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CHROMATIC SUMS FOR ROOTED PLANAR TRIANGULATIONS, IV: THE CASE $\lambda = \infty$.

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Summary. The chromial $P(M, \lambda)$ of a planar near-triangulation M has the leading term $\lambda^{v(M)}$, where $v(M)$ is the number of vertices of M. The problem of finding the number of rooted planar near-triangulations of a given class S, all supposed to have the same number of vertices, can be regarded as a special case of the problem of finding chromatic sums. We can sum $P(M, \lambda)$ over the members of S, divide by the appropriate power of λ and let $\lambda \rightarrow \infty$. We thus get the sum of the coefficient of the leading term of $P(M, \lambda)$ for all $M \in S$, that is we get the number of members of S. This is why we classify such enumerative problems under "the case $\lambda = \infty$". We adopt this point of view in the present paper while enumerating certain kinds of rooted planar triangulations and near-triangulations. Some of the results have been published before.

In an extension of the theory we sum the coefficient of $\lambda^{v(M)-1}$ in $P(M, \lambda)$, this coefficient being minus the number of pairs of adjacent vertices in M. Since our triangulations and near-triangulations may have multiple joins the number of such pairs is not necessarily equal to the number of edges. We give a formula for the average number of such pairs in rooted planar triangulations of $2n$ triangles.

1. On chromials. The chromial $P(G, \lambda)$ of a connected graph G of $v(G)$ vertices can be written as

$$P(G, \lambda) = \sum_{j=0}^{v(G)-1} A_j \lambda^{v(G)-j}, \tag{1}$$

where the A_j are integers. In the theory of chromials it is shown that A_j is non-zero (if G is loopless) for each integer j satisfying $0 \leqq j \leqq v(G) - 1$. Moreover the coefficients A_j alternate in sign, and $A_0 = 1$. (See for example [**2**] or [**4**]).

It is sometimes convenient to replace the above polynomial $P(G, \lambda)$ by

$$Q(G, \mu) = \sum_{j=0}^{v(G)-1} A_j \mu^j. \tag{2}$$

If we write $\mu = 1/\lambda$ we can relate $P(G, \lambda)$ and $Q(G, \mu)$ as follows

$$Q(G, \mu) = \lambda^{-v(G)} P(G, \lambda). \tag{3}$$

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This formula becomes meaningless if $\mu = 0$; but in that case $Q(G, \mu)$ takes the value $A_0 = 1$, by (2).

If G is the graph of a planar map M it is usual to write $P(G, \lambda) = P(M, \lambda)$. Correspondingly we write $Q(G, \mu) = Q(M, \mu)$.

We introduce the generating series

$$(4) \qquad \begin{aligned} \bar{g} &= \bar{g}(X, Y, Z, \mu) \\ &= \sum_M X^{m(M)} Y^{n(M)} Z^{t(M)} Q(M, \mu), \end{aligned}$$

in analogy with Equation (1) of **I** [**5**]. The sum is over all rooted near-triangulations M. We now write

$$(5) \qquad \bar{q} = \bar{q}(X, Z, \mu) = \bar{g}(X, 1, Z, \mu).$$

We also write $\bar{l} = \bar{l}(Y, Z, \mu)$ for the coefficient of X^2 in $\bar{g}$, and we put $\bar{h} = \bar{h}(Z, \mu) = \bar{l}(1, Z, \mu)$. Evidently $\bar{h}$ is the coefficient of X^2 in $\bar{q}$.

We proceed to relate the newly defined series to the series g, q, l and h of **I**. The number of faces of M is clearly $t(M) + 1$ and the number of edges is $(3t(M) + m(M))/2$. Applying the Euler polyhedron formula we find that

$$(6) \qquad v(M) = 1 + (m(M) + t(M))/2.$$

Let w be a square root of λ. Then by (3)

$$P(M, \lambda) = \lambda w^{m(M)+t(M)} Q(M, \mu).$$

Thus

$$\begin{aligned} g &= \sum_M x^{m(M)} y^{n(M)} z^{t(M)} P(M, \lambda) \\ &= \lambda \sum_M (wx)^{m(M)} y^{n(M)} (wz)^{t(M)} Q(M, \mu). \end{aligned}$$

