*Proceedings of the Edinburgh Mathematical Society* (2025) **68**, 526–565
doi:10.1017/S0013091524000919

# NEF VECTOR BUNDLES ON A QUADRIC THREEFOLD WITH FIRST CHERN CLASS TWO

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(Received 17 December 2023)

*Abstract* We classify nef vector bundles on a smooth hyperquadric of dimension three with first Chern class two over an algebraically closed field of characteristic zero. In particular, we see that they are globally generated.

*Keywords:* nef vector bundles; Fano bundles; full strong exceptional collections

*2020 Mathematics subject classification:* Primary 14J60;
Secondary 14J45; 14F08

## 1. Introduction

In [17, § 2 Theorem 2], Peternell–Szurek–Wiśniewski classified nef vector bundles on a smooth hyperquadric $\mathbb{Q}^n$ of dimension $n \geq 3$ with first Chern class $\leq 1$ over an algebraically closed field $K$ of characteristic zero. In [12, Theorem 9.3], we provided a different proof of this classification, which was based on an analysis with a full strong exceptional collection of vector bundles on $\mathbb{Q}^n$.

In this paper, we classify nef vector bundles on a smooth quadric threefold $\mathbb{Q}^3$ with first Chern class two. (In the subsequent paper [14], we classify those on a smooth hyperquadric $\mathbb{Q}^n$ of dimension $n \geq 4$.) The precise statement is as follows.

**Theorem 1.1.** *Let $\mathcal{E}$ be a nef vector bundle of rank $r$ on a smooth hyperquadric $\mathbb{Q}^3$ of dimension 3 over an algebraically closed field $K$ of characteristic zero, and let $\mathcal{S}$ be the spinor bundle on $\mathbb{Q}^3$. Suppose that $\det \mathcal{E} \cong \mathcal{O}(2)$. Then $\mathcal{E}$ is isomorphic to one of the following vector bundles or fits in one of the following exact sequences:*

(1) $\mathcal{O}(2) \oplus \mathcal{O}^{\oplus r-1}$;
(2) $\mathcal{O}(1)^{\oplus 2} \oplus \mathcal{O}^{\oplus r-2}$;
(3) $\mathcal{O}(1) \oplus \mathcal{S} \oplus \mathcal{O}^{\oplus r-3}$;
(4) $0 \to \mathcal{O}(-1) \to \mathcal{O}(1) \oplus \mathcal{O}^{\oplus r} \to \mathcal{E} \to 0$;
(5) $0 \to \mathcal{O}^{\oplus a} \to \mathcal{S}^{\oplus 2} \oplus \mathcal{O}^{\oplus r-4+a} \to \mathcal{E} \to 0$, *where* $a = 0$ *or* $1$, *and the composite of the injection* $\mathcal{O}^{\oplus a} \to \mathcal{S}^{\oplus 2} \oplus \mathcal{O}^{\oplus r-4+a}$ *and the projection* $\mathcal{S}^{\oplus 2} \oplus \mathcal{O}^{\oplus r-4+a} \to \mathcal{O}^{\oplus r-4+a}$ *is zero;*
(6) $0 \to \mathcal{S}(-1) \oplus \mathcal{O}(-1) \to \mathcal{O}^{\oplus r+3} \to \mathcal{E} \to 0$;
(7) $0 \to \mathcal{O}(-1)^{\oplus 2} \to \mathcal{O}^{\oplus r+2} \to \mathcal{E} \to 0$;
(8) $0 \to \mathcal{O}(-2) \to \mathcal{O}^{\oplus r+1} \to \mathcal{E} \to 0$;
(9) $0 \to \mathcal{O}(-2) \to \mathcal{O}(-1)^{\oplus 4} \to \mathcal{O}^{\oplus r+3} \to \mathcal{E} \to 0$.

Note that this list is effective: in each case exists an example. For example, if we denote by $\mathcal{N}$ a null correlation bundle on $\mathbb{P}^3$, then $\pi_p^*(\mathcal{N}(1))$ belongs to Case (9) of Theorem 1.1, where $\pi_p : \mathbb{Q}^3 \to \mathbb{P}^3$ is the projection from a point $p \in \mathbb{P}^4 \setminus \mathbb{Q}^3$. (Similarly, $\pi_p^*(\Omega_{\mathbb{P}^3}(2))$ belongs to Case (9) of Theorem 1.1.) Under the stronger assumption that $\mathcal{E}$ is globally generated, Ballico–Huh–Malaspina provided a classification of $\mathcal{E}$ on $\mathbb{Q}^3$ with $c_1 = 2$ in [3] and [2].

Note also that the projectivization $\mathbb{P}(\mathcal{E})$ of the bundle $\mathcal{E}$ in Theorem 1.1 is a Fano manifold of dimension $r+2$, i.e. the bundle $\mathcal{E}$ in Theorem 1.1 is a Fano bundle on $\mathbb{Q}^3$ of rank $r$. As a related result, Langer classified smooth Fano 4-folds with adjunction theoretic scroll structure over $\mathbb{Q}^3$ in [10, Theorem 7.2].

Our basic strategy and framework for describing $\mathcal{E}$ in Theorem 1.1 is to give a minimal locally free resolution of $\mathcal{E}$ in terms of some twists of the full strong exceptional collection

$$(\mathcal{O}, \mathcal{S}, \mathcal{O}(1), \mathcal{O}(2))$$

of vector bundles (see [12] for more details).

The content of this paper is as follows. In § 2, we briefly recall Bondal's theorem [1, Theorem 6.2] and its related notions and results required in the proof of Theorem 1.1. In particular, we recall some finite-dimensional algebra $A$ and fix some symbols, e.g. $G$, $P_i$ and $S_i$, related to $A$ and to finitely generated right $A$-modules. We also recall the classification [13, Theorem 1.1] of nef vector bundles on a smooth quadric surface $\mathbb{Q}^2$ with Chern class $(2, 2)$ in Theorem 2.3. In § 3, we recall some basic properties of the spinor bundle $\mathcal{S}$ on $\mathbb{Q}^3$. In § 4, we state Hirzebruch–Riemann–Roch formulas for vector bundles $\mathcal{E}$ on $\mathbb{Q}^3$ with $c_1 = 2$ and for $\mathcal{S}^\vee \otimes \mathcal{E}$. In § 5, we show some key lemmas required later in the proof of Theorem 1.1. In § 6, we provide a lower bound for the third Chern class of a nef vector bundle $\mathcal{E}$, if $h^0(\mathcal{E}(-D)) \neq 0$ for some effective divisor $D$. In § 7, we provide the set-up for the proof of Theorem 1.1. The proof of Theorem 1.1 is carried out in § 8–19, according to which case of Theorem 2.3 $\mathcal{E}|_{\mathbb{Q}^2}$ belongs to.

### 1.1. Notation and conventions

Throughout this paper, we work over an algebraically closed field $K$ of characteristic zero. Basically, we follow the standard notation and terminology in algebraic geometry.

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We denote by $\mathbb{Q}^3$ a smooth quadric threefold over $K$, by $\mathbb{Q}^2$ a smooth quadric surface over $K$ and by

- $\mathcal{S}$ the spinor bundle on $\mathbb{Q}^3$.

Note that we follow Kapranov's convention [9, p. 499]; our spinor bundle $\mathcal{S}$ is globally generated, and it is the dual of that of Ottaviani's [16]. For a coherent sheaf $\mathcal{F}$, we denote by $c_i(\mathcal{F})$ the $i$th Chern class of $\mathcal{F}$ and by $\mathcal{F}^\vee$ the dual of $\mathcal{F}$. In particular,

- $c_i$ stands for $c_i(\mathcal{E})$ of the nef vector bundle $\mathcal{E}$ we are dealing with.

For a vector bundle $\mathcal{E}$, $\mathbb{P}(\mathcal{E})$ denotes $\operatorname{Proj} S(\mathcal{E})$, where $S(\mathcal{E})$ denotes the symmetric algebra of $\mathcal{E}$. The tautological line bundle

- $\mathcal{O}_{\mathbb{P}(\mathcal{E})}(1)$ is also denoted by $H(\mathcal{E})$.

Let $A^*\mathbb{Q}^3$ be the Chow ring of $\mathbb{Q}^3$. We denote

- by $H$ a hyperplane section of $\mathbb{Q}^3$ and by $h$ its class in $A^1\mathbb{Q}^3$: $A^1\mathbb{Q}^3 = \mathbb{Z}h$;
- by $L$ a line in $\mathbb{Q}^3$ and by $l$ its class in $A^2\mathbb{Q}^3$: $A^2\mathbb{Q}^3 = \mathbb{Z}l$.

Note that $h^2 = 2l$. Via the map $\deg : A^3\mathbb{Q}^3 \cong \mathbb{Z}$, we identify elements $A^3\mathbb{Q}^3$ with its corresponding integer; thus, we have $h^3 = 2$ and $hl = 1$. For any closed subscheme $Z$ in $\mathbb{Q}^3$, $\mathcal{I}_Z$ denotes the ideal sheaf of $Z$ in $\mathbb{Q}^3$; for a point $p \in \mathbb{Q}^3$, $\mathcal{I}_p$ denotes the ideal sheaf of $p \in \mathbb{Q}^3$ and $k(p)$ denotes the residue field of $p \in \mathbb{Q}^3$. For coherent sheaves $\mathcal{F}$ and $\mathcal{G}$, we set

- $\operatorname{ext}^q(\mathcal{F},\mathcal{G}) = \dim \operatorname{Ext}^q(\mathcal{F},\mathcal{G})$;
- $\operatorname{hom}(\mathcal{F},\mathcal{G}) = \dim \operatorname{Hom}(\mathcal{F},\mathcal{G})$.

Finally we refer to [11] for the definition and basic properties of nef vector bundles.

## 2. Preliminaries

Throughout this paper, $G_0$, $G_1$, $G_2$, $G_3$ denote respectively $\mathcal{O}$, $\mathcal{S}$, $\mathcal{O}(1)$, $\mathcal{O}(2)$ on $\mathbb{Q}^3$. An important and well-known fact [9, Theorem 4.10] of the collection $(G_0, G_1, G_2, G_3)$ is that it is a full strong exceptional collection in $D^b(\mathbb{Q}^3)$, where $D^b(\mathbb{Q}^3)$ denotes the bounded derived category of (the abelian category of) coherent sheaves on $\mathbb{Q}^3$. Here we use the term 'collection' to mean 'family', not 'set'. Thus, an exceptional collection is also called an exceptional sequence. We refer to [7] for the definition of a full strong exceptional sequence.

Denote by $G$ the direct sum $\bigoplus_{i=0}^3 G_i$ of $G_0$, $G_1$, $G_2$ and $G_3$, and by $A$ the endomorphism ring $\operatorname{End}(G)$ of $G$. The ring $A$ is a finite-dimensional $K$-algebra, and $G$ is a left $A$-module. Note that $\operatorname{Ext}^q(G,\mathcal{F})$ is a finitely generated right $A$-module for a coherent sheaf $\mathcal{F}$ on $\mathbb{Q}^3$. We denote by $\operatorname{mod} A$ the category of finitely generated right $A$-modules and by $D^b(\operatorname{mod} A)$ the bounded derived category of $\operatorname{mod} A$. Let $p_i : G \to G_i$ be the projection, and $\iota_i : G_i \hookrightarrow G$ the inclusion. Set $e_i = \iota_i \circ p_i$. Then $e_i \in A$. Set

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$$P_i = e_i A.$$

Then $A \cong \oplus_i P_i$ as right $A$-modules, and $P_i$'s are projective right $A$-modules. We see that $P_i \otimes_A G \cong G_i$. Any finitely generated right $A$-module $V$ has an ascending filtration

$$0 = V^{\leq -1} \subset V^{\leq 0} \subset V^{\leq 1} \subset V^{\leq 2} \subset V^{\leq 3} = V$$

by right $A$-submodules, where $V^{\leq i}$ is defined to be $\bigoplus_{j \leq i} V e_j$. Set $\operatorname{Gr}^i V = V^{\leq i}/V^{\leq i-1}$ and

$$S_i = \operatorname{Gr}^i P_i.$$

Then $\operatorname{Gr}^i S_i \cong K$ as $K$-vector spaces, $\operatorname{Gr}^j S_i = 0$ for any $j \neq i$, and $S_i$ is a simple right $A$-module. If we set $m_i = \dim_K \operatorname{Gr}^i V$, then $\operatorname{Gr}^i V \cong S_i^{\oplus m_i}$ as right $A$-modules.

It follows from Bondal's theorem [1, Theorem 6.2] that

$$\operatorname{RHom}(G, \bullet) : D^b(\mathbb{Q}^3) \to D^b(\operatorname{mod} A)$$

is an exact equivalence, and its quasi-inverse is

$$\bullet \otimes^{\mathrm{L}}{}_A G : D^b(\operatorname{mod} A) \to D^b(\mathbb{Q}^3).$$

For a coherent sheaf $\mathcal{F}$ on $\mathbb{Q}^3$, this fact can be rephrased in terms of a spectral sequence [15, Theorem 1]:

$$E_2^{p,q} = \mathcal{T}or_{-p}^{A}(\operatorname{Ext}^q(G, \mathcal{F}), G) \Rightarrow E^{p+q} = \begin{cases} \mathcal{F} & \text{if} \quad p+q=0 \\ 0 & \text{if} \quad p+q \neq 0, \end{cases} \tag{2.1}$$

which is called the Bondal spectral sequence. Note that $E_2^{p,q}$ is the $p$th cohomology sheaf $\mathcal{H}^p(\operatorname{Ext}^q(G, \mathcal{F}) \otimes^{\mathrm{L}}{}_A G)$ of the complex $\operatorname{Ext}^q(G, \mathcal{F}) \otimes^{\mathrm{L}}{}_A G$. When we compute the spectral sequence, we consider the ascending filtration on the right $A$-module $\operatorname{Ext}^q(G, \mathcal{F})$ and apply the following

**Lemma 2.1.** *We have*

$$S_3 \otimes^{\mathrm{L}}{}_A G \cong \mathcal{O}(-1)[3]; \tag{2.2}$$

$$S_2 \otimes^{\mathrm{L}}{}_A G \cong T_{\mathbb{P}^4}(-2)|_{\mathbb{Q}^3}[2]; \tag{2.3}$$

$$S_1 \otimes^{\mathrm{L}}{}_A G \cong \mathcal{S}^{\vee}[1] \cong \mathcal{S}(-1)[1]; \tag{2.4}$$

$$S_0 \otimes^{\mathrm{L}}{}_A G \cong \mathcal{O}, \tag{2.5}$$

*where $T_{\mathbb{P}^4}$ denotes the tangent bundle of $\mathbb{P}^4$.*

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**Proof.** Since $\mathrm{RHom}(G, \mathcal{O}(-1)[3]) \cong S_3$, we obtain (2.2). Note that we have an isomorphism $\mathrm{RHom}(G, \mathcal{S}^\vee[1]) \cong S_1$ by [12, Lemma 8.2 (1)]. Hence we have (2.4). It is easy to see that the last isomorphism (2.5) holds. To see (2.3), first note that we have the following exact sequence:

$$0 \to \mathcal{O}(-2) \to \mathcal{O}(-1) \otimes H^0(\mathcal{O}(1))^\vee \to T_{\mathbb{P}^4}(-2)|_{\mathbb{Q}^3} \to 0.$$

Serre duality shows that

$$H^3(\mathcal{O}(-4)) \to H^3(\mathcal{O}(-3)) \otimes H^0(\mathcal{O}(1))^\vee$$

is dual of the canonical isomorphism

$$H^0(\mathcal{O}) \otimes H^0(\mathcal{O}(1)) \to H^0(\mathcal{O}(1)).$$

Hence $H^q(T_{\mathbb{P}^4}(-4)|_{\mathbb{Q}^3}) = 0$ for all $q$. Moreover, $h^q(\mathcal{S}^\vee(-i)) = 0$ for $i = 0, 1, 2$ and all $q$. Therefore, we conclude that $\mathrm{RHom}(G, T_{\mathbb{P}^4}(-2)|_{\mathbb{Q}^3})$ is isomorphic to $S_2[-2]$. □

**Remark 2.2.** As the referee pointed out, Lemma 2.1 shows that

$$(\mathcal{O}(-1), T_{\mathbb{P}^4}(-2)|_{\mathbb{Q}^3}, \mathcal{S}^\vee, \mathcal{O}) \tag{2.6}$$

is the left dual exceptional collection of $(G_0, G_1, G_2, G_3)$ (see [1] and [5] for the definition and the characterization of the left dual exceptional collection). Moreover, the full exceptional collection above is strong by [4, Proposition 3.3] (or by showing directly that $\mathrm{Ext}^q(T_{\mathbb{P}^4}(-2)|_{\mathbb{Q}^3}, \mathcal{S}^\vee) = 0$ for any $q > 0$ through the Euler exact sequence).

Our proof of Theorem 1.1 relies on the following theorem [13, Theorem 1.1]:

**Theorem 2.3.** *Let $\mathcal{E}$ be a nef vector bundle of rank $r$ on a smooth quadric surface $\mathbb{Q}^2$ over an algebraically closed field $K$ of characteristic zero. Suppose that $\det \mathcal{E} \cong \mathcal{O}(2,2)$. Then $\mathcal{E}$ is isomorphic to one of the following vector bundles or fits in one of the following exact sequences:*

(1) $\mathcal{O}(2,2) \oplus \mathcal{O}^{\oplus r-1}$;
(2) $\mathcal{O}(2,1) \oplus \mathcal{O}(0,1) \oplus \mathcal{O}^{\oplus r-2}$;
$\mathcal{O}(1,2) \oplus \mathcal{O}(1,0) \oplus \mathcal{O}^{\oplus r-2}$;
(*We do not exhibit the cases obtained by replacing $(a,b)$ with $(b,a)$ in the following:*)
(3) $\mathcal{O}(1,1)^{\oplus 2} \oplus \mathcal{O}^{\oplus r-2}$;
(4) $0 \to \mathcal{O} \xrightarrow{\iota} \mathcal{O}(1,1) \oplus \mathcal{O}(1,0) \oplus \mathcal{O}(0,1) \oplus \mathcal{O}^{\oplus r-2} \to \mathcal{E} \to 0$;
(5) $0 \to \mathcal{O}(-1,-1) \to \mathcal{O}(1,1) \oplus \mathcal{O}^{\oplus r} \to \mathcal{E} \to 0$;
(6) $0 \to \mathcal{O}^{\oplus 2} \to \mathcal{O}(1,0)^{\oplus 2} \oplus \mathcal{O}(0,1)^{\oplus 2} \oplus \mathcal{O}^{\oplus r-2} \to \mathcal{E} \to 0$;
(7) $0 \to \mathcal{O}(-1,-1) \oplus \mathcal{O}(-1,0) \oplus \mathcal{O}(0,-1) \to \mathcal{O}^{\oplus r+3} \to \mathcal{E} \to 0$;
(8) $0 \to \mathcal{O}(-1,-2) \to \mathcal{O}(1,0) \oplus \mathcal{O}^{\oplus r} \to \mathcal{E} \to 0$;
(9) $0 \to \mathcal{O}(-1,-1)^{\oplus 2} \to \mathcal{O}^{\oplus r+2} \to \mathcal{E} \to 0$;
(10) $0 \to \mathcal{O}(-2,-2) \to \mathcal{O}^{\oplus r+1} \to \mathcal{E} \to 0$;

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(11) $0 \to \mathcal{O}(-2,-2) \to \mathcal{O}^{\oplus r+1} \to \mathcal{E} \to k(p) \to 0$;
(12) $0 \to \mathcal{O}(-2,-2) \to \mathcal{O}^{\oplus r} \to \mathcal{E} \to \mathcal{O} \to 0$;
(13) $0 \to \mathcal{O}(-1,-1)^{\oplus 4} \to \mathcal{O}^{\oplus r} \oplus \mathcal{O}(-1,0)^{\oplus 2} \oplus \mathcal{O}(0,-1)^{\oplus 2} \to \mathcal{E} \to 0$.

