). Our main goal in this section is to see that both questions have negative answers.

In the sequel $A$ is isomorphic to $K\Delta/\langle\rho\rangle$, where $\Delta$ is a finite quiver and $\rho$ is an adequate set of relations for $A$ in $K\Delta$. It will be assumed that no subrelation of an $r \in \rho$ is in the ideal $\langle\rho\rangle$ (i.e. if $r = \lambda_1 p_1 + \cdots + \lambda_t p_t \in \rho$, then no proper subset $\{i_1, \ldots, i_s\}$

of $\{1, \ldots, t\}$ exists such that $\lambda_{i_1} p_{i_1} + \cdots + \lambda_{i_s} p_{i_s} \in \langle\rho\rangle$). Note that every path in $\Delta$ may be also viewed as an element of the free monoid on the set $\Delta_1$ of arrows. We introduce now a key concept.

**Definition.** In the above situation, we shall call a *monoid associated* with *the representation* $(\Delta, \rho)$ and denote it by $\Sigma(\Delta, \rho)$ to the monoid given by:

(a) Generators: the arrows of $\Delta$.

(b) Relations: for each $r = \lambda_1 p_1 + \cdots + \lambda_t p_t \in \rho$, with $t \geq 2$ and $\lambda_1, \ldots, \lambda_t \in K^*$, we put $p_i \equiv p_j$ for all $i, j = 1, \ldots, t$.

We have now an immediate result.

**Proposition 2.1.** *Every split basic finite-dimensional algebra A given by the quiver with relations* $(\Delta, \rho)$ *admits a strongly adequate grading by* $\Sigma = \Sigma(\Delta, \rho)$ *or* $\bar{\Sigma}$, *its abelianization, in which the degree of a nonzero path p of length* $\geq 1$ *is exactly the class* $\bar{p}$ *of p in* $\Sigma$ *or* $\bar{\Sigma}$.

**Proof.** By assigning degree $\bar{p}$ to every path $p$ of length $\geq 1$ and degree 1 to every path of length 0, we get a $\Sigma$-grading on the path algebra $K\Delta$. If $r = \lambda_1 p_1 + \cdots + \lambda_t p_t \in \rho$ then, by definition of $\Sigma$, the degrees of $p_1, \ldots, p_t$ are all the same. Hence the ideal $\langle\rho\rangle$ of $K\Delta$ is generated by homogeneous elements and is thereby a graded ideal of $K\Delta$. Now $A \cong K\Delta/\langle\rho\rangle$ inherits an adequate $\Sigma$-grading satisfying the requirements. If now $v \in \bar{\Sigma}$ and $p\colon \Sigma \to \bar{\Sigma}$ is the canonical projection, we put $A_v = \Sigma\{A_\sigma/\sigma \in \Sigma$ and $p(\sigma) = v\}$ and get a strongly adequate $\bar{\Sigma}$-grading for $A$ as desired. □

**Example 2.2.** (i) If $A$ is the local algebra $K\langle X, Y\rangle/\langle\rho\rangle$, where $\rho = \{XY - 2Y^3X, X^2 + 3Y^4, Y^5$, all monomials of degree 6$\}$, assuming char $K \neq 2, 3$ then $\Sigma(\Delta, \rho)$ is given by two generators $x, y$ and relations $xy \equiv y^3x, x^2 \equiv y^4$, not caring about the zero relation $Y^5$. The algebra $A$ has a basis $\{1, x, y, x^2, xy, y, y^2, y^3, yx, y^2x\}$ and the degree of each element of that basis is the same element, but viewed as an element of $\Sigma$.

(ii) Every monomial algebra, by definition, admits a representation by quiver and zero-relations. The associated monoid is then the free monoid on the set $\Delta_1$ of arrows. That grading is precisely the one given by Burgess in [**5**].

(iii) We should observe that $\Sigma(\Delta, \rho)$ depends completely on the set $\rho$ of generators of the ideal $\langle\rho\rangle$ of $K\Delta$. For instance, the adequate ideal $I$ of $K\langle X, Y\rangle$ generated modulo $(X, Y)^3$ by $\{XY, YX\}$ is also generated modulo $(X, Y)^3$ by $\{XY, XY - YX\}$ While the first choice of $\rho$ gives that $\Sigma(\Delta, \rho)$ equals the free commutative monoid on two variables, the second one gives the monoid with two generators $x, y$ subject to the relation $xy = xy$.