Let us write

$$(7) \qquad X = wx, \quad Y = y, \quad Z = wz.$$

Then we have

$$(8) \qquad g(x, y, z, \lambda) = \lambda \bar{g}(X, Y, Z, \mu)$$

and therefore

$$(9) \qquad q(x, z, \lambda) = \lambda \bar{q}(X, Z, \mu).$$

Now $l(y, z, \lambda)$ is the coefficient of x^2 in $g(x, y, z, \lambda)$, which is λ times the coefficient of X^2 in $g(x, y, z, \lambda)$. Hence, by (8),

$$(10) \qquad l(y, z, \lambda) = \lambda^2 \bar{l}(Y, Z, \mu),$$

$$(11) \qquad h(z, \lambda) = \lambda^2 \bar{h}(Z, \mu).$$

Substituting from these formulae into the chromatic equation, Equation (13) of **I**, we obtain

$$w^{-1}X\bar{g}\lambda = w^{-3}X^3Y\lambda(\lambda - 1) + \lambda Yw^{-1}Z\bar{g}\bar{q} + Yw^{-1}Z(\lambda\bar{g} - w^{-2}X^2\lambda^2\bar{l}) - w^{-3}X^2Y^2Z \triangle (\lambda\bar{g}),$$

that is

$$(12) \qquad X\bar{g} = (1 - \mu)X^3Y + YZ\bar{g}\bar{q} + YZ(\bar{g} - X^2\bar{l}) - \mu X^2Y^2Z \triangle (\bar{g}).$$

Since $Y = y$ there is no need to reinterpret the operator $\triangle$.

2. The case $\mu = 0$. We now put $\mu = 0$ in (12). This has the great advantage of eliminating the term in $\triangle(\bar{g})$. We find

$$(13) \qquad X\bar{g} = X^3Y + YZ\bar{g}\bar{q} + YZ(\bar{g} - X^2\bar{l}).$$

Putting $Y = 1$ in this we obtain

$$(14) \quad X\bar{q} = X^3 + Z\bar{q}^2 + Z(\bar{q} - X^2\bar{h}),\ Z\bar{q}^2 + (Z - X)\bar{q} + (X^3 - X^2Z\bar{h}) = 0.$$

This equation resembles those for $l(y, z, \tau + 1)$ and $l(y, z, 3)$, encountered in **II** and **III** [**5**]. We can solve it in the same way. Thus

$$(15) \qquad (2Z\bar{q} + Z - X)^2 = D,$$

where

$$(16) \qquad D = (Z - X)^2 - 4ZX^3 + 4Z^2X^2\bar{h}.$$

In analogy with the two other cases we can show that there is a power series ξ in Z such that

$$(17) \qquad 2Z\bar{q}(\xi, Z, 0) + Z - \xi = 0.$$

We then have

$$(18) \qquad \{D\}_{X=\xi} = 0, \qquad [\partial D/\partial X]_{X=\xi} = 0.$$

Equations (18) are equivalent to the two following:

$$(19) \qquad Z^2 - 2Z\xi + \xi^2 - 4Z\xi^3 + 4Z^2\xi^2\bar{h} = 0,$$

$$(20) \qquad -Z + \xi - 6Z\xi^2 + 4Z^2\xi\bar{h} = 0.$$

Eliminating $\bar{h}$ we find

$$Z^2 - Z\xi + 2Z\xi^3 = 0$$

or, since Z is not identically zero,

$$(21) \qquad Z = \xi(1 - 2\xi^2).$$

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Substituting this result in (20) we find that

$$\bar{h} = \frac{1 - 3\xi^2}{(1 - 2\xi^2)^2}. \tag{22}$$

Substituting for Z and $\bar{h}$ in (16) we find that

$$\begin{aligned} D &= \xi^2(1 - 2\xi^2)^2 - 2\xi(1 - 2\xi^2)X + X^2 \\ &\quad + 4\xi^2(1 - 3\xi^2)X^2 - 4\xi(1 - 2\xi^2)X^3, \\ D &= (1 - 2\xi^2)(X - \xi)^2(1 - 2\xi^2 - 4\xi X). \end{aligned} \tag{23}$$

We can now solve (15) for $\bar{q}$.

$$\begin{aligned} \bar{q} &= \frac{X - \xi + 2\xi^3}{2\xi(1 - 2\xi^2)} \pm \frac{(X - \xi)(1 - 2\xi^2)}{2\xi(1 - 2\xi^2)} \left\{1 - \frac{4\xi X}{1 - 2\xi^2}\right\}^{\frac{1}{2}} \\ &= \frac{\xi X}{1 - 2\xi^2} + \frac{(X - \xi)}{2\xi} \left\{1 - \left[1 - \frac{4\xi X}{1 - 2\xi^2}\right]^{\frac{1}{2}}\right\}. \end{aligned}$$

The negative sign is chosen to avoid negative powers of Z when the expression is expanded in terms of X and Z.