## 3. Some basic properties of the spinor bundle $\mathcal{S}$ on $\mathbb{Q}^3$

We recall some basic facts and properties of the spinor bundle $\mathcal{S}$ on $\mathbb{Q}^3$ in our notation (see Ottaviani's result [16] and [12, Theorem 8.1]). First we have an exact sequence

$$0 \to \mathcal{S}^\vee \to \mathcal{O}^{\oplus 4} \to \mathcal{S} \to 0 \tag{3.1}$$

by [16, Theorem 2.8 (1)]. The restriction $\mathcal{S}|_{\mathbb{Q}^2}$ of $\mathcal{S}$ to a smooth hyperplane section $\mathbb{Q}^2$ of $\mathbb{Q}^3$ is isomorphic to $\mathcal{O}(1,0) \oplus \mathcal{O}(0,1)$, and $h^0(\mathcal{S}) = 4$. We have $\det \mathcal{S} = \mathcal{O}(1)$, and thus the canonical isomorphism

$$\mathcal{S}^\vee(1) \cong \mathcal{S}. \tag{3.2}$$

The zero locus $(s)_0$ of every non-zero element $s$ of $H^0(\mathcal{S})$ is a line $l$ in $\mathbb{Q}^3$. Thus $c_1(\mathcal{S}) \cap [\mathbb{Q}^3] = h$ and $c_2(\mathcal{S}) \cap [\mathbb{Q}^3] = l$. We have $h^q(\mathcal{S}) = 0$ for any $q > 0$ and $h^q(\mathcal{S}(-i)) = 0$ for all $q$ if $i = 1, 2$ or 3.

**Lemma 3.1.** *The natural map*

$$H^0(\mathcal{S}) \otimes H^0(\mathcal{S}) \to H^0(\mathcal{O}(1))$$

*sending $s \otimes t$ to $s \wedge t$ is surjective.*

**Proof.** Without loss of generality, we may assume that $\mathbb{Q}^3$ is defined by an equation $X_{01}^2 - X_{02}X_{13} + X_{03}X_{12} = 0$, where $[X_{01} : X_{02} : X_{03} : X_{12} : X_{13}]$ is the homogeneous coordinates of $\mathbb{P}^4$. We may also regard $\mathbb{Q}^3$ as a smooth hyperplane section $H \cap \mathbb{Q}^4$ of a smooth hyperquadric $\mathbb{Q}^4$ defined by an equation $X_{01}X_{23} - X_{02}X_{13} + X_{03}X_{12} = 0$, where $X_{ij}$ $(0 \le i < j \le 3)$ are homogeneous coordinates of $\mathbb{P}^5$, and $H$ is the hyperplane defined by $X_{01} = X_{23}$. Note that $\mathbb{Q}^4$ is the image of the Grassmannian $G(1,3)$ parametrizing lines in $\mathbb{P}^3$ by the Plücker embedding $\iota$. If we represent a point in $G(1,3)$ by a matrix $\begin{bmatrix} x_{10} & x_{11} & x_{12} & x_{13} \\ x_{20} & x_{21} & x_{22} & x_{23} \end{bmatrix}$, then $\iota^* X_{ij} = \begin{vmatrix} x_{1i} & x_{1j} \\ x_{2i} & x_{2j} \end{vmatrix}$. We will identify $\mathbb{Q}^4$ with $G(1,3)$ via $\iota$. Let $H^0(\mathbb{P}^3, \mathcal{O}(1)) \otimes \mathcal{O}_{G(1,3)} \to \mathcal{Q}$ be the universal quotient bundle on $G(1,3)$, which sends homogeneous coordinates $x_j$ of $\mathbb{P}^3$ to global sections $s_j$ of $\mathcal{Q}$ represented by $\begin{bmatrix} x_{1j} \\ x_{2j} \end{bmatrix}$. Recall that $\mathcal{S}$ is the restriction of $\mathcal{U}$ to the hyperplane section $H \cap \mathbb{Q}^4 = \mathbb{Q}^3$. By abuse of notation, we will denote by $s_j$ the restriction of $s_j$ to $\mathbb{Q}^3$. Since $h^0(\mathcal{S}) = 4$, $H^0(\mathcal{S})$ is spanned by $s_0, s_1, s_2, s_3$. Moreover, $H^0(\mathcal{O}(1))$ is spanned by $X_{i,j} = s_i \wedge s_j$, where $(i,j) = (0,1), (0,2), (0,3), (1,2)$ and $(1,3)$. This completes the proof. □

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## 4. Hirzebruch–Riemann–Roch formulas

Let $\mathcal{E}$ be a vector bundle of rank $r$ on $\mathbb{Q}^3$. Since the tangent bundle $T$ of $\mathbb{Q}^3$ fits in an exact sequence

$$0 \to T \to T_{\mathbb{P}^4}|_{\mathbb{Q}^3} \to \mathcal{O}_{\mathbb{Q}^3}(2) \to 0,$$

the Chern polynomial $c_t(T)$ of $T$ is

$$\frac{(1+ht)^5}{1+2ht} = 1 + 3ht + 4h^2t^2 + 2h^3t^3,$$

where $h$ denotes $c_1(\mathcal{O}_{\mathbb{Q}^3}(1))$. Then the Hirzebruch–Riemann–Roch formula implies that

$$\chi(\mathcal{E}) = r + \frac{13}{12}c_1h^2 + \frac{3}{4}(c_1^2 - 2c_2)h + \frac{1}{6}(c_1^3 - 3c_1c_2 + 3c_3),$$

where we set $c_i = c_i(\mathcal{E})$. To compute $\chi(\mathcal{E}(t))$, note that

$$\begin{aligned} c_1(\mathcal{E}(t)) &= c_1 + rth; \\ c_2(\mathcal{E}(t)) &= c_2 + (r-1)tc_1h + \binom{r}{2}t^2h^2; \\ c_3(\mathcal{E}(t)) &= c_3 + (r-2)tc_2h + \binom{r-1}{2}t^2c_1h^2 + \binom{r}{3}t^3h^3. \end{aligned}$$

Since $h^3 = 2$, we infer that

$$\begin{aligned} \chi(\mathcal{E}(t)) =& \frac{r}{3}t^3 + \frac{1}{2}(c_1h^2 + 3r)t^2 + \frac{1}{2}\{3c_1h^2 + (c_1^2 - 2c_2)h + \frac{13}{3}r\}t \\ &+ r + \frac{13}{12}c_1h^2 + \frac{3}{4}(c_1^2 - 2c_2)h + \frac{1}{6}(c_1^3 - 3c_1c_2 + 3c_3). \end{aligned} \tag{4.1}$$

Since $c_1(\mathcal{E}) = dh$ for some integer $d$, the formula above can be written as

$$\begin{aligned} \chi(\mathcal{E}(t)) =& \frac{r}{6}(2t+3)(t+2)(t+1) + dt^2 + (d^2+3d)t - c_2ht \\ &+ \frac{d}{6}(2d^2 + 9d + 13) + \frac{1}{2}\{c_3 - (d+3)c_2h\}. \end{aligned} \tag{4.2}$$

In this paper, we are dealing with the case $d = 2$:

$$\chi(\mathcal{E}(t)) = \frac{r}{6}(2t+3)(t+2)(t+1) + 2t^2 + 10t + 13 - c_2ht + \frac{1}{2}\{c_3 - 5c_2h\}. \tag{4.3}$$

In particular,

$$\chi(\mathcal{E}(-1)) = 5 - \frac{3}{2}c_2h + \frac{1}{2}c_3; \tag{4.4}$$

$$\chi(\mathcal{E}(-2)) = 1 - \frac{1}{2}c_2h + \frac{1}{2}c_3. \tag{4.5}$$

Next we will compute $\chi(\mathcal{S}^\vee \otimes \mathcal{E}(t))$. Recall that $c_1(\mathcal{S}) = h$ and that $c_1(\mathcal{S})c_2(\mathcal{S}) = 1$. Note also that

$$\begin{aligned}
\operatorname{rank} \mathcal{S}^\vee \otimes \mathcal{E} &= 2r; \\
c_1(\mathcal{S}^\vee \otimes \mathcal{E}) &= 2c_1 - rh; \\
c_2(\mathcal{S}^\vee \otimes \mathcal{E}) &= 2c_2 - (2r-1)c_1h + c_1^2 + \binom{r}{2}h^2 + rc_2(\mathcal{S}); \\
c_3(\mathcal{S}^\vee \otimes \mathcal{E}) &= 2c_3 - 2(r-1)c_2h + (r-1)^2c_1h^2 + 2(r-1)c_1c_2(\mathcal{S}) \\
&\quad +2c_1c_2 - (r-1)c_1^2h - \frac{1}{3}r(r^2-1).
\end{aligned}$$

The formula (4.1) together with the formulas above implies the following formula:

$$\begin{aligned}
\chi(\mathcal{S}^\vee \otimes \mathcal{E}(t)) &= \frac{2}{3}rt^3 + (c_1h^2 + 2r)t^2 + \{2c_1h^2 + (c_1^2 - 2c_2)h + \frac{4}{3}r\}t \\
&\quad + \frac{7}{6}c_1h^2 + c_1^2h - 2c_2h + \frac{1}{3}c_1^3 + c_3 - c_1c_2 - c_1c_2(\mathcal{S}).
\end{aligned}$$

Since $c_1 = dh$, the formula above becomes the following formula:

$$\begin{aligned}
\chi(\mathcal{S}^\vee \otimes \mathcal{E}(t)) &= \frac{2}{3}rt(t+1)(t+2) + 2dt^2 + 2d(d+2)t \\
&\quad + \frac{2}{3}d(d+1)(d+2) - (2t+d+2)c_2h + c_3.
\end{aligned} \tag{4.6}$$

For the case $d = 2$, we have

$$\chi(\mathcal{S}^\vee \otimes \mathcal{E}(t)) = \frac{2}{3}rt(t+1)(t+2) + 4(t+2)^2 - 2(t+2)c_2h + c_3. \tag{4.7}$$

In particular,

$$\chi(\mathcal{S}^\vee \otimes \mathcal{E}(-1)) = 4 - 2c_2h + c_3. \tag{4.8}$$

## 5. Key lemmas

**Lemma 5.1.** *We have the following exact sequence on* $\mathbb{Q}^3$*:*

$$0 \to T_{\mathbb{P}^4}(-2)|_{\mathbb{Q}^3} \to \mathcal{S}^\vee \otimes \operatorname{Hom}(T_{\mathbb{P}^4}(-2)|_{\mathbb{Q}^3}, \mathcal{S}^\vee)^\vee \to \Omega_{\mathbb{P}^4}(1)|_{\mathbb{Q}^3} \to 0, \tag{5.1}$$

*where the injection* $T_{\mathbb{P}^4}(-2)|_{\mathbb{Q}^3} \to \mathcal{S}^\vee \otimes \operatorname{Hom}(T_{\mathbb{P}^4}(-2)|_{\mathbb{Q}^3}, \mathcal{S}^\vee)^\vee$ *is the coevaluation morphism. Moreover,* $\dim \operatorname{Hom}(T_{\mathbb{P}^4}(-2)|_{\mathbb{Q}^3}, \mathcal{S}^\vee)^\vee = 4$.

**Proof.** The following simplified proof is due to the referee. As we have seen in Remark 2.2,

$$(\mathcal{O}(-1), T_{\mathbb{P}^4}(-2)|_{\mathbb{Q}^3}, \mathcal{S}^\vee, \mathcal{O}) \tag{5.2}$$

is a full strong exceptional collection of $D^b(\mathbb{Q}^3)$. Since this is strong, the right mutation $\mathbf{R}_{\mathcal{S}^\vee}(T_{\mathbb{P}^4}(-2)|_{\mathbb{Q}^3})$ of $T_{\mathbb{P}^4}(-2)|_{\mathbb{Q}^3}$ over $\mathcal{S}^\vee$ fits in the following distinguished triangle:

$$T_{\mathbb{P}^4}(-2)|_{\mathbb{Q}^3} \to \mathcal{S}^\vee \otimes \mathrm{Hom}(T_{\mathbb{P}^4}(-2)|_{\mathbb{Q}^3}, \mathcal{S}^\vee)^\vee \to \mathbf{R}_{\mathcal{S}^\vee}(T_{\mathbb{P}^4}(-2)|_{\mathbb{Q}^3}) \to .$$

Now consider the mutated full exceptional collection

$$(\mathcal{O}(-1), \mathcal{S}^\vee, \mathbf{R}_{\mathcal{S}^\vee}(T_{\mathbb{P}^4}(-2)|_{\mathbb{Q}^3}), \mathcal{O}). \tag{5.3}$$

Note here that

$$\mathrm{Ext}^q(\mathcal{S}^\vee, \mathbf{R}_{\mathcal{S}^\vee}(T_{\mathbb{P}^4}(-2)|_{\mathbb{Q}^3})) = 0 \text{ for } q \neq 0. \tag{5.4}$$

Indeed, by taking $\mathrm{RHom}(\mathcal{S}^\vee, \bullet)$ with the triangle above, we see that $\mathrm{Hom}(T_{\mathbb{P}^4}(-2)|_{\mathbb{Q}^3}, \mathcal{S}^\vee)^\vee$ is isomorphic to $\mathrm{RHom}(\mathcal{S}^\vee, \mathbf{R}_{\mathcal{S}^\vee}(T_{\mathbb{P}^4}(-2)|_{\mathbb{Q}^3}))$. On the other hand, by dualizing the collection (5.2) (and reversing the order) and then twisting it by $\mathcal{O}(-1)$ gives the following full strong exceptional collection:

$$(\mathcal{O}(-1), \mathcal{S}^\vee, \Omega_{\mathbb{P}^4}(1)|_{\mathbb{Q}^3}, \mathcal{O}). \tag{5.5}$$

Comparing two full exceptional collections (5.3) and (5.5), we infer that

$$\langle \mathbf{R}_{\mathcal{S}^\vee}(T_{\mathbb{P}^4}(-2)|_{\mathbb{Q}^3})\rangle = {}^\perp\langle \mathcal{O}(-1), \mathcal{S}^\vee\rangle \cap \langle \mathcal{O}\rangle^\perp = \langle \Omega_{\mathbb{P}^4}(1)|_{\mathbb{Q}^3}\rangle.$$

Thus, we have $\mathbf{R}_{\mathcal{S}^\vee}(T_{\mathbb{P}^4}(-2)|_{\mathbb{Q}^3}) \cong \Omega_{\mathbb{P}^4}(1)|_{\mathbb{Q}^3}[d]$ for some integer $d$, but the vanishing (5.4) implies that $d = 0$, namely

$$\mathbf{R}_{\mathcal{S}^\vee}(T_{\mathbb{P}^4}(-2)|_{\mathbb{Q}^3}) \cong \Omega_{\mathbb{P}^4}(1)|_{\mathbb{Q}^3}.$$

Hence we obtain the desired exact sequence (5.1). If follows immediately from the exact sequence (5.1) that $\dim \mathrm{Hom}(T_{\mathbb{P}^4}(-2)|_{\mathbb{Q}^3}, \mathcal{S}^\vee)^\vee = 4$. □

**Lemma 5.2.** *Let $\varphi : \mathcal{S}^\vee \to \Omega_{\mathbb{P}^4}(1)|_{\mathbb{Q}^3}$ be a morphism of $\mathcal{O}_{\mathbb{Q}^3}$-modules. If $\varphi \neq 0$, then $\varphi$ is injective, and there exists a line $L$ on $\mathbb{Q}^3$ such that the restriction* $\mathrm{Coker}(\varphi)|_L$ *to $L$ of the cokernel* $\mathrm{Coker}(\varphi)$ *of $\varphi$ admits a negative degree quotient.*

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**Proof.** We have an exact sequence

$$0 \to \Omega_{\mathbb{P}^4}(1)|_{\mathbb{Q}^3} \xrightarrow{i} H^0(\mathcal{O}(1)) \otimes \mathcal{O} \to \mathcal{O}(1) \to 0,$$

and the composite $i \circ \varphi$ can be written as

$$i \circ \varphi = \sum_{i=1}^{r} l_i \otimes s_i^\vee$$

for some $l_i \in H^0(\mathcal{O}(1))$ and $s_i \in H^0(\mathcal{S})$, where $s_i^\vee$ denotes the dual of the morphism $\mathcal{O} \to \mathcal{S}$ determined by $s_i$. We may assume that $l_i \neq 0$ for all $i$. By replacing $l_i$ if necessary, we may further assume that $s_1, \ldots, s_r$ are linearly independent. Since $h^0(\mathcal{S}) = 4$, we have $r \leq 4$. Note that $\sum_{i=1}^r l_i s_i^\vee = 0$ in $\mathrm{Hom}(\mathcal{S}^\vee, \mathcal{O}(1))$. Hence $r \geq 2$. Moreover, we have a surjective morphism

$$\psi : \mathrm{Coker}(i \circ \varphi) \to \mathcal{O}(1).$$

Note that the morphism $\mathcal{O}^{\oplus r} \to \mathcal{S}$ determined by $(s_1, \ldots, s_r)$ is generically surjective. Hence we see that $i \circ \varphi$ is injective. Therefore, $\varphi$ is injective and

$$\mathrm{Coker}(\varphi) \cong \mathrm{Ker}(\psi).$$

If $r = 2$, then $\mathrm{Coker}(i \circ \varphi) \cong \mathcal{T} \oplus \mathcal{O}^{\oplus 3}$ for some torsion sheaf $\mathcal{T}$ on $\mathbb{Q}^3$. Since $\mathcal{O}(1)$ is torsion-free, $\psi$ maps $\mathcal{T}$ to zero, and we have a surjective morphism $\bar{\psi} : \mathcal{O}^{\oplus 3} \to \mathcal{O}(1)$. On the other hand, $\bar{\psi} : \mathcal{O}^{\oplus 3} \to \mathcal{O}(1)$ cannot be surjecitve since three hyperplane sections of $\mathbb{Q}^3$ always meet at a point. This is a contradiction. Hence $r = 3$ or 4. Suppose that $r = 4$. Then it follows from the exact sequence (3.1) that $\mathrm{Coker}(i \circ \varphi) \cong \mathcal{S} \oplus \mathcal{O}$. Note that $\psi$ induces a morphism $\mathcal{S} \to \mathcal{O}(1)$, which factors through $\mathcal{I}_L(1)$ for some line $L$ in $\mathbb{Q}^3$. Since $L$ and a hyperplane in $\mathbb{Q}^3$ meet at a point, $\psi$ cannot be surjective. Hence the case $r = 4$ does not arise, and we have $r = 3$.