Before tackling the two aforementioned questions, we first see how the grading just introduced could serve as an arithmetic test of the global dimension. That follows the trail of a converse of the graded version of the Cartan determinant conjecture, where some results are already known (see, e.g., Corollary 2.13 in [8] and Proposition 1.4 in [5]). For the terminology that appears in the next proposition, we refer to [1], except that we change the order when writing the paths (e.g., the path $\alpha\beta$ means for us $\beta \circ \alpha$, when we view them as maps between projective indecomposable modules). Also, instead of $\Gamma_m^i$ for the $m$-chains of origin $i$, we use $\Delta_m^i$ for the $m$-chains with terminal $i$ (e.g. $\Delta_2^i$ is the set of obstructions ending at $i$).

**Definition.** We shall say that $\Sigma = \Sigma(\Delta, \rho)$ is a *finite word monoid* in the case that, for every $\sigma \in \Sigma$, there are only finitely many words in the free monoid over $\Delta_1$ representing $\sigma$.

**Proposition 2.3.** *Let $A = K\Delta/\langle\rho\rangle$ be an algebra and $\Sigma = \Sigma(\Delta, \rho)$ be the associated monoid. Fix a suitable ordering of the set of paths in $\Delta$ and suppose $\Sigma$ is a finite word monoid and the following condition holds:*

(%) *If $p \in \Delta_m$ and $q \in \Delta_n$ are two chains which define the same element in $\Sigma$, then $m \equiv n \,(\mathrm{mod}\, 2\mathbb{Z})$.*

*If the generalized Cartan matrix $\hat{C}_A \in M_{n\times n}(\mathbb{Z}\Sigma)$ has a right inverse, then gl.dim $A < \infty$.*

**Proof.** We only need to observe that the construction of a projective resolution of a simple $S_i$ given in [1] can be taken to be graded, i.e., one can inductively give a $\Sigma$-grading to each $K_m$ in Theorem 2.7(3) of [1] and make all the involved maps into $\Sigma$-graded maps. More precisely, under such a grading $K_m = \oplus P_1[\sigma]^{v(i,m,l,\sigma)}$, where $v(i, m, l, \sigma)$ denotes the number of elements of $\Delta_m^i$ with origin 1 that yield the element $\sigma \in \Sigma$. The fact that $\Sigma$ is a finite word monoid allows us to construct the incidence ring $\mathbb{Z}[[\Sigma]]$ of $\Sigma$. This is the set of all functions $f: \Sigma \to \mathbb{Z}$, that we write $f = \sum_{\sigma\in\Sigma} f(\sigma)\sigma$, in which multiplication is defined by convolution: $(f \cdot g)(\sigma) = \Sigma\{f(\tau)g(v)/\tau,\ v \in \Sigma$ and $\tau v = \sigma\}$. An adaptation of the proof of Theorem 1.7 for the Anick and Green resolution following the pattern of [5] yields a right inverse $D = (d_{1i})$ for $\hat{C}_A$ in $M_{n\times n}(\mathbb{Z}[[\Sigma]])$, where $d_{1i} = \sum_{\sigma\in\Sigma}(\sum_{m=0}^{\infty}(-1)^m v(i, m, 1, \sigma))\sigma$ (observe that this is an element of $\mathbb{Z}[[\Sigma]]$ because of our finite word assumption on $\Sigma$). Since, by hypothesis, $\hat{C}_A$ is actually right invertible in $\mathbb{Z}\Sigma$, we have that $\sum_{m=0}^{\infty}(-1)^m v(i, m, l, \sigma) = 0$ for all but finitely many $\sigma \in \Sigma$. Our condition (%) implies that the latter is only possible if, for all but finitely many $\sigma \in \Sigma$, $v(i, m, l, \sigma) = 0$ for all $m \geq 0$. Again our finite word assumption on $\Sigma$ guarantees that there is a large enough $n$ such that $v(i, m, l, \sigma) = 0$, for all $\sigma \in \Sigma$ and all $m \geq n$. So the Anick and Green resolution of each $S_i$ is finite and, consequently, gl.dim $A < \infty$. □