It is convenient to make use of a function $\gamma(t)$ defined by

$$\gamma(t) = 1 + t\gamma^2(t), \tag{24}$$

and having no singularity at $t = 0$. It can be expanded by Lagrange's Theorem, or by the Binomial Theorem, as follows.

$$\gamma(t) = \sum_{n=1}^{\infty} \left[\frac{(2n)!}{n!(n + 1)!} t^n\right]. \tag{25}$$

Formula (24) can be rewritten as

$$\gamma(t) = \frac{1 - (1 - 4t)^{\frac{1}{2}}}{2t}. \tag{26}$$

Hence our equation for $\bar{q}$ is equivalent to the following:

$$\bar{q} = \frac{\xi X}{1 - 2\xi^2} + \frac{X(X - \xi)}{1 - 2\xi^2}\gamma\left[\frac{\xi X}{1 - 2\xi^2}\right], \qquad (\mu = 0). \tag{27}$$

3. The coefficients in $\bar{h}(Z, 0)$ and $\bar{q}(X, Z, 0)$. Writing $\theta = \xi^2$ in (21) and (22) we find

$$Z^2\bar{h} = \theta - 3\theta^2, \tag{28}$$

$$d(Z^2\bar{h})/d\theta = 1 - 6\theta, \tag{29}$$

$$\theta = \frac{Z^2}{(1 - 2\theta)^2}. \tag{30}$$

Applying Lagrange's Theorem we deduce that

$$Z^2\bar{h} = \sum_{n=1}^{\infty} \left\{\frac{Z^{2n}}{n!}\left[\left(\frac{d}{d\theta}\right)^{n-1}\left\{\frac{1-6\theta}{(1-2\theta)^{2n}}\right\}\right]_{\theta=0}\right\},$$

$$= \sum_{n=1}^{\infty} \left\{\frac{Z^{2n}}{n!}\left[\left(\frac{d}{d\theta}\right)^{n-1}\left\{\frac{3}{(1-2\theta)^{2n-1}} - \frac{2}{(1-2\theta)^{2n}}\right\}\right]_{\theta=0}\right\},$$

$$= \sum_{n=1}^{\infty} \left\{\frac{Z^{2n}}{n!}\left[\frac{3\cdot 2^{n-1}(3n-3)!}{(2n-2)!} - \frac{2\cdot 2^{n-1}\cdot(3n-2)!}{(2n-1)!}\right]\right\}$$

$$= \sum_{n=1}^{\infty} \left\{\frac{2^{n-1}(3n-3)!Z^{2n}}{n!(2n-1)!}\right\},$$

(31) $$\bar{h} = \sum_{n=0}^{\infty} \left\{\frac{2^n(3n)!Z^{2n}}{(n+1)!(2n+1)!}\right\}, \qquad (\mu = 0).$$

For $n > 0$ the coefficient $\bar{h}_{2n}$ of Z^{2n} in $\bar{h}$ is the number of rooted planar triangulations with $2n$ faces, or dually the number of rooted non-separable trivalent planar maps with $2n$ vertices. Formulae equivalent to (31) appear in [**1**] and [**3**].

An application of Stirling's Theorem to (31) gives

(32) $$\bar{h}_{2n} = \frac{1}{4}\left(\frac{3}{\pi}\right)^{\frac{1}{2}} n^{-5/2}\left(\frac{27}{2}\right)^n, \qquad (\mu = 0).$$

Formulae (31) and (32) can be used in determinations of averages. Thus if we wish to find the average number of 3-colourings for rooted planar triangulations of $2n$ faces we have only to divide the h_{2n} of **III**, § 4, by $\bar{h}_{2n}$. This gives the average exactly. If we need only an asymptotic approximation we can use Formula (33) of **III** with Formula (32) of the present paper and obtain

$$12\sqrt{3}(16/27)^n.$$

It seems likely, though it has not yet been proved, that the average number of 3-colourings for unrooted planar triangulations with $2n$ faces is given asymptotically by the same expression.

We now turn to the study of $\bar{q}$, with $\mu = 0$. From (25) and (27) we have

$$\bar{q} = \frac{X\xi}{(1-2\xi^2)} + \left\{\frac{X^2}{(1-2\xi^2)} - \frac{X\xi}{(1-2\xi^2)}\right\} \sum_{n=0}^{\infty} \left\{\frac{(2n)!X^n\xi^n}{n!(n+1)!(1-2\xi^2)^n}\right\}.$$

The coefficient of X in this formula is found to be zero, and the coefficient of X^2 is

$$(1-3\xi^2)/(1-2\xi^2)^2,$$

in agreement with (22). For larger values of n we can write the coefficient $\bar{q}_n$ of X^n in $\bar{q}$ as follows:

(33) $$\bar{q}_n = \frac{(2n-4)!\xi^{n-2}}{(n-2)!(n-1)!(1-2\xi^2)^{n-1}} - \frac{(2n-2)!\xi^n}{(n-1)!n!(1-2\xi^2)^n}.$$