Now it follows from the exact sequence (3.1) that the cokernel of the morphism determined by ${}^t(s_1^\vee, s_2^\vee, s_3^\vee) : \mathcal{S}^\vee \to \mathcal{O}^{\oplus 3}$ is isomorphic to the cokernel of some non-zero morphism $\mathcal{O} \to \mathcal{S}$, and hence it is isomorphic to $\mathcal{I}_M(1)$ for some line $M$ on $\mathbb{Q}^3$. Therefore, $\mathrm{Coker}(i \circ \varphi) \cong \mathcal{I}_M(1) \oplus \mathcal{O}^{\oplus 2}$, and we have the following exact sequence:

$$0 \to \mathrm{Coker}(\varphi) \to \mathcal{I}_M(1) \oplus \mathcal{O}^{\oplus 2} \xrightarrow{\psi} \mathcal{O}(1) \to 0. \tag{5.6}$$

Let $\mathbb{Q}^2$ be a general hyperplane section of $\mathbb{Q}^3$ containing $M$. We may assume that $M$ is a divisor of type $(1, 0)$ of $\mathbb{Q}^2$. Then $\mathcal{I}_M(1)$ fits in the following exact sequence:

$$0 \to \mathcal{O}_{\mathbb{Q}^3} \to \mathcal{I}_M(1) \to \mathcal{O}_{\mathbb{Q}^2}(0, 1) \to 0.$$

By pulling back the sequence above to a line $L$ of type $(0, 1)$ in $\mathbb{Q}^2$, we obtain the following exact sequence:

$$\mathcal{O}_L \to \mathcal{I}_M(1) \otimes \mathcal{O}_L \to \mathcal{O}_L \to 0.$$

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The image of $\mathcal{O}_L \to \mathcal{I}_M(1) \otimes \mathcal{O}_L$ is the torsion part of $\mathcal{I}_M(1) \otimes \mathcal{O}_L$. Therefore, $\psi \otimes 1_L$ factors through $\mathcal{O}_L^{\oplus 3}$ and induces a surjection $\mathcal{O}_L^{\oplus 3} \to \mathcal{O}_L(1)$. Hence $\mathrm{Coker}(\varphi) \otimes \mathcal{O}_L$ has $\mathcal{O}_L(-1) \oplus \mathcal{O}_L$ as a quotient. □

Lemma 5.3 will be applied to $\psi_a$ in (12.4) and (12.7) and plays a crucial role in our proof of Theorem 1.1.

**Lemma 5.3.** *For any positive integer $a$ and for any morphism $\psi_a : T_{\mathbb{P}^4}(-2)|_{\mathbb{Q}^3} \to \mathcal{S}^{\vee \oplus a}$, there exists a line $L$ in $\mathbb{Q}^3$ such that the cokernel* $\mathrm{Coker}(\psi_a)$ *of $\psi_a$ has $\mathcal{O}_L(-1)$ as a quotient. In case $a=1$, there is a one-to-one correspondence between lines $L$ in $\mathbb{Q}^3$ and non-zero morphisms $\psi_1 : T_{\mathbb{P}^4}(-2)|_{\mathbb{Q}^3} \to \mathcal{S}^\vee$ up to scalar, and the correspondence is given by the following exact sequence:*

$$0 \to \mathcal{O}(-1)^{\oplus 2} \to T_{\mathbb{P}^4}(-2)|_{\mathbb{Q}^3} \xrightarrow{\psi_1} \mathcal{S}^\vee \to \mathcal{O}_L(-1) \to 0. \tag{5.7}$$

**Proof.** The following brilliant proof is due to the referee. This proof is much shorter than the original and enlightens the meaning of the exact sequence (5.7) more clearly.

Denote by $\mathrm{Quot}(\mathcal{S}^\vee)$ the Quot-scheme parametrizing quotient sheaves of $\mathcal{S}^\vee$. Then we have a morphism

$$\Psi : \mathbb{P}(\mathrm{Hom}(T_{\mathbb{P}^4}(-2)|_{\mathbb{Q}^3}, \mathcal{S}^\vee)^\vee) \to \mathrm{Quot}(\mathcal{S}^\vee)$$

sending $[\psi_1]$ to $\mathrm{Coker}(\psi_1)$. Note that for any line $L \subset \mathbb{Q}^3$ we have $\mathcal{S}^\vee|_L \cong \mathcal{O}_L(-1) \oplus \mathcal{O}_L$ so that $\mathcal{S}^\vee$ admits $\mathcal{O}_L(-1)$ as a quotient. Note also that the Hilbert polynomial $\chi(\mathcal{O}_L(t-1))$ of $\mathcal{O}_L(-1)$ is $t$. Let $Z$ be the Hilbert scheme parametrizing lines in $\mathbb{Q}^3$. Then we have an inclusion

$$Z \hookrightarrow \mathrm{Quot}^t(\mathcal{S}^\vee)$$

sending $[L]$ to $\mathcal{O}_L(-1)$, where $\mathrm{Quot}^t(\mathcal{S}^\vee)$ is the Quot-scheme parametrizing quotients of $\mathcal{S}^\vee$ with Hilbert polynomial $t$. It is well-known that $Z \cong \mathbb{P}^3$. Note also that

$$\mathbb{P}(\mathrm{Hom}(T_{\mathbb{P}^4}(-2)|_{\mathbb{Q}^3}, \mathcal{S}^\vee)^\vee) \cong \mathbb{P}^3$$

by Lemma 5.1. We will show that $\Psi$ is an isomorphism onto $Z$.

We first claim that the image $\mathrm{Im}\,\Psi$ of $\Psi$ is $Z$. To see this, we first apply to $\mathcal{O}_L(-1)$ for any line $L \subset \mathbb{Q}^3$ the Bondal spectral sequence (2.1). We have the following:

$$\mathrm{ext}^q(\mathcal{O}, \mathcal{O}_L(-1)) = 0 \text{ for any } q;$$

$$\mathrm{ext}^q(\mathcal{S}, \mathcal{O}_L(-1)) = h^q(\mathcal{O}_L(-2) \oplus \mathcal{O}_L(-1)) = \begin{cases} 1 & \text{if } q = 1 \\ 0 & \text{if } q \neq 1 \end{cases}; \tag{5.8}$$

$$\mathrm{ext}^q(\mathcal{O}(1), \mathcal{O}_L(-1)) = h^q(\mathcal{O}_L(-2)) = \begin{cases} 1 & \text{if } q = 1 \\ 0 & \text{if } q \neq 1 \end{cases};$$

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$$\operatorname{ext}^q(\mathcal{O}(2), \mathcal{O}_L(-1)) = h^q(\mathcal{O}_L(-3)) = \begin{cases} 2 & \text{if } q = 1 \\ 0 & \text{if } q \neq 1 \end{cases}.$$

Thus, $\operatorname{Ext}^3(G, \mathcal{E}) = 0$, $\operatorname{Ext}^2(G, \mathcal{E}) = 0$, $\operatorname{Hom}(G, \mathcal{E}) = 0$, and $\operatorname{Ext}^1(G, \mathcal{E})$ has a filtration $S_1 \subset F \subset \operatorname{Ext}^1(G, \mathcal{E})$ of right $A$-modules such that the following sequences are exact:

$$0 \to F \to \operatorname{Ext}^1(G, \mathcal{E}) \to S_3^{\oplus 2} \to 0;$$

$$0 \to S_1 \to F \to S_2 \to 0.$$

These exact sequences induce the following distinguished triangles by Lemma 2.1:

$$F \otimes^{\mathrm{L}}{}_A G \to \operatorname{Ext}^1(G, \mathcal{E}) \otimes^{\mathrm{L}}{}_A G \to \mathcal{O}(-1)^{\oplus 2}[3] \to;$$

$$\mathcal{S}^\vee[1] \to F \otimes^{\mathrm{L}}{}_A G \to T_{\mathbb{P}^4}(-2)|_{\mathbb{Q}^3}[2] \to .$$

By taking cohomologies, we obtain the following exact sequences:

$$0 \to E_2^{-3,1} \to \mathcal{O}(-1) \to \mathcal{H}^{-2}(F \otimes^{\mathrm{L}}{}_A G) \to E_2^{-2,1} \to 0;$$

$$0 \to \mathcal{H}^{-2}(F \otimes^{\mathrm{L}}{}_A G) \to T_{\mathbb{P}^4}(-2)|_{\mathbb{Q}^3} \xrightarrow{\psi_L} \mathcal{S}^\vee \to E_2^{-1,1} \to 0.$$

Moreover, we see that $E_2^{p,q} = 0$ unless $q = 1$ and that $E_2^{p,1} = 0$ unless $p = -3, -2$ or $-1$. Hence we infer that $E_2^{-3,1} = 0$, that $E_2^{-2,1} = 0$ and that $E_2^{-1,1} \cong \mathcal{O}_L(-1)$. Therefore, $\mathcal{O}_L(-1)$ is resolved as

$$0 \to \mathcal{O}(-1)^{\oplus 2} \to T_{\mathbb{P}^4}(-2)|_{\mathbb{Q}^3} \xrightarrow{\psi_L} \mathcal{S}^\vee \to \mathcal{O}_L(-1) \to 0 \tag{5.9}$$

in terms of the full strong exceptional collection (2.6). This implies that the image $\operatorname{Im}\Psi$ of $\Psi$ contains $Z$. Since the source of $\Psi$ has the same dimension as $Z$, we conclude that $\operatorname{Im}\Psi = Z$.

Next we show that $\Psi$ is injective. Note that the exact sequence (5.9) splits into the following two exact sequences:

$$0 \to \mathcal{K} \to \mathcal{S}^\vee \to \mathcal{O}_L(-1) \to 0; \tag{5.10}$$

$$0 \to \mathcal{O}(-1)^{\oplus 2} \to T_{\mathbb{P}^4}(-2)|_{\mathbb{Q}^3} \to \mathcal{K} \to 0. \tag{5.11}$$

Since we have (5.8), the exact sequence (5.10) shows that $\mathcal{K}$ is the left mutation of $\mathcal{O}_L(-1)$ over $\mathcal{S}^\vee$. Moreover it follows from (5.11) that $\mathcal{O}(-1)^{\oplus 2}$ is the left mutation of $\mathcal{K}$ over $T_{\mathbb{P}^4}(-2)|_{\mathbb{Q}^3}$, since

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$$K \cong \mathrm{RHom}(T_{\mathbb{P}^4}(-2)|_{\mathbb{Q}^3}, T_{\mathbb{P}^4}(-2)|_{\mathbb{Q}^3}) \cong \mathrm{RHom}(T_{\mathbb{P}^4}(-2)|_{\mathbb{Q}^3}, K).$$

Therefore, $\psi_L$ in (5.9) is uniquely determined by $L$ up to scalar. Hence $\Psi$ is injective.

Finally, if the composite of the morphism $\psi_a$ and some projection $\mathcal{S}^{\vee \oplus a} \to \mathcal{S}^\vee$ is zero, then $\mathrm{Coker}(\psi_a)$ admits $\mathcal{S}^\vee$ as a quotient, and the assertion follows. Hence we may assume that the composite cannot be zero for any projection $\mathcal{S}^{\vee \oplus a} \to \mathcal{S}^\vee$. Then the cokernel of the composite has $\mathcal{O}_L(-1)$ as a quotient, and so does $\mathrm{Coker}(\psi_a)$. □

Since the analyses of $\mathrm{Coker}(\psi_a)$ in case $a \geq 2$ in the original proof of Lemma 5.3 are indispensable for the proof of Lemma 5.4, we also provide that part of the proof as it is. Recall here that, for a coherent sheaf $\mathcal{F}$ of codimension $\geq p+1$ on a non-singular projective variety $X$, we have $c_i(\mathcal{F}) = 0$ for all $1 \leq i \leq p$ (see, e.g., [6, Example 15.3.6]).

**Proof. The original proof of Lemma 5.3 in case** $a \geq 2$ If the composite of the morphism $\psi_a$ and some projection $\mathcal{S}^{\vee \oplus a} \to \mathcal{S}^\vee$ is zero, then $\mathrm{Coker}(\psi_a)$ admits $\mathcal{S}^\vee$ as a quotient, and the assertion follows. Hence we may assume that the composite cannot be zero for any projection $\mathcal{S}^{\vee \oplus a} \to \mathcal{S}^\vee$, and this implies that $a \leq 4$ by Lemma 5.1.

If $a = 4$, then Lemma 5.1 shows that $\mathrm{Coker}(\psi_4) \cong \Omega_{\mathbb{P}^4}(1)|_{\mathbb{Q}^3}$, and the assertion follows.

If $a = 3$, then $\psi_3$ can be regarded as the composite of the coevaluation morphism

$$\psi_4 : T_{\mathbb{P}^4}(-2)|_{\mathbb{Q}^3} \to \mathcal{S}^\vee \otimes \mathrm{Hom}(T_{\mathbb{P}^4}(-2)|_{\mathbb{Q}^3}, \mathcal{S}^\vee)^\vee$$

and some projection $\mathcal{S}^\vee \otimes \mathrm{Hom}(T_{\mathbb{P}^4}(-2)|_{\mathbb{Q}^3}, \mathcal{S}^\vee)^\vee \to \mathcal{S}^{\vee \oplus 3}$. Let $\mathcal{S}^\vee \to \mathcal{S}^{\vee \oplus 4}$ be the kernel of this projection, and let $\varphi$ be the composite of the inclusion $\mathcal{S}^\vee \to \mathcal{S}^{\vee \oplus 4}$ and the surjection $\mathcal{S}^{\vee \oplus 4} \to \Omega_{\mathbb{P}^4}(1)|_{\mathbb{Q}^3}$ in (5.1). Then

$$\mathrm{Coker}(\psi_3) \cong \mathrm{Coker}(\varphi) \tag{5.12}$$

and $\mathrm{Ker}(\psi_3) \cong \mathrm{Ker}(\varphi)$ by the snake lemma. Since $\mathrm{Hom}(\mathcal{S}^\vee, T_{\mathbb{P}^4}(-2)|_{\mathbb{Q}^3}) = 0$, $\varphi$ cannot be zero by (5.1). Lemma 5.2 then shows that $\varphi$ is injective and that the restriction $\mathrm{Coker}(\varphi)|_L$ to some line $L$ on $\mathbb{Q}^3$ admits a negative degree quotient. Hence the assertion holds, and $\psi_3$ is injective.

Suppose that $a = 2$. Then we can regard $\psi_2$ as the composite of some $\psi_3 : T_{\mathbb{P}^4}(-2)|_{\mathbb{Q}^3} \to \mathcal{S}^{\vee \oplus 3}$ and some projection $\mathcal{S}^{\vee \oplus 3} \to \mathcal{S}^{\vee \oplus 2}$. Let $\mathcal{S}^\vee \to \mathcal{S}^{\vee \oplus 3}$ be the kernel of this projection. Note here that we have an exact sequence

$$0 \to T_{\mathbb{P}^4}(-2)|_{\mathbb{Q}^3} \xrightarrow{\psi_3} \mathcal{S}^{\vee \oplus 3} \to \mathrm{Coker}(\varphi) \to 0.$$

Denote by $\varphi_1 : \mathcal{S}^\vee \to \mathrm{Coker}(\varphi)$ the composite of the inclusion $\mathcal{S}^\vee \to \mathcal{S}^{\vee \oplus 3}$ and the surjection $\mathcal{S}^{\vee \oplus 3} \to \mathrm{Coker}(\varphi)$. Then $\varphi_1$ cannot be zero, since $\mathrm{Hom}(\mathcal{S}^\vee, T_{\mathbb{P}^4}(-2)|_{\mathbb{Q}^3}) = 0$. Moreover, the snake lemma implies that

$$\mathrm{Coker}(\psi_2) \cong \mathrm{Coker}(\varphi_1) \quad \text{and that} \quad \mathrm{Ker}(\psi_2) \cong \mathrm{Ker}(\varphi_1).$$

Recall the inclusion $i : \operatorname{Coker}(\varphi) \hookrightarrow \mathcal{I}_M(1) \oplus \mathcal{O}^{\oplus 2}$ in (5.6) and consider the composite $i \circ \varphi_1$. We have the following exact sequence:

$$0 \to \operatorname{Coker}(\varphi_1) \to \operatorname{Coker}(i \circ \varphi_1) \to \mathcal{O}(1) \to 0. \tag{5.13}$$

Let $i \circ \varphi_1$ be equal to $(t^\vee, s_1^\vee, s_2^\vee)$, where $t^\vee \in \operatorname{Hom}(\mathcal{S}^\vee, \mathcal{I}_M(1))$, $t \in H^0(\mathcal{S}(1))$, $s_1^\vee$, $s_2^\vee \in \operatorname{Hom}(\mathcal{S}^\vee, \mathcal{O})$ and $s_1$, $s_2 \in H^0(\mathcal{S})$. Since we have an exact sequence (5.6), we have $t^\vee + h_1 s_1^\vee + h_2 s_2^\vee = 0$ for some $h_1$, $h_2 \in H^0(\mathcal{O}(1))$. Now we have two cases:

(1) $s_1$ and $s_2$ are linearly independent;
(2) $s_1$ and $s_2$ are linearly dependent.

(1) If $s_1$ and $s_2$ are linearly independent, then $\varphi_1$ is injective, and $\operatorname{Coker}(i \circ \varphi_1)$ has rank one. Thus we see that $\operatorname{Coker}(\varphi_1)$ is a torsion sheaf. Moreover, we claim that $\operatorname{Coker}(\varphi_1)$ is pure by [8, Prop. 1.1.6]: first note that $\mathcal{E}xt^q_{\mathbb{Q}^3}(\operatorname{Coker}(\varphi), \omega_{\mathbb{Q}^3}) = 0$ for all $q \geq 2$; thus $\mathcal{E}xt^q_{\mathbb{Q}^3}(\operatorname{Coker}(\varphi_1), \omega_{\mathbb{Q}^3}) = 0$ for all $q \geq 2$, and hence $\operatorname{Coker}(\varphi_1)$ satisfies the generalized Serre's condition $S_{1,1}$ in [8, Section 1.1]. Now we compute the Chern polynomial of $\operatorname{Coker}(\varphi_1)$. First note that $c_t(\operatorname{Coker}(\varphi)) = c_t(\Omega_{\mathbb{P}^4}(1)|_{\mathbb{Q}^3})/c_t(\mathcal{S}^\vee) = 1 + lt^2 - t^3$. Hence

$$c_t(\operatorname{Coker}(\varphi_1)) = c_t(\operatorname{Coker}(\varphi))/c_t(\mathcal{S}^\vee) = 1 + ht + 2lt^2.$$

Since $\operatorname{Coker}(\varphi_1)$ is a torsion sheaf, this implies that $\operatorname{Coker}(\varphi_1)$ is supported on a hyperplane section $H$ of $\mathbb{Q}^3$, and the length of $\operatorname{Coker}(\varphi_1)$ at the generic point of $H$ is one. Since $\operatorname{Coker}(\varphi_1)$ is pure, this implies that $\operatorname{Coker}(\varphi_1)$ is of the form $\mathcal{I}_{Z,H}(D)$, where $D$ is a divisor on $H$ and $\mathcal{I}_{Z,H}$ denotes the ideal sheaf of some zero-dimensional closed subscheme $Z$ in $H$. Note here that $c_t(\mathcal{O}_H) = 1 + ht + 2lt^2 + 2t^3$, that $c_t(\mathcal{O}_L) = (c_t(\mathcal{S}^\vee)/c_t(\mathcal{O}(-1)))^{-1} = 1 - lt^2 - t^3$ and that $c_t(k(p)) = 1 + 2t^3$, where $k(p)$ is the residue field at a point $p$ (see also [6, Example 15.3.1] for the formula $c_t(k(p)) = 1 + 2t^3$). Hence we see that $[D] = 0 \cdot l$ in $A^2\mathbb{Q}^3$. Moreover, if $D$ is of type $(d, -d)$, then $c_t(\mathcal{I}_{Z,H}(D)) = 1 + ht + 2lt^2 + (2 - 2d^2 - 2 \operatorname{length} Z)t^3$. Hence $(d, \operatorname{length} Z) = (0, 1)$ or $(\pm 1, 0)$. Therefore, $\operatorname{Coker}(\varphi_1)$ is isomorphic to either $\mathcal{I}_{p,H}$ or $\mathcal{O}_H(d, -d)$ where $d = \pm 1$. Thus the assertion holds.