**Remark 2.4.** Unfortunately the conditions that we need to impose on the presentation of the algebra to guarantee (%) are very restrictive. It is clearly verified when $\rho$ consists of zero relations, in which case Proposition 2.3 is just Proposition 1.4 in [5]. We shall indicate a nonmonomial situation where (%) holds. If $p, q \in \Delta_2$ are two obstructions such that $p = p^1u$, $q = uq^2$ and $p^1uq^2 \in \Delta_3$, we shall say that $p$ and $q$ *overlap adequately*. In such a case, the length of $u$ will be called the *length of the* *overlapping*. Suppose that $\rho$ consists of homogeneous relations of length $l$ and, moreover, all obstructions $p \in \Delta_2$ have length $l$. If the length of adequate overlappings is a constant, say $t$, then one inductively gets that every $q \in \Delta_m$ has length $l+(m-2)(l-t)$. Then, since the generating relations of $\Sigma(\Delta, \rho)$ are given by paths of equal length, (%) holds (even with $m = n$). For instance, the proposition can be applied to the algebra given by $2 \underset{\gamma}{\overset{\alpha}{\rightleftarrows}} 1 \underset{\delta}{\overset{\beta}{\rightleftarrows}} 3$, with $\rho = \{\alpha\delta, \beta\delta, \beta\gamma, \gamma\alpha - \delta\beta\}$, for, when we consider the antilexicographical order, we have $\Delta_2 = \{\alpha\delta, \beta\delta, \beta\gamma, \gamma\alpha\}$ and the length of adequate overlappings is constantly 1. Notice that the lexicographical order gives $\Delta_2 = \{\alpha\delta, \beta\delta, \beta\gamma, \delta\beta, \alpha\gamma\alpha, \gamma\alpha\gamma\}$.

We come now to answer our two intriguing questions. In both cases the crucial point is the following result of Auslander:

**Lemma 2.5.** ([**2**, Theorem 10.3]) *If $B$ is a finite-dimensional algebra of Loewy length $\ell = \ell(B)$ and $G$ is the $B$-module $B \oplus B/J^{\ell-1} \oplus \ldots \oplus B/J^2 \oplus B/J$, then $A = End_B(G)$ is a (finite-dimensional) algebra of finite global dimension.*

We are now ready for an answer to our first question. In the sequel $B$ will always be a local algebra $B$ and $A = End_B(G)$ as above. For the purpose of distinction, we shall denote by $\Gamma$ and $\Delta$ the quivers of $B$ and $A$, while the adequate sets of relations will be denoted by $\mu$ and $\rho$, respectively.

**Proposition 2.6.** *There are finite-dimensional algebras of finite global dimension admitting no strongly adequate grading by a commutative aperiodic monoid.*

**Proof.** Suppose that every f.d. algebra $A$ with finite global dimension admits such a grading, whence, in particular, every algebra $A = End_B(G)$, for $B$ and $G$ as in the above lemma. We know that $B$ is recoverable from $A$ in the form $B = eAe$, for an idempotent $e$, which we can assume to be homogeneous of degree 1. Now $B_\sigma = eA_\sigma e$, for every $\sigma \in \Sigma$, yields a strongly adequate $\Sigma$-grading on $B$. So the existence of our desired grading for algebras with finite global dimension implies that existence for every f.d. algebra. To finish the proof we only need to show an example of an f.d. (local) algebra that admits no such grading. Take for $B$ the "ungradable" local algebra of [**4**], i.e. $B = K\langle X, Y\rangle/(X^3, XY, YX^2, X^2 - Y^3, YX - Y^3)$. By a result analogous to Theorem 2.1 in [**10**], $B$ admits a strongly adequate grading by the commutative monoid $\Sigma$ if there is a weight function $d: \Gamma_1 = \{X, Y\} \to \Sigma$ and a change of variable $f: K\langle X, Y\rangle \to K\langle X, Y\rangle$ such that the ideal $I'$ of $K\langle X, Y\rangle$ generated by $\{f(X)^3, f(X)f(Y), f(Y)f(X)^2, f(X)^2 - f(Y)^3, f(Y)f(X) - f(Y)^3\} \cup \{$all paths in $\Gamma$ of length 4$\}$ is an adequate $d$-homogeneous ideal for $B$ in $K\langle X, Y\rangle$. But then, if $\mu'$ is a $d$-homogeneous set of generators of $I'$, $d$ extends to a morphism of monoids $\pi: \Sigma(\Gamma, \mu') \to \Sigma$, that induces another one $\bar{\pi}: \bar{\Sigma}(\Gamma, \mu') \to \Sigma$, such that the support of $A = \oplus_{\sigma \in \Sigma} A_\sigma$ is contained in $\operatorname{Im} \bar{\pi}$ and $\operatorname{Ker} \bar{\pi} = 1$. The aperiodicity of $\Sigma$ implies that of $\bar{\Sigma}(\Gamma, \mu')$. By using now Corollary 5 in [**12**], it is now a routine though cumbersome task to check that, for every adequate ideal $I'$ as above and every set of generators $\mu'$ of $I'$, $\bar{\Sigma}(\Gamma, \mu')$ is never aperiodic. Hence $\Sigma$ cannot be aperiodic. □