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Using (21) and writing $\theta = \xi^2$ as before we rewrite this as

$$\bar{q}_n = \frac{(2n-4)!Z^{n-2}}{(n-2)!(n-1)!(1-2\theta)^{2n-3}} - \frac{(2n-2)!Z^n}{(n-1)!n!(1-2\theta)^{2n}}. \tag{34}$$

To eliminate θ we require the expansion of $(1 - 2\theta)^{-k}$, where k is a positive integer, as a power series in Z. We use (30) in an application of Lagrange's Theorem and obtain

$$(1-2\theta)^{-k} = 1 + \sum_{m=1}^{\infty} \left\{ \frac{Z^{2m}}{m!} \left[\left(\frac{d}{d\theta} \right)^{m-1} \left\{ \frac{(-k)(-2)}{(1-2\theta)^{2m+k+1}} \right\} \right]_{\theta=0} \right\},$$

or

$$(1-2\theta)^{-k} = k \sum_{m=0}^{\infty} \frac{2^m(3m+k-1)!Z^{2m}}{m!(2m+k)!}. \tag{35}$$

Using (34) and (35) we find that for $n > 2$ the coefficient of $X^n Z^{n+2j}$ in $\bar{q}$, where j may take all integral values from -1 upward, is

$$\frac{(2n-3)!2^{j+1}(3j+2n-1)!}{(n-2)!(n-1)!(j+1)!(2j+2n-1)!} - \frac{2\cdot(2n-2)!2^j(3j+2n-1)!}{(n-1)!(n-1)!j!(2j+2n)!}.$$

If $j = -1$ the second quotient is to be interpreted as zero. This formula can be written more simply as

$$\frac{2^{j+2}(2n-3)!(3j+2n-1)!}{(n-2)!(n-2)!(j+1)!(2n+2j)!}.$$

We now rewrite this formula more in accordance with the conventions of **I**, first noting that it is valid also when $n = 2$, by (31). Let $\bar{q}_{t,k}$ denote the coefficient of $Z^t X^{t-2k+2}$ in $\bar{q}$. By Equation (16) of **I** this coefficient can be non-zero only if $0 \leqq k \leqq t/2$. We substitute $k - 1$ for j and $t - 2k + 2$ for n, thus obtaining

$$\bar{q}_{t,k} = \frac{2^k(2t-4k+2)!(2t-k)!}{(t-2k)!(t-2k+1)!k!(2t-2k+2)!}. \tag{36}$$

An equivalent formula was given by R. C. Mullin in [**1**].

4. The chromatic sum $\bar{l}$ in the case $\mu = 0$. We go on to consider the sum $\bar{l}(Y, Z, 0)$. The author does not know of any earlier treatment of this in the literature. For $\mu = 0$ the coefficient $\bar{l}_{t,s}$ of $Y^{s+1}Z^t$ can be interpreted, provided $t > 0$, as the number of rooted planar triangulations with t faces and with a root-vertex of valency s. It seems likely that this coefficient will sometimes be needed in determinations of averages, though the author has no immediate application of this kind in mind. We therefore find an exact explicit formula for this coefficient.

In what follows $\mu = 0$. We obtain an equation for $\bar{l}$ by modifying the method of **III**, section 5. First we rewrite our Formula (13) as

$$(X - YZ\bar{q} - YZ)\bar{g} = X^3Y - X^2YZ\bar{l}. \tag{37}$$

We introduce a power series η in Y and Z defined by the following equation

$$\eta - YZ\bar{q}(\eta, Z, 0) - YZ = 0. \tag{38}$$

The coefficients in η can be found, using (36), by equating coefficients for successive powers of Z in (38). Thus the coefficient of Z^0 is zero and the coefficient of Z^1 is Y. Thus the formal power series η is well defined. From (37) and (38) we find

$$\eta = Z\bar{l}. \tag{39}$$

By substituting η for X in (14) we obtain

$$Z\bar{q}^2(\eta, Z, 0) + (Z - \eta)\bar{q}(\eta, Z, 0) + (\eta^3 - \eta^2 Z\bar{h}) = 0. \tag{40}$$

Elimination of $\bar{q}(\eta, Z, 0)$ between (38) and (40) gives

$$Z(\eta - YZ)^2 + YZ(Z - \eta)(\eta - YZ) + Y^2Z^2(\eta^3 - \eta^2 Z\bar{h}) = 0,$$
$$Y^2Z\eta^3 + (1 - Y - Y^2Z^2\bar{h})\eta^2 - YZ(1 - Y)\eta = 0.$$