(2) If $s_1$ and $s_2$ are linearly dependent, by replacing $s_i$ and $h_i$ if necessary, we may assume that $s_2 = 0$, and we have $t^\vee + h_1 s_1^\vee = 0$. Set $\varphi_1' := (t^\vee, s_1^\vee) : \mathcal{S}^\vee \to \mathcal{I}_M(1) \oplus \mathcal{O}_{\mathbb{Q}^3}$. Then $\operatorname{Coker}(i \circ \varphi_1) \cong \operatorname{Coker}(\varphi_1') \oplus \mathcal{O}_{\mathbb{Q}^3}$ and $\operatorname{Ker}(\varphi_1) \cong \operatorname{Ker}(\varphi_1')$. Note that $\varphi_1' \neq 0$ since $\varphi_1 \neq 0$. Hence $s_1 \neq 0$. Let $L$ be the zero locus $(s_1)_0$ of $s_1$. Then the composite of $\varphi_1'$ and the inclusion $\mathcal{I}_M(1) \oplus \mathcal{O}_{\mathbb{Q}^3} \to \mathcal{O}(1) \oplus \mathcal{O}_{\mathbb{Q}^3}$ factors through the morphism $(-h_1, 1) : \mathcal{O} \to \mathcal{O}(1) \oplus \mathcal{O}_{\mathbb{Q}^3}$, and we have the following commutative diagram with exact rows:

$$\begin{array}{ccccccccc}
 & & \mathcal{S}^\vee & \xrightarrow{s_1^\vee} & \mathcal{O}_{\mathbb{Q}^3} & \longrightarrow & \mathcal{O}_L & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle \varphi_1'} & & \downarrow{\scriptstyle (-h_1,1)} & & \downarrow{\scriptstyle -\bar{h}_1} & & \\
0 & \longrightarrow & \mathcal{I}_M(1) \oplus \mathcal{O}_{\mathbb{Q}^3} & \longrightarrow & \mathcal{O}(1) \oplus \mathcal{O}_{\mathbb{Q}^3} & \longrightarrow & \mathcal{O}_M(1) & \longrightarrow & 0
\end{array} \tag{5.14}$$

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We see that $\mathrm{Im}(\varphi_1') \cong \mathcal{I}_L$ and that $\mathrm{Ker}(\varphi_1') \cong \mathcal{O}(-1)$. We claim here that $\bar{h}_1 \neq 0$. Assume, to the contrary, that $\bar{h}_1 = 0$. Then the snake lemma implies that $\mathrm{Coker}(\varphi_1')$ fits in the following exact sequence:

$$0 \to \mathcal{O}_L \to \mathrm{Coker}(\varphi_1') \to \mathcal{O}(1) \to \mathcal{O}_M(1) \to 0.$$

Since $\mathcal{O}_L$ is a torsion sheaf, the surjection $\mathrm{Coker}(\varphi_1') \oplus \mathcal{O}_{\mathbb{Q}^3} \to \mathcal{O}(1)$ induces a surjection $\mathcal{I}_M(1) \oplus \mathcal{O}_{\mathbb{Q}^3} \to \mathcal{O}(1)$. On the other hand, the morphism $\mathcal{I}_M(1) \oplus \mathcal{O}_{\mathbb{Q}^3} \to \mathcal{O}(1)$ cannot be surjective since a line $M$ and a hyperplane meets at least at one point. This is a contradiction. Hence $\bar{h}_1 \neq 0$, and thus $L = M$. Moreover, the commutative diagram (5.14) induces the following exact sequence by the snake lemma:

$$0 \to \mathrm{Coker}(\varphi_1') \to \mathcal{O}(1) \to k(p) \to 0,$$

where $p = (\bar{h}_1)_0$. Therefore, $\mathrm{Coker}(\varphi_1') = \mathcal{I}_p(1)$. The exact sequence (5.13), i.e. the sequence

$$0 \to \mathrm{Coker}(\varphi_1) \to \mathcal{I}_p(1) \oplus \mathcal{O}_{\mathbb{Q}^3} \to \mathcal{O}(1) \to 0$$

then shows that $\mathrm{Coker}(\varphi_1) = \mathcal{I}_p$. Thus the assertion also holds if $s_1$ and $s_2$ are linearly dependent. □

Lemma 5.4 will be applied to $\pi$ in (12.8) and plays a crucial role in the proof of Theorem 1.1.

**Lemma 5.4.** *Let $\psi_a : T_{\mathbb{P}^4}(-2)|_{\mathbb{Q}^3} \to \mathcal{S}^{\vee \oplus a}$ be a morphism of $\mathcal{O}_{\mathbb{Q}^3}$-modules where $a$ is a positive integer, and let $\pi : \mathcal{O}_{\mathbb{Q}^3}(-1) \to \mathrm{Coker}(\psi_a)$ be a morphism of $\mathcal{O}_{\mathbb{Q}^3}$-modules. If* $\mathrm{Coker}(\pi)$ *does not admit a negative degree quotient, then $a = 1$,* $\mathrm{Coker}(\pi) = 0$ *and* $\mathrm{Ker}(\pi)$ *is isomorphic to $\mathcal{I}_L(-1)$ for some line $L$ in $\mathbb{Q}^3$.*

**Proof.** We may assume that $\pi \neq 0$.

Suppose that $\mathrm{Coker}(\psi_a)$ admits $\mathcal{S}^\vee$ as a quotient; let $p : \mathrm{Coker}(\psi_a) \to \mathcal{S}^\vee$ be the surjection. Note that $\mathrm{Coker}(\pi)$ admits $\mathrm{Coker}(p \circ \pi)$ as a quotient. If $p \circ \pi = 0$, then $\mathrm{Coker}(p \circ \pi) \cong \mathcal{S}^\vee$, and if $p \circ \pi \neq 0$, then $\mathrm{Coker}(p \circ \pi) \cong \mathcal{I}_L$ for some line $L$ in $\mathbb{Q}^3$. Therefore, the restriction of $\mathrm{Coker}(\pi)$ to a line admits a negative degree quotient.

In the following, we assume that $\mathrm{Coker}(\psi_a)$ does not admit $\mathcal{S}^\vee$ as a quotient. Hence $a \leq 4$ by Lemma 5.1.

Suppose that $a = 4$. Then $\mathrm{Coker}(\psi_4) \cong \Omega_{\mathbb{P}^4}(1)|_{\mathbb{Q}^3}$ by Lemma 5.1. Since $\Omega_{\mathbb{P}^4}(1)|_L \cong \mathcal{O}_L(-1) \oplus \mathcal{O}_L^{\oplus 3}$ for any line $L$ in $\mathbb{Q}^3$, if $\mathrm{Coker}(\pi)|_L$ does not admit a negative degree quotient for any line $L$ in $\mathbb{Q}^3$, we see that $\mathrm{Coker}(\pi)|_L \cong \mathcal{O}_L^{\oplus 3}$ for any line $L$ in $\mathbb{Q}^3$. This implies that $\mathrm{Coker}(\pi) \cong \mathcal{O}_{\mathbb{Q}^3}^{\oplus 3}$ by [18, (3.6.1) Lemma]. Thus $\Omega_{\mathbb{P}^4}(1)|_{\mathbb{Q}^3} \cong \mathcal{O}(-1) \oplus \mathcal{O}^{\oplus 3}$, which contradicts $H^0(\Omega_{\mathbb{P}^4}(1)|_{\mathbb{Q}^3}) = 0$. Therefore, $\mathrm{Coker}(\pi)|_L$ admits a negative degree quotient for some line $L$ in $\mathbb{Q}^3$.

Suppose that $a = 3$. Recall that $\mathrm{Coker}(\psi_3) \cong \mathrm{Coker}(\varphi)$ in (5.12). Recall also the inclusion $i : \mathrm{Coker}(\varphi) \hookrightarrow \mathcal{I}_M(1) \oplus \mathcal{O}^{\oplus 2}$ in (5.6) and consider the composite $i \circ \pi$. We have the following exact sequence:

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$$0 \to \mathrm{Coker}(\pi) \to \mathrm{Coker}(i \circ \pi) \xrightarrow{\rho} \mathcal{O}(1) \to 0. \tag{5.15}$$

Let $i \circ \pi$ be equal to $(t, g_1, g_2)$, where $t \in \mathrm{Hom}(\mathcal{O}(-1), \mathcal{I}_M(1)) \cong H^0(\mathcal{I}_M(2))$, $g_1$, $g_2 \in \mathrm{Hom}(\mathcal{O}(-1), \mathcal{O}) \cong H^0(\mathcal{O}(1))$. Since we have an exact sequence (5.6), we have $t + h_1 g_1 + h_2 g_2 = 0$ for some $h_1$, $h_2 \in H^0(\mathcal{O}(1))$. Now we have two cases:

(1) $g_1$ and $g_2$ are linearly independent;
(2) $g_1$ and $g_2$ are linearly dependent.

(1) If $g_1$ and $g_2$ are linearly independent, then the cokernel of the morphism $(g_1, g_2) : \mathcal{O}(-1) \to \mathcal{O}^{\oplus 2}$ is of the form $\mathcal{I}_C(1)$, where $C$ is the conic defined by $g_1$ and $g_2$. Hence $\mathrm{Coker}(i \circ \pi)$ fits in the following exact sequence:

$$0 \to \mathcal{I}_M(1) \to \mathrm{Coker}(i \circ \pi) \to \mathcal{I}_C(1) \to 0.$$

Now consider the composite of the injection $\mathcal{I}_M \to \mathrm{Coker}(i \circ \pi)(-1)$ and the surjection $\rho(-1) : \mathrm{Coker}(i \circ \pi)(-1) \to \mathcal{O}$. The composite is nothing but the inclusion $\mathcal{I}_M \hookrightarrow \mathcal{O}$ and its cokernel is $\mathcal{O}_M$. Thus the surjection $\rho(-1)$ induces a surjection $\bar{\rho}(-1) : \mathcal{I}_C \to \mathcal{O}_M$. This implies that $C \cap M = \emptyset$. Moreover $\mathrm{Coker}(\pi)(-1) \cong \mathrm{Ker}(\bar{\rho}(-1)) \cong \mathcal{I}_{C \sqcup M}$. Hence $\mathrm{Coker}(\pi) \cong \mathcal{I}_{C \sqcup M}(1)$. Note that the conic $C$ and the line $M$ can be joined by a line $L$ in $\mathbb{Q}^3$. Indeed, any hyperplane section $H$ containing $M$ intersects $C$ at some point $p$, and the point $p$ and $M$ can be joined by a line $L$ in $H$. Now we see that $\mathrm{Coker}(\pi)|_L$ admits a negative degree quotient.

(2) If $g_1$ and $g_2$ are linearly dependent, by replacing $g_i$ and $h_i$ if necessary, we may assume that $g_2 = 0$, and we have $t + h_1 g_1 = 0$. Set $\pi_1' := (t, g_1) : \mathcal{O}(-1) \to \mathcal{I}_M(1) \oplus \mathcal{O}_{\mathbb{Q}^3}$. Then $\mathrm{Coker}(i \circ \pi) \cong \mathrm{Coker}(\pi') \oplus \mathcal{O}_{\mathbb{Q}^3}$. Note that $\pi' \neq 0$ since $\pi \neq 0$. Hence $g_1 \neq 0$. Let $H$ be the hyperplane defined by $g_1$. Then we have the following commutative diagram with exact rows:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathcal{O}(-1) & \xrightarrow{g_1} & \mathcal{O}_{\mathbb{Q}^3} & \longrightarrow & \mathcal{O}_H & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle \pi'} & & \downarrow{\scriptstyle (-h_1,1)} & & \downarrow{\scriptstyle -\bar{h}_1} & & \\ 0 & \longrightarrow & \mathcal{I}_M(1) \oplus \mathcal{O}_{\mathbb{Q}^3} & \longrightarrow & \mathcal{O}(1) \oplus \mathcal{O}_{\mathbb{Q}^3} & \longrightarrow & \mathcal{O}_M(1) & \longrightarrow & 0 \end{array} \tag{5.16}$$

We claim here that $\bar{h}_1 \neq 0$. Assume, to the contrary, that $\bar{h}_1 = 0$. Then the snake lemma shows that we have the following exact sequence:

$$0 \to \mathcal{O}_H \to \mathrm{Coker}(\pi') \to \mathcal{O}(1) \to \mathcal{O}_M(1) \to 0.$$

Since $\mathcal{O}_H$ is a torsion sheaf, the surjection $\rho : \mathrm{Coker}(\pi') \oplus \mathcal{O}_{\mathbb{Q}^3} \to \mathcal{O}(1)$ sends $\mathcal{O}_H$ to zero, and thus $\rho$ induces a surjection $\mathcal{I}_M(1) \oplus \mathcal{O}_{\mathbb{Q}^3} \to \mathcal{O}(1)$. On the other hand, the morphism $\mathcal{I}_M(1) \oplus \mathcal{O}_{\mathbb{Q}^3} \to \mathcal{O}(1)$ cannot be surjective since a line $M$ and a hyperplane meets at least at one point. This is a contradiction. Hence $\bar{h}_1 \neq 0$. Then the kernel of the morphism $-\bar{h}_1 : \mathcal{O}_H \to \mathcal{O}_M(1)$ is $\mathcal{O}_H(-M)$ and the cokernel of $-\bar{h}_1$ is $k(p)$ for some point $p \in M$.

Hence the commutative diagram (5.16) induces the following exact sequence by the snake lemma:

$$0 \to \mathcal{O}_H(-M) \to \operatorname{Coker}(\pi') \to \mathcal{O}(1) \to k(p) \to 0.$$

Since $\mathcal{O}_H(-M)$ is a torsion sheaf, the surjection $\rho : \operatorname{Coker}(\pi') \oplus \mathcal{O}_{\mathbb{Q}^3} \to \mathcal{O}(1)$ sends $\mathcal{O}_H(-M)$ to zero, and thus the inclusion $\mathcal{O}_H(-M) \hookrightarrow \operatorname{Coker}(\pi') \oplus \mathcal{O}_{\mathbb{Q}^3}$ induces an inclusion $\mathcal{O}_H(-M) \hookrightarrow \operatorname{Coker}(\pi)$. The exact sequence (5.15) induces the following exact sequence:

$$0 \to \operatorname{Coker}(\pi)/\mathcal{O}_H(-M) \to \mathcal{I}_p(1) \oplus \mathcal{O}_{\mathbb{Q}^3} \to \mathcal{O}(1) \to 0.$$

This shows that $\operatorname{Coker}(\pi)/\mathcal{O}_H(-M) = \mathcal{I}_p$.

Suppose that $a = 2$. As we have seen in the original proof of Lemma 5.3, $\operatorname{Coker}(\psi_2)$ is isomorphic to $\operatorname{Coker}(\varphi_1)$, and $\operatorname{Coker}(\varphi_1)$ is one of the following: $\mathcal{I}_{p,H}$; $\mathcal{O}_H(d,-d)$ where $d = \pm 1$; $\mathcal{I}_p$. If $\operatorname{Coker}(\varphi_1) = \mathcal{I}_{p,H}$, then $\operatorname{Coker}(\pi)$ admits $\mathcal{O}_C(-p)$ as a quotient, where $C$ is a conic on $H$. If $\operatorname{Coker}(\varphi_1) = \mathcal{O}_H(d,-d)$ with $d = \pm 1$, then $\operatorname{Coker}(\pi)$ admits $\mathcal{O}_L(-1)$ as a quotient, where $L$ is a line on $H$. If $\operatorname{Coker}(\varphi_1) = \mathcal{I}_p$, then $\operatorname{Coker}(\pi)$ admits $\mathcal{I}_{p,H}$ as a quotient. Hence the assertion follows if $a = 2$.

Suppose that $a = 1$. Then $\operatorname{Coker}(\psi_1) \cong \mathcal{O}_L(-1)$ by Lemma 5.3. Since $\pi \neq 0$, the morphism $\pi : \mathcal{O}(-1) \to \mathcal{O}_L(-1)$ is surjective, and $\operatorname{Ker}(\pi) \cong \mathcal{I}_L(-1)$. This completes the proof. □

## 6. A lower bound for the third Chern class

Note that

$$c_3 \geq 2c_1c_2 - c_1^3 \tag{6.1}$$

for a nef vector bundle $\mathcal{E}$ on a complete threefold $X$, since $H(\mathcal{E})^{r+2} = c_3 - 2c_1c_2 + c_1^3 \geq 0$ for a nef line bundle $H(\mathcal{E})$. If there exists an injection $\mathcal{L} \to \mathcal{E}$ from a line bundle $\mathcal{L}$, then we have a lower bound, which is better if $\mathcal{L} \cong \mathcal{O}(D)$ for some effective divisor $D$, as the following lemma shows:

**Lemma 6.1.** *Let $\mathcal{E}$ be a nef vector bundle of rank $r$ on a complete variety $X$ of dimension three. Let $\mathcal{L}$ be a line bundle on $X$ such that $H^0(\mathcal{E} \otimes \mathcal{L}^{-1}) \neq 0$. Then we have the following inequality:*

$$c_3 \geq 2c_1c_2 - c_1^3 + (c_1^2 - c_2)c_1(\mathcal{L}).$$

**Proof.** The following short proof is due to the referee. Let $p : \mathbb{P}(\mathcal{E}) \to X$ be the projection. Then $H^0(H(\mathcal{E}) \otimes p^*\mathcal{L}^{-1}) \cong H^0(\mathcal{E} \otimes \mathcal{L}^{-1}) \neq 0$. Hence $H(\mathcal{E})^{r+1}(H(\mathcal{E}) - p^*c_1(\mathcal{L})) \geq 0$. This yields the desired inequality. □

Lemma 6.1 will be applied to $\mathcal{E}$ in § 12.1.

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## 7. Set-up for the proof of Theorem 1.1

Let $\mathcal{E}$ be a nef vector bundle of rank $r$ on $\mathbb{Q}^3$ with $c_1 = 2h$. It follows from [12, Lemma 4.1 (1)] that

$$h^q(\mathcal{E}(t)) = 0 \text{ for } q > 0 \text{ and } t \geq 0. \tag{7.1}$$

Moreover, if $H(\mathcal{E})^{r+2} = c_3 - 2c_1c_2 + c_1^3 = c_3 - 4c_2h + 16 > 0$, then

$$h^q(\mathcal{E}(-1)) = 0 \text{ for } q > 0 \tag{7.2}$$

by [12, Lemma 4.1 (2)]. Note here that

$$c_3 \geq 0 \tag{7.3}$$

by [11, Theorem 8.2.1], since $\mathcal{E}$ is nef. Hence we see that

$$h^q(\mathcal{E}(-1)) = 0 \text{ for } q > 0 \text{ if } c_2h \leq 3. \tag{7.4}$$

It follows from [12, Lemma 4.3] that

$$\operatorname{Ext}^q(\mathcal{S}, \mathcal{E}(2)) = 0 \text{ for } q > 0. \tag{7.5}$$

The exact sequence (3.1) together with the isomorphism (3.2) implies that $\mathcal{S}^\vee \otimes \mathcal{E}(2)$ fits in an exact sequence

$$0 \to \mathcal{S}^\vee \otimes \mathcal{E}(1) \to \mathcal{E}(1)^{\oplus 4} \to \mathcal{S}^\vee \otimes \mathcal{E}(2) \to 0.$$

It then follows from (7.1) and (7.5) that

$$\operatorname{Ext}^q(\mathcal{S}, \mathcal{E}(1)) = 0 \text{ for } q \geq 2. \tag{7.6}$$

If $h^0(\mathcal{E}(-2)) \neq 0$, then $\mathcal{E} \cong \mathcal{O}(2) \oplus \mathcal{O}^{\oplus r-1}$ by [12, Proposition 5.1 and Remark 5.3]. Thus, we will always assume that

$$h^0(\mathcal{E}(-2)) = 0 \tag{7.7}$$

in the following. It follows from Theorem 2.3 that

$$h^q(\mathcal{E}|_{\mathbb{Q}^2}) = 0 \text{ for } q \geq 2. \tag{7.8}$$

Moreover

$$h^1(\mathcal{E}|_{\mathbb{Q}^2}) = \begin{cases} 1 & \text{if } \mathcal{E}|_{\mathbb{Q}^2} \text{ belongs to Case (11) of Theorem 2.3;} \\ 0 & \text{otherwise.} \end{cases} \tag{7.9}$$

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The vanishing (7.1) then shows that

$$h^3(\mathcal{E}(-1)) = 0. \tag{7.10}$$

Moreover

$$h^2(\mathcal{E}(-1)) = 0 \,\text{unless}\, \mathcal{E}|_{\mathbb{Q}^2} \text{ belongs to Case (11) of Theorem 2.3.} \tag{7.11}$$

It follows from Theorem 2.3 that

$$h^q(\mathcal{E}(-1)|_{\mathbb{Q}^2}) = 0 \,\text{for}\, q \geq 2. \tag{7.12}$$

The vanishing (7.10) then shows that

$$h^3(\mathcal{E}(-2)) = 0. \tag{7.13}$$

The exact sequence (3.1) together with (3.2) also induces the following exact sequence

$$0 \to \mathcal{S}^\vee \otimes \mathcal{E}(-1) \to \mathcal{E}(-1)^{\oplus 4} \to \mathcal{S}^\vee \otimes \mathcal{E} \to 0. \tag{7.14}$$

This exact sequence (7.14) and an exact sequence

$$0 \to \mathcal{S}^\vee \otimes \mathcal{E}(-1) \to \mathcal{S}^\vee \otimes \mathcal{E} \to \mathcal{S}^\vee \otimes \mathcal{E}|_{\mathbb{Q}^2} \to 0 \tag{7.15}$$

will be used to compute $\mathrm{Ext}^q(\mathcal{S}, \mathcal{E})$.