In order to answer our second question we shall first develop a method of constructing a presentation by quiver and relations of the algebra $A = End_B(G)$, when $B$ is a diagram algebra in the sense of [7]. We do it just for the case that $B$ is local; that is enough for our purposes, but the reader will have no difficulty in generalizing the construction for arbitrary $B$. Let us start with a presentation $B = K\Gamma/\langle\mu\rangle$, where $\mu$ consists of paths and differences of paths. We abuse notation and use the same letter for the paths and for the elements of $B$ that they represent. First of all, for every path $p$ in $\Gamma$ and positive integers $1 \leq k, m \leq \ell(B)$ such that $J(B)^k p \subseteq J(B)^m$, we denote by $p_{km}$ ($e_{km}$ in case $e = 1$ is the vertex) the element of $A$ induced by the map $B/J(B)^k \to B/J(B)^m$ that takes $1 + J(B)^k$ onto $p + J(B)^m$. It is well-known (see [3, p. 220]) that $B$ has a multiplicative basis $\mathfrak{B}$ consisting of paths in $\Gamma$. For each pair $(k, m)$ as above, $\mathcal{G}_{km} = \{p_{km}/p \in \mathfrak{B},\ J(B)^m p \neq 0$ in $B$ and $J(B)^k p \subseteq J(B)^m\}$ is a $K$-generating set of $Hom_B(B/J(B)^k, B/J(B)^m)$. Clearly, the vertices of the quiver $\Delta$ of $A$ can be taken to be the idempotents $e_{kk}$, where $k \in \{1, \ldots, \ell = \ell(B)\}$. Secondly, for each $1 \leq k,\ m \leq \ell(B)$, $e_{kk}Ae_{mm} \cong Hom_B(B/J(B)^k, B/J(B)^m)$ and we consider the subset $\Delta(k, m)$ of $\mathcal{G}_{km}$ consisting of those $p_{km}$'s which cannot be expressed as nontrivial products of $q_{st}$'s, for other paths $q \in \mathfrak{B}$ and positive integers $s, t$ (here a product of the $q_{st}$'s is called *trivial* if some $q_{st}$ equals $e_{rr}$, for some $r$). Unrigorously, the elements of the different $\Delta(k, m)$'s will be called *irreducible endomorphisms* of $G$. It is not difficult to see, using the multiplicity of $\mathfrak{B}$, that $\Delta(k, m)$ is a basis of $e_{kk}J(A)e_{mm}$ modulo $e_{kk}J(A)^2e_{mm}$ and so it can be taken as the set of arrows between $e_{kk}$ and $e_{mm}$ in $\Delta$. It is clear now that a product of irreducible endomorphisms $(p^1)_{k_1k_2} \cdots (p^r)_{k_rk_{r+1}}$ is zero iff $p^1 \cdots p^r \in J(B)^{k_{r+1}}$. This is the source of all the zero relations to be considered for $A$. Finally, if $p \in \mathfrak{B}$, $p_{km} \neq 0$ in $A$ and $(p^1)_{k_1k_2} \cdots (p^r)_{k_rk_{r+1}} = p_{km} = (q^1)_{m_1m_2} \cdots (q^s)_{m_sm_{s+1}}$ are two different factorizations of it as a product of irreducible endomorphisms of $G$, then $(p^1)_{k_1k_2} \cdots (p^r)_{k_rk_{r+1}} - (q^1)_{m_1m_2} \cdots (q^s)_{m_sm_{s+1}}$ is a nonzero relation for $A$ in $K\Delta$. It is not hard to see that we only need to consider those nonzero relations in order to get an adequate set $\rho$ of relations for $A$ in $K\Delta$. One can finally implement the procedure by dropping the redundant relations. We have proved:

**Lemma 2.7.** *The above procedure yields a presentation* $(\Delta, \rho)$ *of* $A = End_B(G)$ *by quiver and relations.*

We are now ready for an answer to our second question:

**Proposition 2.8.** *There are algebras of finite global dimension that are not gradable in a semisimple way, but admit a strongly adequate grading by a commutative aperiodic monoid.*

**Proof.** Let $B = K\Gamma/\langle\mu\rangle$ an arbitrary local diagram algebra and let $(\Delta, \rho)$ the presentation by quiver and relations given above. Our next goal is to show the existence of a surjective homomorphism of monoids $f\colon \Sigma(\Delta,\rho)\to\Sigma(\Gamma,\mu)$, which in turn induces one (also surjective) $\bar f\colon \bar\Sigma(\Delta,\rho)\to\bar\Sigma(\Gamma,\mu)$. Indeed, we define $f(p_{km})=p$, if $p_{km}$ is an irreducible element of $\mathscr{G}_{km}$ and $p$ is a path in $\Gamma$ of length $\geq 1$, and $f(p_{km})=1$ if $p=e$ ($e_{km}$ is irreducible in $\mathscr{G}_{km}$ only in case $k=m+1$) (here, of course, we are identifying the paths of $\Gamma$ with the elements of $\Sigma(\Gamma,\mu)$ that they represent). If $(p^1)_{k_1k_2}\cdots(p^r)_{k_rk_{r+1}}-(q^1)_{m_1m_2}\cdots(q^s)_{m_sm_{s+1}}$ is a nonzero relation for $A$ in $K\Delta$ as described above (hence $k_1=m_1$ and $k_{r+1}=m_{s+1}$) then $p^1\cdots p^r-q^1\cdots q^s\in J(B)^m$, where $m=k_{r+1}=m_{s+1}$. But, due to the multiplicative condition of $\mathfrak{B}$ and the fact that $p^1\cdots p^r$ and $q^1\cdots q^s$ are both elements of $\mathfrak{B}$, we have that $p^1\cdots p^r-q^1\cdots q^s\in J(B)^m$ iff either both $p^1\cdots p^r$ and $q^1\cdots q^s$ belong to $J(B)^m$ or $p^1\cdots p^r-q^1\cdots q^s=0$ in $B$. The fact that $(p^1)_{k_1k_2}\cdots(p^r)_{k_rk_{r+1}}=p_{km}=(q^1)_{m_1m_2}\cdots(q^s)_{m_sm_{s+1}}$, for some $p_{km}\neq 0$, excludes the first possibility. Hence $(p^1)_{k_1k_2}\cdots(p^r)_{k_rk_{r+1}}=(q^1)_{m_1m_2}\cdots(q^s)_{m_sm_{s+1}}$ in $\Sigma(\Delta,\rho)$ implies $f[(p^1)_{k_1k_2}\cdots(p^r)_{k_rk_{r+1}}]=p^1\cdots p^r=q^1\cdots q^s=f[(q^1)_{m_1m_2}\cdots(q^s)_{m_sm_{s+1}}]$ in $\Sigma(\Gamma,\mu)$. So $f$ is a well-defined map. It is clear that $f$ preserves multiplication and the 1. Since $f(\alpha_{km})=\alpha$, for every arrow $\alpha\in\Gamma_1$, $\Gamma_1\subseteq Im f$ and so $f$ is surjective. We should notice now that $E=\operatorname{Ker} f$ is the submonoid of $\Sigma(\Delta,\rho)$ generated by $\{e_{km}/k\geq m\}$, which is a free monoid with basis $\{e_{k+1,k}/k=1,\dots,\ell(B)-1\}$. From that, and the fact that the defining relations of $\Sigma(\Delta,\rho)$ give rise, via $f$, to relations in $\Sigma(\Gamma,\mu)$ each term of which is of length at least 2, follows easily that the induced morphism of commutative monoids $\bar f\colon\bar\Sigma(\Delta,\rho)\to\bar\Sigma(\Gamma,\mu)$ is surjective and has as kernel the abelianization $\bar E$ of $E$, which is a free commutative monoid. From that we immediately get that $\bar\Sigma(\Delta,\rho)$ is aperiodic if $\bar\Sigma(\Gamma,\mu)$ is.