Since η is not identically zero it follows from (39) that

$$Y^2Z^2\bar{l}^2 + (1 - Y - Y^2Z^2\bar{h})\bar{l} - Y(1 - Y) = 0, \quad (\mu = 0). \tag{41}$$

This equation can also be written in the forms

$$Y^2Z^2\bar{l}(\bar{l} - \bar{h}) + (1 - Y)(\bar{l} - Y) = 0, \qquad (\mu = 0), \tag{42}$$

and

$$\triangle\,(\bar{l}) = 1/Y^2Z^2 - 1/YZ^2\bar{l}, \qquad (\mu = 0). \tag{43}$$

Now let us write the equation in terms of the parameter θ. We have

$$Y^2\theta(1 - 2\theta)^2\bar{l}^2 + (1 - Y - Y^2\theta(1 - 3\theta))\bar{l} - Y(1 - Y) = 0,$$

that is

$$\{2Y^2\theta(1 - 2\theta)^2\bar{l} + 1 - Y - Y^2\theta(1 - 3\theta)\}^2 = D_1, \tag{44}$$

where

$$D_1 = (1 - Y - Y^2\theta(1 - 3\theta))^2 + 4Y^3(1 - Y)\theta(1 - 2\theta)^2. \tag{45}$$

Other forms of the last equation are

$$D_1 = 1 - 2Y + Y^2(1 - 2\theta + 6\theta^2) + Y^3(6\theta - 22\theta^2 + 16\theta^3) + Y^4(-4\theta + 17\theta^2 - 22\theta^3 + 9\theta^4),$$

$$D_1 = (1 - Y + \theta Y)^2((1 - \theta Y)^2 - 4\theta(1 - 2\theta)Y^2). \tag{46}$$

Since

$$1 - Y - \theta(1 - 3\theta)Y^2 = (1 - \theta Y)(1 - Y + \theta Y) - 2\theta(1 - 2\theta)Y^2$$

we can deduce from (44) and (45) that

$$\bar{l} = \frac{2\theta(1-2\theta)Y^2}{2\theta(1-2\theta)^2Y^2} + \frac{(1-Y+\theta Y)(1-\theta Y)}{2\theta(1-2\theta)^2Y^2}\left\{-1 \pm \left[1 - \frac{4\theta(1-2\theta)Y^2}{(1-\theta Y)^2}\right]^{\frac{1}{2}}\right\}.$$

We choose the sign to avoid negative powers of Y in the final expansion of $\bar{l}$. We may now write

$$(47) \qquad \bar{l} = \frac{1}{1-2\theta} - \frac{(1-Y+\theta Y)}{(1-2\theta)(1-\theta Y)}\gamma\left\{\frac{\theta(1-2\theta)Y^2}{(1-\theta Y)^2}\right\},$$

with of course $\mu = 0$.

As a partial expansion of (47) we have

$$\bar{l} = \frac{1}{1-2\theta} + \theta^{-1}\left\{\frac{1}{1-\theta Y} - \frac{1-\theta}{1-2\theta}\right\}\sum_{n=0}^{\infty}\left\{\frac{(2n)!\theta^n(1-2\theta)^nY^{2n}}{n!(n+1)!(1-\theta Y)^{2n}}\right\}.$$

This formula gives the coefficient of Y^0 in $\bar{l}$ as zero, which is correct. It gives the coefficient C_s of Y^{s+1}, where $s \geqq 0$, as the coefficient of Y^{s+1} in

$$\sum_{n=0}^{\infty}\sum_{r=0}^{\infty}\left\{\frac{(2n)!\theta^{n-1}}{n!(n+1)!}\left[\frac{(1-2\theta)^n(2n+r)!\theta^rY^{r+2n}}{(2n)!r!} - \frac{(1-\theta)(1-2\theta)^{n-1}(2n-1+r)!\theta^rY^{r+2n}}{(2n-1)!r!}\right]\right\},$$

that is in

$$\sum_{n=0}^{\infty}\sum_{r=0}^{\infty}\left\{\left(\frac{(1-2\theta)^{n-1}\theta^{n-1+r}(2n-1+r)!Y^{r+2n}}{n!(n+1)!r!}\right)(r-2\theta(n+r))\right\}.$$

From this we infer that

$$C_s = \sum_{n=0}^{[s/2]}\left\{\frac{(1-2\theta)^{n-1}\theta^{s-n}s!}{n!(n+1)!(s-2n)!}\right\} - 2\sum_{n=0}^{[(s+1)/2]}\left\{\frac{(1-2\theta)^{n-1}\theta^{s+1-n}s!(s+1-n)}{n!(n+1)!(s+1-2n)!}\right\},$$

$$(48) \quad C_s = \sum_{n=0}^{[s/2]}\left\{\frac{s!Z^{2s-2n}}{n!(n+1)!(s-2n)!(1-2\theta)^{2s-3n+1}}\right\} - 2\sum_{n=0}^{[(s+1)/2]}\left\{\frac{s!Z^{2s-2n+2}(s-n+1)}{n!(n+1)!(s-2n+1)!(1-2\theta)^{2s-3n+3}}\right\}.$$