## 8. The case where $\mathcal{E}|_{\mathbb{Q}^2}$ belongs to Case (1) of Theorem 2.3

The assumption (7.7) implies that this case does not arise. Indeed, if $\mathcal{E}|_{\mathbb{Q}^2} \cong \mathcal{O}(2,2) \oplus \mathcal{O}^{\oplus r-1}$, then $h^q(\mathcal{E}(-1)|_{\mathbb{Q}^2}) = 0$ for $q > 0$. Moreover $c_2 h = 0$. Hence $h^q(\mathcal{E}(-1)) = 0$ for $q > 0$ by (7.4). This implies that $h^q(\mathcal{E}(-2)) = 0$ for $q \geq 2$. The assumption (7.7) then shows that

$$0 \geq -h^1(\mathcal{E}(-2)) = \chi(\mathcal{E}(-2)) = 1 + \frac{1}{2}c_3$$

by (4.5). This contradicts (7.3). Hence this case does not arise.

## 9. The case where $\mathcal{E}|_{\mathbb{Q}^2}$ belongs to Case (2) of Theorem 2.3

Suppose that

$$\mathcal{E}|_{\mathbb{Q}^2} \cong \mathcal{O}(2,1) \oplus \mathcal{O}(0,1) \oplus \mathcal{O}^{\oplus r-2}.$$

Then $h^0(\mathcal{E}(-1)|_{\mathbb{Q}^2}) = 2$ and $h^q(\mathcal{E}(-1)|_{\mathbb{Q}^2}) = 0$ for $q > 0$. Moreover $c_2 h = 2$. Hence

$$h^q(\mathcal{E}(-1)) = 0 \,\text{for}\, q > 0$$

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by (7.4). It then follows from (4.4) and (7.3) that $h^0(\mathcal{E}(-1)) = \chi(\mathcal{E}(-1)) = 2 + \frac{1}{2}c_3 \geq 2$. On the other hand, we have $h^0(\mathcal{E}(-1)) \leq h^0(\mathcal{E}(-1)|_{\mathbb{Q}^2}) = 2$ by (7.7). Therefore, the restriction map $H^0(\mathcal{E}(-1)) \to H^0(\mathcal{E}(-1)|_{\mathbb{Q}^2})$ is an isomorphism,

$$h^0(\mathcal{E}(-1)) = 2 \,\text{and}\; c_3 = 0.$$

Hence we see that

$$h^q(\mathcal{E}(-2)) = 0 \,\text{for all}\; q.$$

Since $\mathcal{E}(-2)|_{\mathbb{Q}^2} \cong \mathcal{O}(0,-1) \oplus \mathcal{O}(-2,-1) \oplus \mathcal{O}(-2,-2)^{\oplus r-2}$, we have $h^q(\mathcal{E}(-2)|_{\mathbb{Q}^2}) = 0$ for $q < 2$ and $h^2(\mathcal{E}(-2)|_{\mathbb{Q}^2}) = r - 2$. Therefore

$$h^q(\mathcal{E}(-3)) = 0 \,\text{for}\; q < 3 \,\text{and}\; h^3(\mathcal{E}(-3)) = r - 2.$$

Next we will compute $\operatorname{Ext}^q(\mathcal{S}, \mathcal{E}(-1))$. Since

$$\mathcal{S}^\vee \otimes \mathcal{E}(t)|_{\mathbb{Q}^2} \cong (\mathcal{O}(-1,0) \oplus \mathcal{O}(0,-1)) \otimes (\mathcal{O}(2+t, 1+t) \oplus \mathcal{O}(t, 1+t) \oplus \mathcal{O}(t,t)^{\oplus r-2}),$$

we see that $h^q(\mathcal{S}^\vee \otimes \mathcal{E}(t)|_{\mathbb{Q}^2}) = 0$ for $q > 0$ and $t \geq 0$. Hence it follows from (7.6) that

$$\operatorname{Ext}^q(\mathcal{S}, \mathcal{E}(-1)) = 0 \,\text{for}\; q \geq 2.$$

Since $c_2 h = 2$ and $c_3 = 0$, the formula (4.8) shows that

$$h^0(\mathcal{S}^\vee \otimes \mathcal{E}(-1)) = h^1(\mathcal{S}^\vee \otimes \mathcal{E}(-1)).$$

Set $a = h^0(\mathcal{S}^\vee \otimes \mathcal{E}(-1))$. Note that $\mathcal{S}^\vee \otimes \mathcal{E}(-1)$ fits in an exact sequence

$$0 \to \mathcal{S}^\vee \otimes \mathcal{E}(-2) \to \mathcal{E}(-2)^{\oplus 4} \to \mathcal{S}^\vee \otimes \mathcal{E}(-1) \to 0$$

by (3.1) and (3.2). Since $h^q(\mathcal{E}(-2)) = 0$ for all $q$, this exact sequence shows that

$$h^q(\mathcal{S}^\vee \otimes \mathcal{E}(-2)) = \begin{cases} 0 & \text{if } q = 0, 3 \\ a & \text{otherwise.} \end{cases}$$

On the other hand, we have an exact sequence

$$0 \to \mathcal{S}^\vee \otimes \mathcal{E}(-2) \to \mathcal{S}^\vee \otimes \mathcal{E}(-1) \to (\mathcal{S}^\vee \otimes \mathcal{E}(-1))|_{\mathbb{Q}^2} \to 0. \tag{9.1}$$

Since

$$\mathcal{S}^\vee \otimes \mathcal{E}(-1)|_{\mathbb{Q}^2} \cong (\mathcal{O}(-1,0) \oplus \mathcal{O}(0,-1)) \otimes (\mathcal{O}(1,0) \oplus \mathcal{O}(-1,0) \oplus \mathcal{O}(-1,-1)^{\oplus r-2}),$$

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we see that

$$h^q(\mathcal{S}^\vee \otimes \mathcal{E}(-1)|_{\mathbb{Q}^2}) = \begin{cases} 1 & \text{if } q = 0, 1 \\ 0 & \text{if } q = 2, 3. \end{cases}$$

Hence the exact sequence (9.1) implies that $a = 1$.

We apply to $\mathcal{E}(-1)$ the Bondal spectral sequence (2.1). We have $\mathrm{Ext}^3(G, \mathcal{E}(-1)) \cong S_3^{\oplus r-2}$, $\mathrm{Ext}^2(G, \mathcal{E}(-1)) = 0$ and $\mathrm{Ext}^1(G, \mathcal{E}(-1)) \cong S_1$. Moreover, $\mathrm{Hom}(G, \mathcal{E}(-1))$ fits in an exact sequence

$$0 \to S_0^{\oplus 2} \to \mathrm{Hom}(G, \mathcal{E}(-1)) \to S_1 \to 0.$$

Now Lemma 2.1 shows that $E_2^{p,3} = 0$ unless $p = -3$, that $E_2^{-3,3} \cong \mathcal{O}(-1)^{\oplus r-2}$, that $E_2^{p,2} = 0$ for all $p$, that $E_2^{p,1} = 0$ unless $p = -1$, that $E_2^{-1,1} \cong \mathcal{S}(-1)$ and that a distinguished triangle

$$\mathcal{O}^{\oplus 2} \to \mathrm{Hom}(G, \mathcal{E}(-1)) \otimes^{\mathrm{L}}{}_A G \to \mathcal{S}(-1)[1] \to$$

exists. Hence we have the following exact sequence:

$$0 \to E_2^{-1,0} \to \mathcal{S}(-1) \to \mathcal{O}^{\oplus 2} \to E_2^{0,0} \to 0. \tag{9.2}$$

Note here that $E_2^{-1,0} \cong E_\infty^{-1,0} = 0$. Hence we see that $E_2^{0,0}$ is a non-zero torsion sheaf. On the other hand, $\mathcal{E}(-1)$ has $E_2^{0,0}$ as a subsheaf, so that $E_2^{0,0}$ must be torsion-free. This is a contradiction. Therefore, this case does not arise.

## 10. The case where $\mathcal{E}|_{\mathbb{Q}^2}$ belongs to Case (3) of Theorem 2.3

Suppose that $\mathcal{E}|_{\mathbb{Q}^2} \cong \mathcal{O}(1,1)^{\oplus 2} \oplus \mathcal{O}^{\oplus r-2}$. Then $c_2 \cdot h = 2$. Hence $h^q(\mathcal{E}(-1)) = 0$ for $q > 0$ by (7.4). Since $h^q(\mathcal{E}(-1)|_{\mathbb{Q}^2}) = 0$ for $q > 0$, this implies that $h^q(\mathcal{E}(-2)) = 0$ for $q \geq 2$. The assumption (7.7) together with (4.5) and (7.3) shows that

$$0 \geq -h^1(\mathcal{E}(-2)) = \chi(\mathcal{E}(-2)) = \frac{1}{2}c_3 \geq 0.$$

Hence $h^1(\mathcal{E}(-2)) = 0$ and $c_3 = 0$. Thus $h^0(\mathcal{E}(-1)) = h^0(\mathcal{E}(-1)|_{\mathbb{Q}^2}) = 2$. Since $h^q(\mathcal{E}(-2)) = 0$ for any $q$, we see that $h^q(\mathcal{E}(-3)) = h^{q-1}(\mathcal{E}(-2)|_{\mathbb{Q}^2})$ for all $q$. Hence $h^q(\mathcal{E}(-3)) = 0$ unless $q = 3$ and $h^3(\mathcal{E}(-3)) = r - 2$. Since

$$\mathcal{S}^\vee \otimes \mathcal{E}(t)|_{\mathbb{Q}^2} \cong (\mathcal{O}(-1,0) \oplus \mathcal{O}(0,-1)) \otimes (\mathcal{O}(1+t,1+t)^{\oplus 2} \oplus \mathcal{O}(t,t)^{\oplus r-2}),$$

we see that $h^q(\mathcal{S}^\vee \otimes \mathcal{E}(t)|_{\mathbb{Q}^2}) = 0$ for $q > 0$ and $t \geq -1$. Hence it follows from (7.6) that $\mathrm{Ext}^q(\mathcal{S}, \mathcal{E}(-t)) = 0$ for $q \geq 2$ and $t = 0, 1, 2$. Since the exact sequence (3.1) together with (3.2) induces an exact sequence

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$$0 \to \mathcal{S}^\vee \otimes \mathcal{E}(-2) \to \mathcal{E}(-2)^{\oplus 4} \to \mathcal{S}^\vee \otimes \mathcal{E}(-1) \to 0,$$

the vanishing $h^1(\mathcal{E}(-2)) = 0$ implies that $h^1(\mathcal{S}^\vee \otimes \mathcal{E}(-1)) = 0$. Since $h^0(\mathcal{S}^\vee \otimes \mathcal{E}(-1)|_{\mathbb{Q}^2}) = 0$, this implies that $h^1(\mathcal{S}^\vee \otimes \mathcal{E}(-2)) = 0$. Hence $h^0(\mathcal{S}^\vee \otimes \mathcal{E}(-1)) = h^0(\mathcal{S}^\vee \otimes \mathcal{E}(-1)|_{\mathbb{Q}^2}) = 0$. We apply to $\mathcal{E}(-1)$ the Bondal spectral sequence (2.1). We see that $\mathrm{Hom}(G, \mathcal{E}(-1)) \cong S_0^{\oplus 2}$, that $\mathrm{Ext}^q(G, \mathcal{E}(-1)) = 0$ for $q = 1, 2$ and that $\mathrm{Ext}^3(G, \mathcal{E}(-1)) \cong S_3^{\oplus r-2}$. Hence $E_2^{p,q} = 0$ unless $q = 0$ or $q = 3$, $E_2^{p,0} = 0$ unless $p = 0$, $E_2^{0,0} = \mathcal{O}^{\oplus 2}$, $E_2^{p,3} = 0$ unless $p = -3$ and $E_2^{-3,3} = \mathcal{O}(-1)^{\oplus r-2}$ by Lemma 2.1. Therefore, $\mathcal{E}(-1)$ fits in an exact sequence

$$0 \to \mathcal{O}^{\oplus 2} \to \mathcal{E}(-1) \to \mathcal{O}(-1)^{\oplus r-2} \to 0.$$

Hence $\mathcal{E} \cong \mathcal{O}(1)^{\oplus 2} \oplus \mathcal{O}^{\oplus r-2}$. This is Case (2) of Theorem 1.1.

## 11. The case where $\mathcal{E}|_{\mathbb{Q}^2}$ belongs to Case (4) of Theorem 2.3

Suppose that $\mathcal{E}|_{\mathbb{Q}^2}$ fits in an exact sequence

$$0 \to \mathcal{O} \to \mathcal{O}(1,1) \oplus \mathcal{O}(1,0) \oplus \mathcal{O}(0,1) \oplus \mathcal{O}^{\oplus r-2} \to \mathcal{E}|_{\mathbb{Q}^2} \to 0.$$

Then $c_2 h = 3$. Hence $h^q(\mathcal{E}(-1)) = 0$ for $q > 0$ by (7.4). Note that $h^q(\mathcal{E}(-1)|_{\mathbb{Q}^2}) = 0$ for $q > 0$ and that $h^0(\mathcal{E}(-1)|_{\mathbb{Q}^2}) = 1$. Hence $h^q(\mathcal{E}(-2)) = 0$ for $q \geq 2$. The assumption (7.7) together with (4.5) and (7.3) shows that

$$0 \geq -h^1(\mathcal{E}(-2)) = \chi(\mathcal{E}(-2)) = -\frac{1}{2} + \frac{1}{2}c_3 \geq -\frac{1}{2}.$$

Hence $h^1(\mathcal{E}(-2)) = 0$ and $c_3 = 1$. Now that $h^q(\mathcal{E}(-2)) = 0$ for any $q$, we have $h^q(\mathcal{E}(-3)) = h^{q-1}(\mathcal{E}(-2)|_{\mathbb{Q}^2})$ for any $q$. Set $a = h^1(\mathcal{E}(-2)|_{\mathbb{Q}^2})$. Then $a = 0$ or $1$, and $h^2(\mathcal{E}(-2)|_{\mathbb{Q}^2}) = r - 3 + a$. Hence we see that $h^q(\mathcal{E}(-3)) = 0$ for $q \leq 1$, that $h^2(\mathcal{E}(-3)) = a$ and that $h^3(\mathcal{E}(-3)) = r - 3 + a$. Moreover, the assumption (7.7) implies that $h^0(\mathcal{E}(-1)) = h^0(\mathcal{E}(-1)|_{\mathbb{Q}^2}) = 1$. Since $\mathcal{E}|_{\mathbb{Q}^2}(-2,-1)$ fits in an exact sequence

$$\begin{aligned} 0 \to \mathcal{O}(-2,-1) \to \mathcal{O}(-1,0) \oplus \mathcal{O}(-1,-1) \oplus \mathcal{O}(-2,0) \oplus \quad & \mathcal{O}(-2,-1)^{\oplus r-2} \\ & \to \mathcal{E}|_{\mathbb{Q}^2}(-2,-1) \to 0, \end{aligned}$$

we see that $h^q(\mathcal{E}|_{\mathbb{Q}^2}(-2,-1)) = 0$ unless $q = 1$. Hence $h^q(\mathcal{S}^\vee \otimes \mathcal{E}(-1)|_{\mathbb{Q}^2}) = 0$ unless $q = 1$. Note that $h^q(\mathcal{S}^\vee \otimes \mathcal{E}(t)|_{\mathbb{Q}^2}) = 0$ for $t \geq 0$ and $q \geq 1$. Hence it follows from (7.6) that $\mathrm{Ext}^q(\mathcal{S}, \mathcal{E}(-t)) = 0$ for $q \geq 2$ and $t = 0, 1$. Note that $\mathcal{S}^\vee \otimes \mathcal{E}(-2)$ is a subbundle of $\mathcal{E}(-2)^{\oplus 4}$ by (3.1). Since $h^0(\mathcal{E}(-2)) = 0$, this implies that $h^0(\mathcal{S}^\vee \otimes \mathcal{E}(-2)) = 0$. Since we have an exact sequence

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$$0 \to \mathcal{S}^\vee \otimes \mathcal{E}(-2) \to \mathcal{S}^\vee \otimes \mathcal{E}(-1) \to \mathcal{S}^\vee \otimes \mathcal{E}(-1)|_{\mathbb{Q}^2} \to 0$$

and $h^0(\mathcal{S}^\vee \otimes \mathcal{E}(-1)|_{\mathbb{Q}^2}) = 0$, we infer that $h^0(\mathcal{S}^\vee \otimes \mathcal{E}(-1)) = 0$. Now, from (4.8), it follows that

$$-h^1(\mathcal{S}^\vee \otimes \mathcal{E}(-1)) = \chi(\mathcal{S}^\vee \otimes \mathcal{E}(-1)) = 4 - 2 \cdot 3 + 1 = -1.$$

We apply to $\mathcal{E}(-1)$ the Bondal spectral sequence (2.1). We have the following isomorphisms: $\mathrm{Ext}^3(G, \mathcal{E}(-1)) \cong S_3^{\oplus r-3+a}$; $\mathrm{Ext}^2(G, \mathcal{E}(-1)) \cong S_3^{\oplus a}$; $\mathrm{Ext}^1(G, \mathcal{E}(-1)) \cong S_1$; $\mathrm{Hom}(G, \mathcal{E}(-1)) \cong S_0$. Lemma 2.1 then shows that $E_2^{p,q} = 0$ unless $(p,q) = (-3,3)$, $(-3,2)$, $(-1,1)$ or $(0,0)$, that $E_2^{-3,3} = \mathcal{O}(-1)^{\oplus r-3+a}$, that $E_2^{-3,2} = \mathcal{O}(-1)^{\oplus a}$, that $E_2^{-1,1} = \mathcal{S}(-1)$ and that $E_2^{0,0} = \mathcal{O}$. Hence $E_3^{-3,2} = 0$ and $E_3^{-1,1}$ fits in the following exact sequence:

$$0 \to \mathcal{O}(-1)^{\oplus a} \to \mathcal{S}(-1) \to E_3^{-1,1} \to 0.$$

Moreover $\mathcal{E}(-1)$ has a filtration $\mathcal{O} \subset F(\mathcal{E}(-1)) \subset \mathcal{E}(-1)$ such that $F(\mathcal{E}(-1))$ fits in the following exact sequences:

$$0 \to F(\mathcal{E}(-1)) \to \mathcal{E}(-1) \to \mathcal{O}(-1)^{\oplus r-3} \to 0;$$

$$0 \to \mathcal{O} \to F(\mathcal{E}(-1)) \to E_3^{-1,1} \to 0.$$

In particular, we see that $F(\mathcal{E}(-1))$ is a vector bundle, since so is $\mathcal{E}(-1)$. On the other hand, since $\mathrm{Ext}^1(\mathcal{S}(-1), \mathcal{O}) = 0$, $F(\mathcal{E}(-1))$ fits in the following exact sequence:

$$0 \to \mathcal{O}(-1)^{\oplus a} \to \mathcal{O} \oplus \mathcal{S}(-1) \to F(\mathcal{E}(-1)) \to 0.$$

This implies that $a = 0$. Indeed, if $a = 1$, then $F(\mathcal{E}(-1))$ cannot be a vector bundle, since the intersection of a line and a hyperplane section cannot be empty. Therefore $F(\mathcal{E}(-1)) \cong \mathcal{O} \oplus \mathcal{S}(-1)$, and thus $\mathcal{E} \cong \mathcal{O}(1) \oplus \mathcal{S} \oplus \mathcal{O}^{\oplus r-3}$. This is Case (3) of Theorem 1.1.