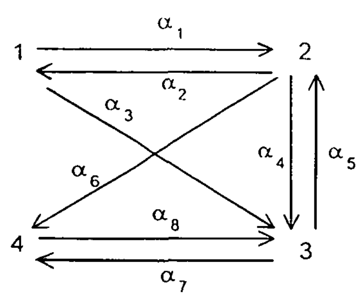
In case $B$ is not gradable in a semisimple way $A$ cannot be either (cf. [**4**]). So, in order to end the proof, we only need to give an example of local diagram algebra $B=K\Gamma/(\mu)$, which is not gradable in a semisimple way and such that $\bar\Sigma(\Gamma,\mu)$ is aperiodic. That is the content of next lemma. □

**Lemma 2.9.** *Let $B=K\langle X,Y\rangle/(X^2,Y^3,YX-XY^2)$. Then $B$ is not gradable in a semisimple way, but $\bar\Sigma(\Gamma,\mu)$ is aperiodic.*

**Proof.** By definition, $\bar\Sigma(\Gamma,\mu)$ is the commutative monoid with generators $x,y$ subject to the relations $xy\equiv xy^2$. From that it follows that $x^ry^s=x^uy^v$ in $\bar\Sigma(\Gamma,\mu)$ implies $r=u$. If now $(x^ry^s)^m=(x^ry^s)^n$ then either $r=0$ and $m\neq n$, or $m=n$. But the first possibility is clearly discarded unless $s=0$. Therefore $\bar\Sigma(\Gamma,\mu)$ is aperiodic. By leaving the details as exercises for the reader, we shall sketch the proof of the fact that $B$ is not gradable in a semisimple way. If $B$ were gradable in such a way, according to Theorem 2.1 in [**13**], there would exist a weight function $d\colon\Gamma_1=\{X,Y\}\to\mathbb{Z}$ taking positive values and a change of variable $f\colon K\langle X,Y\rangle\to K\langle X,Y\rangle$ such that the ideal $I'$ of $K\langle X,Y\rangle$ generated by $f(I)+(X,Y)^4$ is $d$-homogeneous and $B\cong K\langle X,Y\rangle/I'$. We then consider the two possibilities: (a) $d(X)=d(Y)$; (b) $d(X)\neq d(Y)$. In the first case $B$ would be gradable by the radical and one can use Proposition 2.5 in [**13**] to get a contradiction. In the second case, by working directly with the change of variable $f$, we see that either $f(X)\equiv aX$, $f(Y)\equiv dY$ or $f(X)\equiv bY$, $f(Y)\equiv cX$ $(\operatorname{mod}(X,Y)^2)$, where $a,b,c,d\in K^*$. That leads to the existence of elements $x',y'\in J(B)\backslash J(B)^2$ such that

$x' \equiv ax$, $y' \equiv dy \pmod{J(B)^2}$ and $y'x' = 0$, in the first situation, while $x' \equiv by$, $y' \equiv cx$ and $x'y' = 0$, in the second situation. In both cases one gets $yx = 0$, which is a contradiction ($x, y$ are the classes of $X, Y$ in $B = K\langle X, Y\rangle/(X^2, Y^3, YX - XY^2)$). □

**Example 2.10.** (1) Let us consider the local diagram algebra $B = K\langle X, Y\rangle/(X^2, Y^3, YX - XY^2)$ of the above lemma, that has Loewy length 4 and a multiplicative basis $\mathfrak{B} = \{1, x, y, xy, y^2, yx\}$. The procedure described before Lemma 2.5 shows that the quiver $\Delta$ of $A = End_B(G)$ has 4 vertices $\{1, 2, 3, 4\}$ corresponding to the idempotents $e_{kk}: G \xrightarrow{\text{proj}} B/J(B)^k \xrightarrow{\text{incl}} G$, $k = 1, \ldots, 4$, and as arrows: $\alpha_1 = y_{12}$ (viewed as endomorphism of $G$, it is the one induced by the map $B/J(B) \to B/J(B)^2$ that takes $1 + J(B)$ onto $y + J(B)^2$), $\alpha_2 = e_{21}$, $\alpha_3 = x_{13}$, $\alpha_4 = y_{23}$, $\alpha_5 = e_{32}$, $\alpha_6 = x_{24}$, $\alpha_7 = y_{34}$ and $\alpha_8 = e_{43}$, where the subindices in the second members of the equalities also denote the origin and extremity of the arrows. Hence the quiver of $A$ is:




If our calculations are correct, the procedure also yields the following adequate set of relations for A:

$$\rho = \{\alpha_1\alpha_2, \alpha_3\alpha_5\alpha_2, \alpha_3\alpha_5\alpha_6, \alpha_1\alpha_4\alpha_7, \alpha_6\alpha_8\alpha_5\alpha_2, \alpha_7\alpha_8\alpha_5\alpha_2, \alpha_2\alpha_3 - \alpha_6\alpha_8,$$

$$\alpha_4\alpha_5 - \alpha_2\alpha_1, \alpha_7\alpha_8 - \alpha_5\alpha_4, \alpha_1\alpha_6 - \alpha_3\alpha_5\alpha_4\alpha_7\}.$$

(2) If now $B = K\langle X, Y\rangle/(X^3, XY, YX^2, X^2 - Y^3, YX - Y^3)$ is the ungradable local algebra given in [4], then our procedure yields for $A = End_B(G)$ the same quiver as the above example, where the arrows represent: $\alpha_1 = y_{12}$, $\alpha_2 = e_{21}$, $\alpha_3 = x_{13}$, $\alpha_4 = y_{23}$, $\alpha_5 = e_{32}$, $\alpha_6 = x_{24}$, $\alpha_7 = y_{34}$ and $\alpha_8 = e_{43}$. As an adequate set of relations for $A$ one gets: $\rho = \{\alpha_1\alpha_2, \alpha_3\alpha_7, \alpha_4\alpha_5\alpha_2, \alpha_3\alpha_5\alpha_4, \alpha_6\alpha_8\alpha_5\alpha_2, \alpha_7\alpha_8\alpha_5\alpha_2, \alpha_4\alpha_5 - \alpha_2\alpha_1, \alpha_2\alpha_3 - \alpha_6\alpha_8, \alpha_7\alpha_8 - \alpha_5\alpha_8, \alpha_1\alpha_6 - \alpha_3\alpha_5\alpha_6, \alpha_1\alpha_6 - \alpha_1\alpha_4\alpha_7\}$. The monoid $\Sigma = \Sigma(\Delta, \rho)$ is generated by the $\alpha_i$'s subject to the relations $\alpha_4\alpha_5 - \alpha_2\alpha_1$, $\alpha_2\alpha_3 - \alpha_6\alpha_8, \alpha_7\alpha_8 - \alpha_5\alpha_4, \alpha_1\alpha_6 - \alpha_3\alpha_5\alpha_6, \alpha_1\alpha_6 - \alpha_1\alpha_4\alpha_7$. By the proof of Proposition 2.5, its abelianization is not aperiodic. This can be explicitly seen. Indeed, $\alpha_5\alpha_6\alpha_7^3\alpha_8^4$ is an idempotent element of $\bar{\Sigma}(\Delta, \rho)$.

(3) In the above two examples we have seen that the arrows in $\Delta$ are of two particular types: (i) $e_{k+1,k}$, for $k = 1, \ldots, \ell(B) - 1$; (ii) $\beta_{km}$, where $\beta$ is an arrow in the quiver $\Gamma$ of $B$ and $J(B)^k\beta \subseteq J(B)^m$ but $J(B)^k\beta \not\subseteq J(B)^{m+1}$ in $B$. It is not difficult to see

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that, for every local diagram algebra $B$, the elements of $A = End_B(G)$ of types (i) or (ii) yield arrows of $\Delta$, but, in general, they are not the only ones. For instance, if $B$ is the commutative algebra $K[X, Y]/(X^3 - Y^4, X^2Y, XY^3)$, then $(x^2)_{14}$ is an irreducible $B$-endomorphism of $G$, so that it yields an arrow in $\Delta$.

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