The coefficient of $Z^{2j-2s+2n}$ in $(1-2\theta)^{-2s+3n-1}$ is non-zero only if $j - s + n \geqq 0$. It is then

$$\frac{(3j-s)!2^{j-s+n}(2s-3n+1)}{(j-s+n)!(2j-n+1)!},$$

by (35). Similarly the coefficient of $Z^{2j-2s+2n-2}$ in $(1-2\theta)^{-2s+3n-3}$ is

$$\frac{(3j-s-1)!2^{j-s+n-1}(2s-3n+3)}{(j-s+n-1)!(2j-n+1)!}$$

if $j - s + n - 1 \geqq 0$, and is zero otherwise.

We may now deduce from (48) that $\bar{l}_{2j,s}$, the coefficient of $Y^{s+1}Z^{2j}$ in $\bar{l}$, is given by the following formula.

$$(49)\quad \bar{l}_{2j,s} = \sum_{n=\max(0,s-j)}^{[s/2]} \left\{\frac{s!(3j-s)!2^{j-s+n}(2s-3n+1)}{n!(n+1)!(s-2n)!(j-s+n)!(2j-n+1)!}\right\} - \sum_{n=\max(0,s-j+1)}^{[(s+1)/2]} \left\{\frac{s!(3j-s-1)!2^{j-s+n}(2s-3n+3)(s-n+1)}{n!(n+1)!(s-2n+1)!(j-s+n-1)!(2j-n+1)!}\right\}.$$

5. Derivatives at $\mu = 0$. Continuing our study of the function g at $\mu = 0$ we apply the operator $\triangle$ to (37).

$$(X - Z\bar{q} - Z)\triangle(\bar{g}) - (Z\bar{q} + Z)\bar{g} = X^3 - X^2Z\bar{l} - X^2Z\triangle(\bar{l}).$$

Hence, by (37),

$$(X - Z\bar{q} - Z)(X - YZ\bar{q} - YZ)\triangle(\bar{g}) - (Z\bar{q} + Z)(X^3Y - X^2YZ\bar{l}) = (X - YZ\bar{q} - YZ)(X^3 - X^2Z\bar{l} - X^2Z\triangle(\bar{l})),$$

$$(50)\quad \triangle(\bar{g}) = \frac{X^3(X - Z\bar{l})}{(X - Z\bar{q} - Z)(X - YZ\bar{q} - YZ)} - \frac{X^2Z\triangle(\bar{l})}{(X - Z\bar{q} - Z)}, \qquad (\mu = 0).$$

Formula (12) is valid for all μ. Let us denote derivatives with respect to μ by primes. Differentiating with respect to μ and then setting $\mu = 0$ we obtain

$$X\bar{g}' = -X^3Y + YZ(\bar{g}\bar{q}' + \bar{g}'\bar{q}) + YZ(\bar{g}' - X^2\bar{l}') - X^2Y^2Z\triangle(\bar{g}),$$

$$(51)\quad (X - YZ\bar{q} - YZ)\bar{g}' = -X^3Y + YZ\bar{g}\bar{q}' - X^2YZ\bar{l}' - X^2Y^2Z\triangle(\bar{g}), \qquad (\mu = 0).$$

Now let us write $Y = 1$ in some of the foregoing equations. From (37),

$$(52)\qquad (X - Z\bar{q} - Z)\bar{q} = X^2(X - Z\bar{h}), \qquad (\mu = 0).$$

From (43) we have

$$(53)\qquad \{\triangle(\bar{l})\}_{Y=1} = (\bar{h} - 1)/Z^2\bar{h}, \qquad (\mu = 0).$$

From (50) and (52),

$$\{\triangle(\bar{g})\}_{Y=1} = \frac{\bar{q}^2X^3(X - Z\bar{h})}{X^4(X - Z\bar{h})^2} - \frac{\bar{q}X^2Z\{\triangle(\bar{l})\}_{Y=1}}{X^2(X - Z\bar{h})}$$

$$= \frac{\bar{q}^2}{X(X - Z\bar{h})} - \frac{\bar{q}X^2Z(\bar{h} - 1)}{X^2Z^2\bar{h}(X - Z\bar{h})}, \text{ by (53)},$$

$$= \frac{\bar{q}^2Z\bar{h} - \bar{q}X\bar{h} + \bar{q}X}{XZ\bar{h}(X - Z\bar{h})}$$

$$= \frac{\bar{h}(-Z\bar{q} - X^3 + X^2Z\bar{h}) + \bar{q}X}{XZ\bar{h}(X - Z\bar{h})},$$