## 12. The case where $\mathcal{E}|_{\mathbb{Q}^2}$ belongs to Case (5) of Theorem 2.3

Suppose that $\mathcal{E}|_{\mathbb{Q}^2}$ fits in an exact sequence

$$0 \to \mathcal{O}(-1,-1) \to \mathcal{O}(1,1) \oplus \mathcal{O}^{\oplus r} \to \mathcal{E}|_{\mathbb{Q}^2} \to 0.$$

Then $c_2 h = 4$. Note that

$$h^q(\mathcal{S}^\vee \otimes \mathcal{E}|_{\mathbb{Q}^2}) = \begin{cases} 4 & \text{if} \quad q = 0 \\ 0 & \text{if} \quad q \neq 0, \end{cases} \tag{12.1}$$

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and that

$$h^q(\mathcal{E}(-1)|_{\mathbb{Q}^2}) = \begin{cases} 1 & \text{if} \quad q = 0, 1 \\ 0 & \text{if} \quad q \neq 0, 1. \end{cases} \tag{12.2}$$

Hence we have

$$h^0(\mathcal{E}(-1)) \leq 1$$

by (7.7).

### 12.1. Suppose that $h^0(\mathcal{E}(-1)) = 1$.

Lemma 6.1 then shows that $c_3 \geq 4$. Hence $H^q(\mathcal{E}(-1))$ vanishes for $q > 0$ by (7.2). The formula (4.4) then shows that

$$h^0(\mathcal{E}(-1)) = -1 + \frac{1}{2}c_3.$$

Thus we have $c_3 = 4$. We also see that $h^q(\mathcal{E}(-2)) = 0$ unless $q = 2$ and that $h^2(\mathcal{E}(-2)) = 1$ by (12.2) and (7.7). We have $h^0(\mathcal{E}) = r + 5$. Since we have an exact sequence

$$0 \to \mathcal{S}^\vee \otimes \mathcal{E}(-2) \to \mathcal{E}(-2)^{\oplus 4} \to \mathcal{S}^\vee \otimes \mathcal{E}(-1) \to 0,$$

we see that $h^0(\mathcal{S}^\vee \otimes \mathcal{E}(-2)) = 0$ and that $h^3(\mathcal{S}^\vee \otimes \mathcal{E}(-1)) = 0$. Note that $h^0(\mathcal{S}^\vee \otimes \mathcal{E}(-1)|_{\mathbb{Q}^2}) = 0$. Since we have an exact sequence

$$0 \to \mathcal{S}^\vee \otimes \mathcal{E}(-2) \to \mathcal{S}^\vee \otimes \mathcal{E}(-1) \to \mathcal{S}^\vee \otimes \mathcal{E}(-1)|_{\mathbb{Q}^2} \to 0,$$

we infer that $h^0(\mathcal{S}^\vee \otimes \mathcal{E}(-1)) = 0$. Since we have an exact sequence (7.14), we see that $h^q(\mathcal{S}^\vee \otimes \mathcal{E}) = 0$ for $q \geq 2$. The exact sequence (7.15) together with (12.1) shows that $h^2(\mathcal{S}^\vee \otimes \mathcal{E}(-1)) = 0$. Now the formula (4.8) shows that

$$-h^1(\mathcal{S}^\vee \otimes \mathcal{E}(-1)) = \chi(\mathcal{S}^\vee \otimes \mathcal{E}(-1)) = 0,$$

since $c_3 = 4$ and $c_2h = 4$. The exact sequence (7.14) then implies that $h^q(\mathcal{S}^\vee \otimes \mathcal{E}) = 0$ unless $q = 0$ and that $h^0(\mathcal{S}^\vee \otimes \mathcal{E}) = 4$. Since $h^0(\mathcal{E}(-1)) = 1$, we have an injection $\mathcal{O}(1) \to \mathcal{E}$. Let $\mathcal{F}$ be its cokernel: we have the following exact sequence:

$$0 \to \mathcal{O}(1) \to \mathcal{E} \to \mathcal{F} \to 0.$$

We apply to $\mathcal{F}$ the Bondal spectral sequence (2.1). We see that $h^q(\mathcal{F}) = 0$ unless $q = 0$ and that $h^0(\mathcal{F}) = r$. Moreover $h^q(\mathcal{F}(-1)) = 0$ for any $q$, $h^q(\mathcal{F}(-2)) = 0$ unless $q = 2$ and $h^2(\mathcal{F}(-2)) = 1$. Finally, we have $h^q(\mathcal{S}^\vee \otimes \mathcal{F}) = 0$ for all $q$. Therefore $\operatorname{Ext}^q(G, \mathcal{F}) = 0$ for $q = 3$ and 1, $\operatorname{Ext}^2(G, \mathcal{F}) \cong S_3$ and $\operatorname{Hom}(G, \mathcal{F}) \cong S_0^{\oplus r}$. Hence $E_2^{p,q} = 0$ unless $(p \cdot q) =$

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$(-3,2)$ or $(0,0)$, $E_2^{-3,2} = \mathcal{O}(-1)$ and $E_2^{0,0} = \mathcal{O}^{\oplus r}$ by Lemma 2.1. Thus, we have an exact sequence

$$0 \to \mathcal{O}(-1) \to \mathcal{O}^{\oplus r} \to \mathcal{F} \to 0.$$

Therefore $\mathcal{E}$ belongs to Case (4) of Theorem 1.1.

### 12.2. Suppose that $h^0(\mathcal{E}(-1)) = 0$.

Then $h^0(\mathcal{S}^\vee \otimes \mathcal{E}(-1)) = 0$ by (7.14). Note that $H^q(\mathcal{E}|_{\mathbb{Q}^2})$ vanishes for all $q > 0$. Since $h^q(\mathcal{E}) = 0$ for all $q > 0$ by (7.1), we have $h^q(\mathcal{E}(-1)) = 0$ for all $q \geq 2$. Hence (4.4) and (7.3) imply that

$$0 \geq -h^1(\mathcal{E}(-1)) = \chi(\mathcal{E}(-1)) = -1 + \frac{1}{2}c_3 \geq -1.$$

Therefore, $(h^1(\mathcal{E}(-1)), c_3)$ is either $(0,2)$ or $(1,0)$. Since $h^3(\mathcal{E}(-1)) = 0$, we first have $h^3(\mathcal{S}^\vee \otimes \mathcal{E}) = 0$ by (7.14). Secondly, we have $h^3(\mathcal{S}^\vee \otimes \mathcal{E}(-1)) = 0$ by (12.1) and (7.15). Thirdly, we have $h^2(\mathcal{S}^\vee \otimes \mathcal{E}) = 0$ by (7.14) since $h^2(\mathcal{E}(-1)) = 0$. Finally, we have $h^2(\mathcal{S}^\vee \otimes \mathcal{E}(-1)) = 0$ by (12.1) and (7.15). Hence

$$-h^1(\mathcal{S}^\vee \otimes \mathcal{E}(-1)) = \chi(\mathcal{S}^\vee \otimes \mathcal{E}(-1)) = -4 + c_3 \tag{12.3}$$

by (4.8). We apply to $\mathcal{E}$ the Bondal spectral sequence (2.1).

*12.2.1. Suppose that* $(h^1(\mathcal{E}(-1)), c_3) = (0,2)$.

Then $h^1(\mathcal{S}^\vee \otimes \mathcal{E}) = 0$ by (7.14). Moreover $h^1(\mathcal{S}^\vee \otimes \mathcal{E}(-1)) = 2$ by (12.3). Hence we have $h^0(\mathcal{S}^\vee \otimes \mathcal{E}) = 2$ by (7.14). Since $h^q(\mathcal{E}(-1)|_{\mathbb{Q}^2}) = 1$ for $q = 0$, 1 and $h^q(\mathcal{E}(-1)|_{\mathbb{Q}^2}) = 0$ for $q = 2$, 3, we infer that $h^q(\mathcal{E}(-2)) = 1$ for $q = 1$, 2, and that $h^q(\mathcal{E}(-2)) = 0$ unless $q = 1$ or 2. Since $h^0(\mathcal{E}|_{\mathbb{Q}^2}) = r+4$, we see that $h^0(\mathcal{E}) = r+4$. Therefore, we have an exact sequence

$$0 \to S_0^{\oplus r+4} \to \mathrm{Hom}(G, \mathcal{E}) \to S_1^{\oplus 2} \to 0$$

and the following: $\mathrm{Ext}^1(G, \mathcal{E}) \cong S_3$; $\mathrm{Ext}^2(G, \mathcal{E}) \cong S_3$ and $\mathrm{Ext}^3(G, \mathcal{E}) = 0$. Therefore, Lemma 2.1 implies that $E_2^{p,q} = 0$ unless $(p,q) = (-3,1)$, $(-3,2)$, $(-1,0)$ or $(0,0)$, that $E_2^{-3,1} \cong \mathcal{O}(-1)$, that $E_2^{-3,2} \cong \mathcal{O}(-1)$ and that there is an exact sequence

$$0 \to E_2^{-1,0} \to \mathcal{S}(-1)^{\oplus 2} \to \mathcal{O}^{\oplus r+4} \to E_2^{0,0} \to 0.$$

It follows from the Bondal spectral sequence (2.1) that $E_2^{-3,1} \cong E_2^{-1,0}$, that $E_2^{-3,2} \cong E_3^{-3,2}$, that $E_2^{0,0} \cong E_3^{0,0}$ and that there is an exact sequence

$$0 \to E_3^{-3,2} \to E_3^{0,0} \to \mathcal{E} \to 0.$$

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Hence we obtain the following exact sequences:

$$0 \to \mathcal{O}(-1) \to \mathcal{S}(-1)^{\oplus 2} \to \mathcal{O}^{\oplus r+4} \to E_3^{0,0} \to 0;$$

$$0 \to \mathcal{O}(-1) \to E_3^{0,0} \to \mathcal{E} \to 0.$$

The latter exact sequence shows that $E_3^{0,0}$ is a vector bundle since so is $\mathcal{E}$. The former exact sequence then splits into the following two exact sequences with $\mathcal{G}$ a vector bundle of rank three:

$$0 \to \mathcal{O}(-1) \to \mathcal{S}(-1)^{\oplus 2} \to \mathcal{G} \to 0;$$

$$0 \to \mathcal{G} \to \mathcal{O}^{\oplus r+4} \to E_3^{0,0} \to 0.$$

The latter exact sequence shows that the dual $\mathcal{G}^\vee$ of $\mathcal{G}$ is globally generated. The injection $\mathcal{O}(-1) \to \mathcal{S}(-1)^{\oplus 2}$ in the former exact sequence gives rise to two global sections $s_0$, $s_1$ of $\mathcal{S}$, and we infer that $(s_0)_0 \cap (s_1)_0 = \emptyset$ since $\mathcal{G}$ is a vector bundle. Hence $s_0$ and $s_1$ are linearly independent. We also see that $\mathcal{G}^\vee$ fits in the following exact sequence:

$$0 \to \mathcal{G}^\vee \to \mathcal{S}^{\oplus 2} \to \mathcal{O}(1) \to 0.$$

Note that the induced map $H^0(\mathcal{S})^{\oplus 2} \to H^0(\mathcal{O}(1))$ sends $(t_0, t_1)$ to $s_0 \wedge t_0 + s_1 \wedge t_1$, and Lemma 3.1 implies that it is surjective. Therefore $h^0(\mathcal{G}^\vee) = 3$. Since $\mathcal{G}^\vee$ is a globally generated vector bundle of rank three, this implies that $\mathcal{G}^\vee \cong \mathcal{O}^{\oplus 3}$. On the other hand, the exact sequence above shows that $c_1(\mathcal{G}^\vee) = 1$. This is a contradiction. Hence the case $(h^1(\mathcal{E}(-1)), c_3) = (0, 2)$ does not arise.

*12.2.2. Suppose that* $(h^1(\mathcal{E}(-1)), c_3) = (1, 0)$.

Then $h^1(\mathcal{S}^\vee \otimes \mathcal{E}(-1)) = 4$ by (12.3). Set $a := h^0(\mathcal{S}^\vee \otimes \mathcal{E})$. Then $h^1(\mathcal{S}^\vee \otimes \mathcal{E}) = a$ by (7.14). From (12.2), it follows that $h^q(\mathcal{E}(-2)) = 0$ unless $q = 1$ or 2 and that $(h^1(\mathcal{E}(-2)), h^2(\mathcal{E}(-2))) = (1, 0)$ or $(2, 1)$. Note also that $h^0(\mathcal{E}) = r + 3$.

12.2.2.1. Suppose that $(h^1(\mathcal{E}(-2)), h^2(\mathcal{E}(-2))) = (1, 0)$. Then we see that $\mathrm{Ext}^3(G, \mathcal{E}) = 0$, that $\mathrm{Ext}^2(G, \mathcal{E}) = 0$, that $\mathrm{Ext}^1(G, \mathcal{E})$ has a filtration $S_1^{\oplus a} \subset F \subset \mathrm{Ext}^1(G, \mathcal{E})$ of right $A$-modules such that the following sequences are exact:

$$0 \to F \to \mathrm{Ext}^1(G, \mathcal{E}) \to S_3 \to 0;$$

$$0 \to S_1^{\oplus a} \to F \to S_2 \to 0,$$

and that $\mathrm{Hom}(G, \mathcal{E})$ fits in the following exact sequence of right $A$-modules:

$$0 \to S_0^{\oplus r+3} \to \mathrm{Hom}(G, \mathcal{E}) \to S_1^{\oplus a} \to 0.$$

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These exact sequences induce the following distinguished triangles by Lemma 2.1:

$$F \otimes^{\mathrm{L}}{}_A G \to \mathrm{Ext}^1(G, \mathcal{E}) \otimes^{\mathrm{L}}{}_A G \to \mathcal{O}(-1)[3] \to;$$

$$\mathcal{S}(-1)[1]^{\oplus a} \to F \otimes^{\mathrm{L}}{}_A G \to T_{\mathbb{P}^4}(-2)|_{\mathbb{Q}^3}[2] \to;$$

$$\mathcal{O}^{\oplus r+3} \to \mathrm{Hom}(G, \mathcal{E}) \otimes^{\mathrm{L}}{}_A G \to \mathcal{S}(-1)[1]^{\oplus a} \to .$$

By taking cohomologies, we obtain the following exact sequences by (3.2):

$$0 \to E_2^{-3,1} \to \mathcal{O}(-1) \to \mathcal{H}^{-2}(F \otimes^{\mathrm{L}}{}_A G) \to E_2^{-2,1} \to 0;$$

$$0 \to \mathcal{H}^{-2}(F \otimes^{\mathrm{L}}{}_A G) \to T_{\mathbb{P}^4}(-2)|_{\mathbb{Q}^3} \xrightarrow{\psi_a} \mathcal{S}^{\vee \oplus a} \to E_2^{-1,1} \to 0; \tag{12.4}$$

$$0 \to E_2^{-1,0} \to \mathcal{S}^{\vee \oplus a} \to \mathcal{O}^{\oplus r+3} \to E_2^{0,0} \to 0.$$

Moreover, we have the following exact sequences:

$$0 \to E_2^{-2,1} \to E_2^{0,0} \to E_3^{0,0} \to 0;$$

$$0 \to E_2^{-3,1} \to E_2^{-1,0} \to 0;$$

$$0 \to E_3^{0,0} \to \mathcal{E} \to E_2^{-1,1} \to 0.$$

Since $\mathcal{E}$ is nef, $E_2^{-1,1}$ cannot admit negative degree quotients. Hence it follows from Lemma 5.3 that $a=0$. Then $E_2^{-1,1}=0$, $E_2^{-3,1}=E_2^{-1,0}=0$, $E_2^{0,0}=\mathcal{O}^{\oplus r+3}$, and we have the following exact sequence:

$$0 \to \mathcal{O}(-1) \to T_{\mathbb{P}^4}(-2)|_{\mathbb{Q}^3} \to E_2^{-2,1} \to 0.$$

Hence $\mathcal{E}$ fits in the following exact sequence:

$$0 \to \mathcal{O}(-1) \to T_{\mathbb{P}^4}(-2)|_{\mathbb{Q}^3} \to \mathcal{O}^{\oplus r+3} \to \mathcal{E} \to 0. \tag{12.5}$$

Since $T_{\mathbb{P}^4}(-2)|_{\mathbb{Q}^3}$ fits in an exact sequence

$$0 \to \mathcal{O}(-2) \to \mathcal{O}(-1)^{\oplus 5} \to T_{\mathbb{P}^4}(-2)|_{\mathbb{Q}^3} \to 0,$$

the exact sequence (12.5) induces the following exact sequence:

$$0 \to \mathcal{O}(-2) \to \mathcal{O}(-1)^{\oplus 4} \to \mathcal{O}^{\oplus r+3} \to \mathcal{E} \to 0.$$

This is Case (9) of Theorem 1.1.