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by (52). Thus

$$\{\triangle(\bar{g})\}_{Y=1} = \frac{\bar{q} - X^2\bar{h}}{XZ\bar{h}}, \qquad (\mu = 0). \tag{54}$$

From (51) and (54),

$$(X - 2Z\bar{q} - Z)\bar{q}' = -X^3 - X^2Z\bar{h}' - X(\bar{q} - X^2\bar{h})\bar{h}^{-1},$$
$$(2Z\bar{q} + Z - X)\bar{q}' = X^2Z\bar{h}' + X(\bar{q}/\bar{h}), \qquad (\mu = 0). \tag{55}$$

In this equation we substitute the parameter ξ for X and apply (17). We find

$$0 = \xi^2Z\bar{h}' + \xi(\xi - Z)/2Z\bar{h},$$
$$\bar{h}' = -2\xi^3/2\xi^3(1 - 3\xi^2), \qquad (\text{by (21) and (22)}),$$
$$\bar{h}' = -(1 - 3\xi^2)^{-1}, \qquad (\mu = 0). \tag{56}$$

Using (55) and (56) we can express $\bar{q}'$ in terms of X and ξ as follows. From (23) and the ensuing resolution of the ambiguity of sign we have

$$2Z\bar{q} + Z - X = -(1 - 2\xi^2)(X - \xi)\left[1 - \frac{4\xi X}{(1 - 2\xi^2)}\right]^{\frac{1}{2}}.$$

But

$$X^2Z\bar{h}' + X(\bar{q}/\bar{h}) = \frac{-X^2\xi(1 - 2\xi^2)}{(1 - 3\xi^2)} + \frac{X(1 - 2\xi^2)^2}{(1 - 3\xi^2)}\bar{q}$$
$$= \frac{X(1 - 2\xi^2)X(X - \xi)}{(1 - 3\xi^2)}\gamma\left[\frac{\xi X}{(1 - 2\xi^2)}\right], \text{ by (27)},$$
$$= \frac{X(X - \xi)(1 - 2\xi^2)^2}{2\xi(1 - 3\xi^2)}\left\{1 - \left[1 - \frac{4\xi X}{(1 - 2\xi^2)}\right]^{\frac{1}{2}}\right\}.$$

Hence, by division,

$$\bar{q}' = \frac{X(1 - 2\xi^2)}{2\xi(1 - 3\xi^2)}\left\{1 - \left[1 - \frac{4\xi X}{(1 - 2\xi^2)}\right]^{-1/2}\right\}, \tag{57}$$

for $\mu = 0$.

The author has not investigated $l'(Y, Z, 0)$ in detail. It can however be expressed in terms of known functions by substituting the parameter η for X in (51). The expression on the left then vanishes. The expression $\bar{g}(\eta, Y, Z, 0)$ occurs in the resulting formula. However this can be counted as a known function, for it can be shown by an application of l'Hospital's Rule to (37) that

$$\bar{g}(\eta, Y, Z, 0) = \frac{\eta^2(2\eta - \eta Y - YZ)}{Z(1 - Y + \eta^2 Y)}. \tag{58}$$

The effect of substituting η for X in $\triangle(\bar{g}(X, Y, Z, 0)$ can be determined from (50).

6. Pairs of adjacent vertices. There is a graph-theoretical interpretation of the derivatives $\bar{h}'$ and $\bar{q}'$ at $\mu = 0$, given by the following theorem.

(6.1) *Let G be any loopless graph. Then the coefficient $A_1(G)$ of $\lambda^{v(G)-1}$ in $P(G, \lambda)$ is minus the number of pairs of adjacent vertices in G.*

Proof. If G is edgeless $P(G, \lambda) = \lambda^{v(G)}$, and the theorem is trivially true. Assume as an inductive hypothesis that the theorem holds whenever the number $e(G)$ of edges of G is less than some positive integer j, and consider the case $e(G) = j$.

Let A be any edge of G. Let G'_A be the graph obtained from G by deleting A, and G''_A the graph obtained from G by contracting A to a single vertex.

Suppose G to have a second edge joining the ends of A. Then the deletion of A changes neither the chromial nor the number of pairs of adjacent vertices. Since the theorem is true for G'_A by the inductive hypothesis, it is true also for G.

We may now assume that A is the only edge joining its two ends in G. By a well-known and easily proved theorem we have

$$P(G, \lambda) = P(G'_A, \lambda) - P(G''_A, \lambda). \tag{59}$$

We observe that G''_A is loopless. The number of pairs of adjacent vertices is one less in G'_A than in G. But $v(G) - 1$ is the number of vertices of G''_A, and therefore the coefficient of $\lambda^{v(G)-1}$ in $P(G''_A, \lambda)$ is 1. Since the theorem holds for G'_A by the inductive hypothesis it holds also for G, by (59).

The theorem follows in general by induction.

Now for $n > 0$ the sum over all rooted planar triangulations T of $2n$ faces of the number $K(T)$ of pairs of adjacent vertices of T is the coefficient $-\bar{h}'_{2n}$ of Z^{2n} in $-\bar{h}'$, with $\mu = 0$. But

$$\begin{aligned}
-\bar{h}' &= (1 - 3\xi^2)^{-1} = (1 - 3\theta)^{-1} \\
&= 1 + \sum_{n=1}^{\infty} \left\{\frac{Z^{2n}}{n!}\left[\left(\frac{d}{d\theta}\right)^{n-1}\left\{\frac{3}{(1-2\theta)^{2n}(1-3\theta)^2}\right\}\right]_{\theta=0}\right\} \\
&= 1 + 3\sum_{n=1}^{\infty} \left\{\frac{Z^{2n}}{n!}\sum_{j=0}^{n-1}\left\{\frac{(n-1)!}{j!(n-1-j)!}\left[\left(\frac{d}{d\theta}\right)^{j}\left\{\frac{1}{(1-2\theta)^{2n}}\right\}\right.\right.\right. \\
&\qquad\qquad \left.\left.\left.\times\left(\frac{d}{d\theta}\right)^{n-1-j}\left\{\frac{1}{(1-3\theta)^2}\right\}\right]_{\theta=0}\right\}\right\} \\
&= 1 + 3\sum_{n=1}^{\infty}\sum_{j=0}^{n-1}\left\{\frac{Z^{2n}(n-1)!(2n+j-1)!2^j3^{n-1-j}(n-j)!}{n!j!(n-1-j)!(2n-1)!}\right\} \\
&= 1 + 2\sum_{n=1}^{\infty}\left\{\frac{3^nZ^{2n}}{(2n)!}\sum_{j=0}^{n-1}\left\{\frac{(2n+j-1)!(2/3)^j(n-j)}{j!}\right\}\right\}.
\end{aligned}$$

Thus

$$-\bar{h}'_{2n} = \frac{2\cdot 3^n}{(2n)!}\sum_{j=0}^{n-1}\left\{\frac{(2n+j-1)!(2/3)^j(n-j)}{j!}\right\}, \tag{60}$$

provided that $n > 0$.

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For large n we can use an asymptotic formula derived as follows. From (60)

$$-\bar{h}'_{2n} = \frac{2\cdot 3^n(3n-2)!(2/3)^{n-1}J}{(2n)!(n-1)!}$$
$$= \frac{3\cdot 2^n(3n-2)!\,J}{(2n)!(n-1)!},$$

where

$$J = 1 + \frac{2(n-1)(3/2)}{(3n-2)} + \frac{3(n-1)(n-2)(3/2)^2}{(3n-2)(3n-3)} + \frac{4(n-1)(n-2)(n-3)(3/2)^3}{(3n-2)(3n-3)(3n-4)} + \ldots,$$

to n terms.

Each term in the sum J is positive, and less than the corresponding term in the absolutely convergent series

$$H = 1 + 2\cdot 2^{-1} + 3\cdot 2^{-2} + 4\cdot 2^{-3} + \ldots .$$

For large n the sum of the first s terms of J, where s is fixed, can be approximated asymptotically by the sum of the first s terms of H. Since s may be taken as large as we please it follows that

$$J \sim (1-2^{-1})^{-2} = 4,$$

$$-\bar{h}'_{2n} \sim \frac{4\cdot 2^n(3n)!}{3n\cdot(2n)!n!},$$

$$(61)\qquad -\bar{h}'_{2n} \sim \left(\frac{2}{(3\pi)^{\frac{1}{2}}}\right) n^{-3/2}\left(\frac{27}{2}\right)^n, \qquad (\mu = 0),$$

by Stirling's Theorem.

From (32) and (61) we have

$$(62)\qquad \frac{-\bar{h}'_{2n}}{\bar{h}_{2n}} \sim \frac{8n}{3}, \qquad (\mu = 0).$$

Thus for large n the average number of pairs of adjacent vertices in a rooted planar triangulation of $2n$ faces is asymptotically $(8/3)n$. Since the number of edges is $3n$ we can express this by saying that the average number of edges joining two adjacent vertices is asymptotically 9/8. It seems likely that the same result holds for unrooted planar triangulations, but this has still to be proved.

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