12.2.2.2. Suppose that $(h^1(\mathcal{E}(-2)), h^2(\mathcal{E}(-2))) = (2,1)$. Then we see that $\mathrm{Ext}^3(G,\mathcal{E}) = 0$, that $\mathrm{Ext}^2(G,\mathcal{E}) \cong S_3$, that $\mathrm{Ext}^1(G,\mathcal{E})$ has a filtration $S_1^{\oplus a} \subset F \subset \mathrm{Ext}^1(G,\mathcal{E})$ of right $A$-modules such that the following sequences are exact:

$$0 \to F \to \mathrm{Ext}^1(G,\mathcal{E}) \to S_3^{\oplus 2} \to 0;$$

$$0 \to S_1^{\oplus a} \to F \to S_2 \to 0,$$

and that $\mathrm{Hom}(G,\mathcal{E})$ fits in the following exact sequence of right $A$-modules:

$$0 \to S_0^{\oplus r+3} \to \mathrm{Hom}(G,\mathcal{E}) \to S_1^{\oplus a} \to 0.$$

Lemma 2.1 implies that $\mathrm{Ext}^2(G,\mathcal{E}) \otimes^{\mathrm{L}}{}_A G \cong \mathcal{O}(-1)[3]$ and that the three exact sequences above induce the following distinguished triangles:

$$F \otimes^{\mathrm{L}}{}_A G \to \mathrm{Ext}^1(G,\mathcal{E}) \otimes^{\mathrm{L}}{}_A G \to \mathcal{O}(-1)^{\oplus 2}[3] \to;$$

$$\mathcal{S}(-1)[1]^{\oplus a} \to F \otimes^{\mathrm{L}}{}_A G \to T_{\mathbb{P}^4}(-2)|_{\mathbb{Q}^3}[2] \to;$$

$$\mathcal{O}^{\oplus r+3} \to \mathrm{Hom}(G,\mathcal{E}) \otimes^{\mathrm{L}}{}_A G \to \mathcal{S}(-1)[1]^{\oplus a} \to .$$

By taking cohomologies, we see that $E_2^{p,2} = 0$ unless $p = -3$, that $E_2^{-3,2} \cong \mathcal{O}(-1)$, and that we have the following exact sequences by (3.2):

$$0 \to E_2^{-3,1} \to \mathcal{O}(-1)^{\oplus 2} \to \mathcal{H}^{-2}(F \otimes^{\mathrm{L}}{}_A G) \to E_2^{-2,1} \to 0; \tag{12.6}$$

$$0 \to \mathcal{H}^{-2}(F \otimes^{\mathrm{L}}{}_A G) \to T_{\mathbb{P}^4}(-2)|_{\mathbb{Q}^3} \xrightarrow{\psi_a} \mathcal{S}^{\vee \oplus a} \to E_2^{-1,1} \to 0; \tag{12.7}$$

$$0 \to E_2^{-1,0} \to \mathcal{S}^{\vee \oplus a} \to \mathcal{O}^{\oplus r+3} \to E_2^{0,0} \to 0.$$

Moreover, we have the following exact sequences:

$$0 \to E_3^{-3,2} \to E_2^{-3,2} \xrightarrow{\pi} E_2^{-1,1} \to E_3^{-1,1} \to 0; \tag{12.8}$$

$$0 \to E_2^{-2,1} \to E_2^{0,0} \to E_3^{0,0} \to 0;$$

$$0 \to E_2^{-3,1} \to E_2^{-1,0} \to 0;$$

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$$0 \to E_3^{-3,2} \to E_3^{0,0} \to E_4^{0,0} \to 0;$$

$$0 \to E_4^{0,0} \to \mathcal{E} \to E_3^{-1,1} \to 0.$$

Since $\mathcal{E}$ is nef, $E_3^{-1,1}$ cannot admit negative degree quotients. If $a > 0$, it follows from Lemmas 5.4 and 5.3 that $a = 1$, that $E_3^{-1,1} = 0$, that $E_3^{-3,2} \cong \mathcal{I}_L(-1)$ for some line $L \subset \mathbb{Q}^3$, that $E_2^{-1,1} \cong \mathcal{O}_L(-1)$ and that $\mathcal{H}^{-2}(F \otimes^{\mathrm{L}}{}_A G) \cong \mathcal{O}(-1)^{\oplus 2}$. Therefore, $\mathcal{E} \cong E_4^{0,0}$ and the exact sequence (12.6) becomes the following exact sequence:

$$0 \to E_2^{-3,1} \to \mathcal{O}(-1)^{\oplus 2} \to \mathcal{O}(-1)^{\oplus 2} \to E_2^{-2,1} \to 0.$$

Set $\mathcal{O}(-1)^{\oplus b} \cong E_2^{-3,1}$ for some non-negative integer $b \leq 2$. Then $E_2^{-2,1} \cong \mathcal{O}(-1)^{\oplus b}$ and we have the following exact sequences:

$$0 \to \mathcal{O}(-1)^{\oplus b} \to \mathcal{S}^\vee \to \mathcal{O}^{\oplus r+3} \to E_2^{0,0} \to 0;$$

$$0 \to \mathcal{O}(-1)^{\oplus b} \to E_2^{0,0} \to E_3^{0,0} \to 0;$$

$$0 \to \mathcal{I}_L(-1) \to E_3^{0,0} \to \mathcal{E} \to 0.$$

Since $\mathcal{O}^{\oplus r+3}$ is torsion-free and $\mathcal{S}^\vee$ is not isomorphic to $\mathcal{O}^{\oplus 2}$, we see that $b \leq 1$. Note here that $E_3^{0,0}$ is torsion-free, and so is $E_2^{0,0}$. If $b = 1$, we get an exact sequence

$$0 \to \mathcal{I}_M \to \mathcal{O}^{\oplus r+3} \to E_2^{0,0} \to 0$$

for some line $M$ in $\mathbb{Q}^3$. Since we can extend $\mathcal{I}_M \to \mathcal{O}^{\oplus r+3}$ to an injection $\mathcal{O} \to \mathcal{O}^{\oplus r+3}$ by taking double duals, we infer that $E_2^{0,0}$ contains a torsion sheaf $\mathcal{O}_M$. This is a contradiction. Hence $b = 0$, and $E_2^{0,0}$ fits in the following exact sequences:

$$0 \to \mathcal{S}^\vee \to \mathcal{O}^{\oplus r+3} \to E_2^{0,0} \to 0;$$

$$0 \to \mathcal{I}_L(-1) \to E_2^{0,0} \to \mathcal{E} \to 0.$$

Since $\mathcal{I}_L(-1)$ is torsion-free but not locally free, so is $E_2^{0,0}$. Hence the former exact sequence together with (3.1) implies that $E_2^{0,0} \cong \mathcal{I}_M(1) \oplus \mathcal{O}^{\oplus r}$ for some line $M$ in $\mathbb{Q}^3$. This can be shown by the similar argument as in the proof of Lemma 5.2. Indeed, by taking a free basis of $\mathcal{O}^{\oplus r+3}$ suitably, we may assume that the injection $\mathcal{S}^\vee \to \mathcal{O}^{\oplus r+3}$ is written as ${}^t(s_1^\vee, \ldots, s_m^\vee, 0, \ldots, 0)$ for some linearly independent elements $s_1, \ldots, s_m$ of $H^0(\mathcal{S})$, where $s_i^\vee$ denotes the dual of the morphism $\mathcal{O} \to \mathcal{S}$ defined by $s_i$. We have $2 = \operatorname{rank} \mathcal{S}^\vee \leq m \leq h^0(\mathcal{S}) = 4$. Since $E_2^{0,0}$ is torsion-free, we have $3 \leq m$. Since $E_2^{0,0}$ is not

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locally free, it follows from the exact sequence (3.1) that $m \neq 4$. Hence $m = 3$. Moreover, the exact sequence (3.1) shows that if we extend $(s_1, s_2, s_3)$ to a basis $(s_1, s_2, s_3, s_4)$ of $H^0(\mathcal{S})$ then there exists a basis $(t_1, t_2, t_3, t_4)$ of $H^0(\mathcal{S})$ such that $\sum_{i=1}^4 t_i s_i^\vee = 0$ and that the cokernel of the morphism ${}^t(s_1^\vee, s_2^\vee, s_3^\vee)$ is isomorphic to the cokernel of the morphism $t_4 : \mathcal{O} \to \mathcal{S}$. Hence the cokernel of ${}^t(s_1^\vee, s_2^\vee, s_3^\vee)$ is isomorphic to $\mathcal{I}_M(1)$ for some line $M$ on $\mathbb{Q}^3$. Therefore $E_2^{0,0} \cong \mathcal{I}_M(1) \oplus \mathcal{O}^{\oplus r}$. By taking the double dual of the injection $\mathcal{I}_L(-1) \to E_2^{0,0}$ in the latter exact sequence, we obtain a commutative diagram with exact rows

$$\begin{array}{ccccccccc} 0 & \to & \mathcal{I}_L(-1) & \to & E_2^{0,0} & \to & \mathcal{E} & \to & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \to & \mathcal{O}(-1) & \to & \mathcal{O}(1) \oplus \mathcal{O}^{\oplus r} & \to & \mathcal{F} & \to & 0 \end{array}$$

for some coherent sheaf $\mathcal{F}$. Note that $\mathrm{Tor}_q^{\mathcal{O}_p}(\mathcal{F}_p, k(p)) = 0$ for $q \geq 2$ and any point $p$. Since $\mathcal{E}$ is torsion-free, the snake lemma implies that $L = M$ and that we have an exact sequence

$$0 \to \mathcal{O}_L(-1) \to \mathcal{O}_M(1) \to \mathcal{O}_Z \to 0$$

for some closed subscheme $Z$ of length two. Moreover, $\mathcal{E}$ fits in the following exact sequence:

$$0 \to \mathcal{E} \to \mathcal{F} \to \mathcal{O}_Z \to 0.$$

For an associated point $p$ of $Z$, the exact sequence above induces a coherent sheaf $\mathcal{G}$ and the following exact sequence:

$$0 \to \mathcal{E} \to \mathcal{G} \to k(p) \to 0.$$

Since $\mathrm{Tor}_3^{\mathcal{O}_p}(\mathcal{F}_p, k(p)) = 0$, we have $\mathrm{Tor}_3^{\mathcal{O}_p}(\mathcal{G}_p, k(p)) = 0$. Note that $\mathrm{Tor}_q^{\mathcal{O}_p}(\mathcal{E}_p, k(p)) = 0$ for $q \geq 1$. Hence $\mathrm{Tor}_3^{\mathcal{O}_p}(k(p), k(p)) = 0$, which contradicts the fact that $\mathrm{Tor}_3^{\mathcal{O}_p}(k(p), k(p)) = 1$. Therefore, $a$ cannot be positive: $a = 0$. Thus $0 = E_2^{-1,1} = E_3^{-1,1}$, $0 = E_2^{-1,0} = E_2^{-3,1}$, $\mathcal{O}^{\oplus r+3} \cong E_2^{0,0}$, $E_3^{-3,2} \cong E_2^{-3,2} \cong \mathcal{O}(-1)$, $E_4^{0,0} \cong \mathcal{E}$, and we have the following exact sequences:

$$0 \to \mathcal{O}(-1)^{\oplus 2} \to T_{\mathbb{P}^4}(-2)|_{\mathbb{Q}^3} \to E_2^{-2,1} \to 0;$$

$$0 \to E_2^{-2,1} \to \mathcal{O}^{\oplus r+3} \to E_3^{0,0} \to 0;$$

$$0 \to \mathcal{O}(-1) \to E_3^{0,0} \to \mathcal{E} \to 0.$$

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Since $T_{\mathbb{P}^4}(-2)|_{\mathbb{Q}^3}$ fits in an exact sequence

$$0 \to \mathcal{O}(-2) \to \mathcal{O}(-1)^{\oplus 5} \to T_{\mathbb{P}^4}(-2)|_{\mathbb{Q}^3} \to 0,$$

$E_2^{-2,1}$ has a resolution of the following form:

$$0 \to \mathcal{O}(-2) \to \mathcal{O}(-1)^{\oplus 3} \to E_2^{-2,1} \to 0.$$

Therefore, we see that $\mathcal{E}$ belongs to Case (9) of Theorem 1.1.

## 13. The case where $\mathcal{E}|_{\mathbb{Q}^2}$ belongs to Case (6) of Theorem 2.3

Suppose that $\mathcal{E}|_{\mathbb{Q}^2}$ fits in the following exact sequence:

$$0 \to \mathcal{O}^{\oplus 2} \to \mathcal{O}(1,0)^{\oplus 2} \oplus \mathcal{O}(0,1)^{\oplus 2} \oplus \mathcal{O}^{\oplus r-2} \to \mathcal{E}|_{\mathbb{Q}^2} \to 0.$$

Then $h^q(\mathcal{E}(-1)|_{\mathbb{Q}^2}) = 0$ for any $q$, and $c_2h = 4$. Since $h^q(\mathcal{E}|_{\mathbb{Q}^2}) = 0$ for any $q > 0$, the vanishing (7.1) shows that $h^q(\mathcal{E}(-t)) = 0$ for $q \geq 2$ and $t = 1, 2$. The assumption (7.7) together with (4.5) and (7.3) shows that

$$0 \geq -h^1(\mathcal{E}(-2)) = \chi(\mathcal{E}(-2)) = -1 + \frac{1}{2}c_3 \geq -1.$$

Therefore we have two cases: $(h^1(\mathcal{E}(-2)), c_3) = (0, 2)$ or $(1, 0)$. Note here that $h^q(\mathcal{E}(-1)) = h^q(\mathcal{E}(-2))$ for any $q$. In particular, $h^0(\mathcal{E}(-1)) = h^0(\mathcal{E}(-2)) = 0$ by (7.7).

We claim here that $h^q(\mathcal{S}^\vee \otimes \mathcal{E}(t)|_{\mathbb{Q}^2}) = 0$ for $q > 0$ and $t \geq 0$. Indeed, we see that

$$h^q((\mathcal{O}(-1,0) \oplus \mathcal{O}(0,-1)) \otimes (\mathcal{O}(1+t,t)^{\oplus 2} \oplus \mathcal{O}(t,1+t)^{\oplus 2} \oplus \mathcal{O}(t,t)^{\oplus r-3})) = 0$$

for $q > 0$ and $t \geq 0$. Hence we obtain the claim. Then it follows from (7.6) that

$$h^q(\mathcal{S}^\vee \otimes \mathcal{E}(t)) = 0 \text{ for } q \geq 2 \text{ and } t = 0, -1. \tag{13.1}$$

Since $h^0(\mathcal{E}(-1)) = 0$, the exact sequence (7.14) together with (13.1) shows that $h^q(\mathcal{S}^\vee \otimes \mathcal{E}(-1)) = 0$ unless $q = 1$. Hence

$$-h^1(\mathcal{S}^\vee \otimes \mathcal{E}(-1)) = \chi(\mathcal{S}^\vee \otimes \mathcal{E}(-1)) = -4 + c_3 \tag{13.2}$$

by (4.8).

### 13.1. Suppose that $(h^1(\mathcal{E}(-2)), c_3) = (0, 2)$.

Then $h^q(\mathcal{E}(-2)) = 0$ for any $q$. Hence $h^q(\mathcal{E}(-1)) = 0$ for any $q$. Set $a = h^1(\mathcal{E}(-2)|_{\mathbb{Q}^2})$. Then $a \leq 2$ and $h^2(\mathcal{E}(-2)|_{\mathbb{Q}^2}) = r - 4 + a$. Thus $h^2(\mathcal{E}(-3)) = a$, $h^3(\mathcal{E}(-3)) = r - 4 + a$ and $h^q(\mathcal{E}(-3)) = 0$ unless $q = 2$ or 3. It follows from (13.2) that $h^1(\mathcal{S}^\vee \otimes \mathcal{E}(-1)) = 2$. We

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apply to $\mathcal{E}(-1)$ the Bondal spectral sequence (2.1). We have $\mathrm{Ext}^3(G,\mathcal{E}(-1)) \cong S_3^{\oplus r-4+a}$, $\mathrm{Ext}^2(G,\mathcal{E}(-1)) \cong S_3^{\oplus a}$, $\mathrm{Ext}^1(G,\mathcal{E}(-1)) \cong S_1^{\oplus 2}$ and $\mathrm{Hom}(G,\mathcal{E}(-1)) = 0$. Lemma 2.1 then shows that $E_2^{-3,3} \cong \mathcal{O}(-1)^{\oplus r-4+a}$, that $E_2^{-3,2} \cong \mathcal{O}(-1)^{\oplus a}$, that $E_2^{-1,1} \cong \mathcal{S}(-1)^{\oplus 2}$ and that $E_2^{p,q} = 0$ unless $(p,q) = (-3,3)$, $(-3,2)$ or $(-1,1)$. Then $\mathcal{E}(-1)$ fits in the $(-1)$-twist of the following exact sequence:

$$0 \to \mathcal{O}^{\oplus a} \to \mathcal{S}^{\oplus 2} \to \mathcal{E} \to \mathcal{O}^{\oplus r-4+a} \to 0. \tag{13.3}$$

This sequence splits into the following two exact sequences:

$$0 \to \mathcal{O}^{\oplus a} \to \mathcal{S}^{\oplus 2} \to \mathcal{F} \to 0;$$

$$0 \to \mathcal{F} \to \mathcal{E} \to \mathcal{O}^{\oplus r-4+a} \to 0,$$

where $\mathcal{F}$ is a globally generated vector bundle of rank $4-a$. We claim here that $a \leq 1$. Indeed, if $a=2$, then we have the following exact sequences:

$$0 \to \mathcal{O} \to \mathcal{S}^{\oplus 2} \to \mathcal{G} \to 0;$$

$$0 \to \mathcal{O} \to \mathcal{G} \to \mathcal{F} \to 0,$$

where $\mathcal{G}$ is a globally generated vector bundle of rank 3. Since $\mathcal{F}$ is a vector bundle, $\mathcal{G}$ must have a nowhere vanishing global section, and thus $c_3(\mathcal{G}) = 0$. On the other hand, $c_3(\mathcal{G}) = c_3(\mathcal{S}^{\oplus 2}) = 2c_2(\mathcal{S})h = 2$. This is a contradiction. Hence the case $a=2$ does not arise. Now note that $\mathcal{E}$ is isomorphic to $\mathcal{F} \oplus \mathcal{O}^{\oplus r-4+a}$ since $h^1(\mathcal{F}) = 0$. Therefore, $\mathcal{E}$ fits in an exact sequence

$$0 \to \mathcal{O}^{\oplus a} \to \mathcal{S}^{\oplus 2} \oplus \mathcal{O}^{\oplus r-4+a} \to \mathcal{E} \to 0,$$

where the composite of the inclusion $\mathcal{O}^{\oplus a} \to \mathcal{S}^{\oplus 2} \oplus \mathcal{O}^{\oplus r-4+a}$ and the projection $\mathcal{S}^{\oplus 2} \oplus \mathcal{O}^{\oplus r-4+a} \to \mathcal{O}^{\oplus r-4+a}$ is zero. This is Case (5) of Theorem 1.1.

### 13.2. Suppose that $(h^1(\mathcal{E}(-2)), c_3) = (1, 0)$.

Then $h^1(\mathcal{E}(-1)) = 1$. Hence $h^0(\mathcal{E}) = h^0(\mathcal{E}|_{\mathbb{Q}^2}) - 1 = r+3$. It follows from (13.2) that $h^1(\mathcal{S}^\vee \otimes \mathcal{E}(-1)) = 4$. Set $a = h^0(\mathcal{S}^\vee \otimes \mathcal{E})$. Then the exact sequence (7.14) shows that $a \leq 4$, that $h^q(\mathcal{S}^\vee \otimes \mathcal{E}) = 0$ unless $q=0$ or 1 and that $h^1(\mathcal{S}^\vee \otimes \mathcal{E}) = a$. Hence we have $\mathrm{Ext}^q(G,\mathcal{E}) = 0$ for $q=2$ and 3, and $\mathrm{Hom}(G,\mathcal{E})$ fits in an exact sequence

$$0 \to S_0^{\oplus r+3} \to \mathrm{Hom}(G,\mathcal{E}) \to S_1^{\oplus a} \to 0.$$

Moreover, $\mathrm{Ext}^1(G,\mathcal{E})$ has a filtration $S_1^{\oplus a} \subset F \subset \mathrm{Ext}^1(G,\mathcal{E})$ of right $A$-modules such that the following sequences are exact:

$$0 \to F \to \mathrm{Ext}^1(G,\mathcal{E}) \to S_3 \to 0;$$

$$0 \to S_1^{\oplus a} \to F \to S_2 \to 0.$$

Now the structures of right $A$-modules $\mathrm{Ext}^q(G,\mathcal{E})$'s are the same as those of $\mathrm{Ext}^q(G,\mathcal{E})$'s in § 12.2.2.1, and we conclude that $\mathcal{E}$ belongs to Case (9) of Theorem 1.1.

## 14. The case where $\mathcal{E}|_{\mathbb{Q}^2}$ belongs to Case (7) of Theorem 2.3

Suppose that $\mathcal{E}|_{\mathbb{Q}^2}$ fits in the following exact sequence:

$$0 \to \mathcal{O}(-1,-1) \oplus \mathcal{O}(-1,0) \oplus \mathcal{O}(0,-1) \to \mathcal{O}^{\oplus r+3} \to \mathcal{E}|_{\mathbb{Q}^2} \to 0.$$

Then $c_2 h = 5$. It then follows from (6.1) that $c_3 \geq 4$. Note that

$$h^q(\mathcal{S}^\vee \otimes \mathcal{E}|_{\mathbb{Q}^2}) = \begin{cases} 2 & \text{if} \quad q = 0 \\ 0 & \text{if} \quad q \neq 0, \end{cases} \tag{14.1}$$

and that

$$h^q(\mathcal{E}(-1)|_{\mathbb{Q}^2}) = \begin{cases} 1 & \text{if} \quad q = 1 \\ 0 & \text{if} \quad q \neq 1. \end{cases} \tag{14.2}$$

Hence we have

$$h^0(\mathcal{E}(-1)) = 0$$

by (7.7). Note that $H^q(\mathcal{E}|_{\mathbb{Q}^2})$ vanishes for any $q > 0$. Since $h^q(\mathcal{E}) = 0$ for any $q > 0$ by (7.1), we have $h^q(\mathcal{E}(-1)) = 0$ for any $q \geq 2$. Hence it follows from (4.4) that

$$0 \geq -h^1(\mathcal{E}(-1)) = \chi(\mathcal{E}(-1)) = -\frac{5}{2} + \frac{1}{2}c_3 \geq -\frac{1}{2}.$$

Therefore $c_3 = 5$ and $h^1(\mathcal{E}(-1)) = 0$. Now it follows from (7.14) that $h^q(\mathcal{S}^\vee \otimes \mathcal{E}) = h^{q+1}(\mathcal{S}^\vee \otimes \mathcal{E}(-1))$ for any $q$. In particular, $h^0(\mathcal{S}^\vee \otimes \mathcal{E}(-1)) = 0$. Moreover $h^{q+1}(\mathcal{S}^\vee \otimes \mathcal{E}(-1)) = h^{q+1}(\mathcal{S}^\vee \otimes \mathcal{E})$ for $q \geq 1$ by (14.1) and (7.15). Hence $h^q(\mathcal{S}^\vee \otimes \mathcal{E}) = 0$ for $q \geq 1$ and $h^q(\mathcal{S}^\vee \otimes \mathcal{E}(-1)) = 0$ for $q \geq 2$. Therefore

$$-h^1(\mathcal{S}^\vee \otimes \mathcal{E}(-1)) = \chi(\mathcal{S}^\vee \otimes \mathcal{E}(-1)) = -6 + c_3 = -1$$

by (4.8). Thus $h^0(\mathcal{S}^\vee \otimes \mathcal{E}) = 1$. We apply to $\mathcal{E}$ the Bondal spectral sequence (2.1). From (14.2), it follows that $h^q(\mathcal{E}(-2)) = 0$ unless $q = 2$ and that $h^2(\mathcal{E}(-2)) = 1$. Since $h^0(\mathcal{E}|_{\mathbb{Q}^2}) = r+3$, we see that $h^0(\mathcal{E}) = r+3$. Hence we have an exact sequence

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$$0 \to S_0^{\oplus r+3} \to \mathrm{Hom}(G, \mathcal{E}) \to S_1 \to 0,$$

and the following: $\mathrm{Ext}^q(G, \mathcal{E}) = 0$ for $q = 1, 3$; $\mathrm{Ext}^2(G, \mathcal{E}) \cong S_3$. Therefore, Lemma 2.1 implies that $E_2^{p,q} = 0$ unless $(p, q) = (-3, 2)$ or $(0, 0)$, that $E_2^{-3,2} \cong \mathcal{O}(-1)$, and that there is the following exact sequence:

$$0 \to \mathcal{S}(-1) \to \mathcal{O}^{\oplus r+3} \to E_2^{0,0} \to 0.$$

Note that we have the following exact sequence:

$$0 \to E_2^{-3,2} \to E_2^{0,0} \to \mathcal{E} \to 0.$$

Since $\mathrm{Ext}^1(\mathcal{O}(-1), \mathcal{S}(-1)) = 0$, this implies that $\mathcal{E}$ fits in the following exact sequence:

$$0 \to \mathcal{S}(-1) \oplus \mathcal{O}(-1) \to \mathcal{O}^{\oplus r+3} \to \mathcal{E} \to 0.$$

This is Case (6) of Theorem 1.1.

## 15. The case where $\mathcal{E}|_{\mathbb{Q}^2}$ belongs to Case (8) of Theorem 2.3

Suppose that $\mathcal{E}|_{\mathbb{Q}^2}$ fits in the following exact sequence:

$$0 \to \mathcal{O}(-1, -2) \to \mathcal{O}(1, 0) \oplus \mathcal{O}^{\oplus r} \to \mathcal{E}|_{\mathbb{Q}^2} \to 0.$$

Then $c_2 h = 6$. It then follows from (6.1) that $c_3 \geq 8$. Note that

$$h^q(\mathcal{E}(-1)|_{\mathbb{Q}^2}) = \begin{cases} 2 & \text{if} \quad q = 1 \\ 0 & \text{if} \quad q \neq 1, \end{cases} \tag{15.1}$$

and that

$$h^q(\mathcal{S}^\vee \otimes \mathcal{E}|_{\mathbb{Q}^2}) = \begin{cases} 1 & \text{if} \quad q = 0, 1 \\ 0 & \text{if} \quad q \neq 0, 1. \end{cases} \tag{15.2}$$

Hence we have

$$h^0(\mathcal{E}(-1)) = 0$$

by (7.7). Note that $H^q(\mathcal{E}|_{\mathbb{Q}^2})$ vanishes for all $q > 0$. Since $h^q(\mathcal{E}) = 0$ for all $q > 0$ by (7.1), we have $h^q(\mathcal{E}(-1)) = 0$ for all $q \geq 2$. It follows from (4.4) that

$$0 \geq -h^1(\mathcal{E}(-1)) = \chi(\mathcal{E}(-1)) = -4 + \frac{1}{2} c_3 \geq 0.$$

Therefore $c_3 = 8$ and $h^1(\mathcal{E}(-1)) = 0$. Now it follows from (7.14) that $h^q(\mathcal{S}^\vee \otimes \mathcal{E}) = h^{q+1}(\mathcal{S}^\vee \otimes \mathcal{E}(-1))$ for any $q$. In particular, $h^0(\mathcal{S}^\vee \otimes \mathcal{E}(-1)) = 0$. Moreover $h^{q+1}(\mathcal{S}^\vee \otimes$

$\mathcal{E}(-1)) = h^{q+1}(\mathcal{S}^\vee \otimes \mathcal{E})$ for $q \geq 2$ by (7.15) and (15.2). Hence $h^q(\mathcal{S}^\vee \otimes \mathcal{E}) = 0$ for $q \geq 2$ and $h^3(\mathcal{S}^\vee \otimes \mathcal{E}(-1)) = 0$. Hence

$$-h^1(\mathcal{S}^\vee \otimes \mathcal{E}(-1)) + h^2(\mathcal{S}^\vee \otimes \mathcal{E}(-1)) = \chi(\mathcal{S}^\vee \otimes \mathcal{E}(-1)) = -8 + c_3 = 0$$

by (4.8). Set $a = h^0(\mathcal{S}^\vee \otimes \mathcal{E})$. Then $a = h^1(\mathcal{S}^\vee \otimes \mathcal{E}(-1)) = h^2(\mathcal{S}^\vee \otimes \mathcal{E}(-1)) = h^1(\mathcal{S}^\vee \otimes \mathcal{E})$. We see that $a = 1$ by (7.15) and (15.2). We apply to $\mathcal{E}$ the Bondal spectral sequence (2.1). It follows from (15.1) that $h^q(\mathcal{E}(-2))$ vanishes unless $q = 2$ and that $h^2(\mathcal{E}(-2)) = 2$. Since $h^0(\mathcal{E}|_{\mathbb{Q}^2}) = r+2$, we see that $h^0(\mathcal{E}) = r+2$. Therefore, $\mathrm{Ext}^3(G, \mathcal{E}) = 0$, $\mathrm{Ext}^2(G, \mathcal{E}) \cong S_3^{\oplus 2}$, $\mathrm{Ext}^1(G, \mathcal{E}) \cong S_1$ and $\mathrm{Hom}(G, \mathcal{E})$ fits in the following exact sequence:

$$0 \to S_0^{\oplus r+2} \to \mathrm{Hom}(G, \mathcal{E}) \to S_1 \to 0.$$

Therefore, Lemma 2.1 implies that $E_2^{p,q} = 0$ unless $(p, q) = (-3, 2)$, $(-1, 1)$, $(-1, 0)$ or $(0, 0)$, that $E_2^{-3,2} \cong \mathcal{O}(-1)^{\oplus 2}$, that $E_2^{-1,1} \cong \mathcal{S}(-1)$ and that there exists the following exact sequence:

$$0 \to E_2^{-1,0} \to \mathcal{S}(-1) \to \mathcal{O}^{\oplus r+2} \to E_2^{0,0} \to 0.$$

The Bondal spectral sequence implies that $E_2^{-1,0} = 0$, that $E_2^{0,0} \cong E_3^{0,0}$ and that we have the following exact sequences:

$$0 \to E_3^{-3,2} \to \mathcal{O}(-1)^{\oplus 2} \xrightarrow{\varphi} \mathcal{S}(-1) \to E_3^{-1,1} \to 0;$$

$$0 \to E_3^{-3,2} \to E_3^{0,0} \to E_4^{0,0} \to 0;$$

$$0 \to E_4^{0,0} \to \mathcal{E} \to E_3^{-1,1} \to 0.$$

Since $\mathcal{E}$ is nef, $E_3^{-1,1}$ cannot admit a negative degree quotient. Hence $\varphi \neq 0$. Thus, there exists an inclusion $\iota : \mathcal{O}(-1) \to \mathcal{O}(-1)^{\oplus 2}$ such that $\varphi \circ \iota \neq 0$. Now we have a morphism $\bar{\varphi} : \mathcal{O}(-1) \cong \mathrm{Coker}(\iota) \to \mathrm{Coker}(\varphi \circ \iota) \cong \mathcal{I}_L$ for some line $L$ in $\mathbb{Q}^3$ and $\bar{\varphi}$ fits in the following exact sequence:

$$0 \to E_3^{-3,2} \to \mathcal{O}(-1) \xrightarrow{\bar{\varphi}} \mathcal{I}_L \to E_3^{-1,1} \to 0.$$

This shows that $E_3^{-1,1}|_M$ admits a negative degree quotient for some line $M$ in $\mathbb{Q}^3$. This is a contradiction. Therefore, $\mathcal{E}|_{\mathbb{Q}^2}$ cannot belong to Case (8) of Theorem 2.3.

## 16. The case where $\mathcal{E}|_{\mathbb{Q}^2}$ belongs to Case (9) of Theorem 2.3

Suppose that $\mathcal{E}|_{\mathbb{Q}^2}$ fits in the following exact sequence:

$$0 \to \mathcal{O}(-1, -1)^{\oplus 2} \to \mathcal{O}^{\oplus r+2} \to \mathcal{E}|_{\mathbb{Q}^2} \to 0.$$

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Then $c_2h = 6$. It then follows from (6.1) that $c_3 \geq 8$. Note that

$$h^q(\mathcal{E}(-1)|_{\mathbb{Q}^2}) = \begin{cases} 2 & \text{if} \quad q = 1 \\ 0 & \text{if} \quad q \neq 1, \end{cases} \tag{16.1}$$

and that

$$h^q(\mathcal{S}^\vee \otimes \mathcal{E}|_{\mathbb{Q}^2}) = 0 \text{ for all } q. \tag{16.2}$$

Hence we have

$$h^0(\mathcal{E}(-1)) = 0$$

by (7.7). Note that $H^q(\mathcal{E}|_{\mathbb{Q}^2})$ vanishes for all $q > 0$. Since $h^q(\mathcal{E}) = 0$ for all $q > 0$ by (7.1), we have $h^q(\mathcal{E}(-1)) = 0$ for all $q \geq 2$. It follows from (4.4) that

$$0 \geq -h^1(\mathcal{E}(-1)) = \chi(\mathcal{E}(-1)) = -4 + \frac{1}{2}c_3 \geq 0.$$

Therefore $c_3 = 8$ and $h^1(\mathcal{E}(-1)) = 0$. Now it follows from (7.14) that $h^q(\mathcal{S}^\vee \otimes \mathcal{E}) = h^{q+1}(\mathcal{S}^\vee \otimes \mathcal{E}(-1))$ for any $q$. Moreover $h^{q+1}(\mathcal{S}^\vee \otimes \mathcal{E}(-1)) = h^{q+1}(\mathcal{S}^\vee \otimes \mathcal{E})$ for any $q$ by (7.15) and (16.2). Hence $h^q(\mathcal{S}^\vee \otimes \mathcal{E}) = 0$ for any $q$. We apply to $\mathcal{E}$ the Bondal spectral sequence (2.1). It follows from (16.1) that $h^q(\mathcal{E}(-2))$ vanishes unless $q = 2$ and that $h^2(\mathcal{E}(-2)) = 2$. Since $h^0(\mathcal{E}|_{\mathbb{Q}^2}) = r + 2$, we see that $h^0(\mathcal{E}) = r + 2$. Therefore, $\operatorname{Hom}(G, \mathcal{E}) \cong S_0^{\oplus r+2}$, $\operatorname{Ext}^1(G, \mathcal{E}) = 0$, $\operatorname{Ext}^2(G, \mathcal{E}) \cong S_3^{\oplus 2}$ and $\operatorname{Ext}^3(G, \mathcal{E}) = 0$. Therefore, Lemma 2.1 implies that $E_2^{p,q} = 0$ unless $(p, q) = (-3, 2)$ or $(0, 0)$, that $E_2^{-3,2} \cong \mathcal{O}(-1)^{\oplus 2}$ and that $E_2^{0,0} \cong \mathcal{O}^{\oplus r+2}$. It follows from the Bondal spectral sequence that $\mathcal{E}$ fits in the following exact sequence:

$$0 \to \mathcal{O}(-1)^{\oplus 2} \to \mathcal{O}^{\oplus r+2} \to \mathcal{E} \to 0.$$

This is Case (7) of Theorem 1.1.

## 17. The case where $\mathcal{E}|_{\mathbb{Q}^2}$ belongs to Case (10) of Theorem 2.3

Suppose that $\mathcal{E}|_{\mathbb{Q}^2}$ fits in the following exact sequence:

$$0 \to \mathcal{O}(-2, -2) \to \mathcal{O}^{\oplus r+1} \to \mathcal{E}|_{\mathbb{Q}^2} \to 0.$$

Then $c_2h = 8$. It then follows from (6.1) that $c_3 \geq 16$. Note that

$$h^q(\mathcal{E}|_{\mathbb{Q}^2}) = \begin{cases} r+1 & \text{if} \quad q = 0 \\ 1 & \text{if} \quad q = 1 \\ 0 & \text{if} \quad q = 2, \end{cases} \tag{17.1}$$

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that

$$h^q(\mathcal{E}(-1)|_{\mathbb{Q}^2}) = \begin{cases} 4 & \text{if} \quad q = 1 \\ 0 & \text{if} \quad q \neq 1, \end{cases} \tag{17.2}$$

that

$$h^q(\mathcal{S}^\vee \otimes \mathcal{E}(1)|_{\mathbb{Q}^2}) = \begin{cases} 4r + 4 & \text{if} \quad q = 0 \\ 0 & \text{if} \quad q \neq 0, \end{cases} \tag{17.3}$$

and that

$$h^q(\mathcal{S}^\vee \otimes \mathcal{E}|_{\mathbb{Q}^2}) = \begin{cases} 4 & \text{if} \quad q = 1 \\ 0 & \text{if} \quad q \neq 1. \end{cases} \tag{17.4}$$

Hence we have

$$h^0(\mathcal{E}(-1)) = 0$$

by (7.7). Then $h^0(\mathcal{S}^\vee \otimes \mathcal{E}(-1)) = 0$ by (7.14). Since $h^q(\mathcal{E}) = 0$ for all $q > 0$ by (7.1), we have $h^2(\mathcal{E}(-1)) = 1$ and $h^3(\mathcal{E}(-1)) = 0$ by (17.1). It then follows from (4.4) that

$$1 \geq 1 - h^1(\mathcal{E}(-1)) = \chi(\mathcal{E}(-1)) = -7 + \frac{1}{2}c_3 \geq 1.$$

Therefore $c_3 = 16$ and $h^1(\mathcal{E}(-1)) = 0$. Hence $h^0(\mathcal{E}) = r + 1$ since $h^0(\mathcal{E}|_{\mathbb{Q}^2}) = r + 1$ by (17.1). Moreover $h^2(\mathcal{E}(-2)) = 5$ and $h^q(\mathcal{E}(-2)) = 0$ unless $q = 2$ by (17.2). It follows from (7.6) and (17.3) that

$$h^q(\mathcal{S}^\vee \otimes \mathcal{E}) = 0 \text{ for } q \geq 2.$$

Moreover $h^0(\mathcal{S}^\vee \otimes \mathcal{E}) = 0$ since $h^0(\mathcal{S}^\vee \otimes \mathcal{E}|_{\mathbb{Q}^2}) = 0$ by (17.4). Hence it follows from (4.7)

$$-h^1(\mathcal{S}^\vee \otimes \mathcal{E}) = \chi(\mathcal{S}^\vee \otimes \mathcal{E}) = 16 - 4c_2h + c_3 = 0.$$

We apply to $\mathcal{E}$ the Bondal spectral sequence (2.1). We see that $\mathrm{Hom}(G, \mathcal{E}) \cong S_0^{\oplus r+1}$, that $\mathrm{Ext}^q(G, \mathcal{E}) = 0$ for $q = 1, 3$ and that $\mathrm{Ext}^2(G, \mathcal{E})$ fits in the following exact sequence of right $A$-modules:

$$0 \to S_2 \to \mathrm{Ext}^2(G, \mathcal{E}) \to S_3^{\oplus 5} \to 0.$$

Therefore, Lemma 2.1 implies that $E_2^{p,q} = 0$ unless $(p, q) = (-3, 2)$ $(-2, 2)$ or $(0, 0)$, that $E_2^{0,0} \cong \mathcal{O}^{\oplus r+1}$ and that $E_2^{-3,2}$ and $E_2^{-2,2}$ fit in the following exact sequence:

$$0 \to E_2^{-3,2} \to \mathcal{O}(-1)^{\oplus 5} \to T_{\mathbb{P}^4}(-2)|_{\mathbb{Q}^3} \to E_2^{-2,2} \to 0. \tag{17.5}$$

The Bondal spectral sequence induces the following isomorphisms and exact sequences:

$$E_2^{-3,2} \cong E_3^{-3,2};$$

$$E_2^{0,0} \cong E_3^{0,0};$$

$$0 \to E_3^{-3,2} \to E_3^{0,0} \to E_4^{0,0} \to 0;$$

$$0 \to E_4^{0,0} \to \mathcal{E} \to E_2^{-2,2} \to 0.$$

Note here that $E_2^{-2,2}|_L$ cannot admit a negative degree quotient for any line $L \subset \mathbb{Q}^3$ since $\mathcal{E}$ is nef. We will show that $E_2^{-2,2} = 0$; first note that the exact sequence (17.5) induces the following exact sequence:

$$0 \to E_2^{-3,2} \to \mathcal{O}(-1)^{\oplus 5} \oplus \mathcal{O}(-2) \xrightarrow{p} \mathcal{O}(-1)^{\oplus 5} \to E_2^{-2,2} \to 0.$$

Consider the composite of the inclusion $\mathcal{O}(-1)^{\oplus 5} \to \mathcal{O}(-1)^{\oplus 5} \oplus \mathcal{O}(-2)$ and the morphism $p$ above, and let $\mathcal{O}(-1)^{\oplus a}$ be the cokernel of this composite. Then we have the following exact sequence:

$$\mathcal{O}(-2) \xrightarrow{\pi} \mathcal{O}(-1)^{\oplus a} \to E_2^{-2,2} \to 0.$$

We claim here that $a = 0$. Suppose, to the contrary, that $a > 0$. Since $E_2^{-2,2}$ cannot be isomorphic to $\mathcal{O}(-1)^{\oplus a}$, the morphism $\pi$ above is not zero. Therefore, the composite of $\pi$ and some projection $\mathcal{O}(-1)^{\oplus a} \to \mathcal{O}(-1)$ is not zero, whose quotient is of the form $\mathcal{O}_H(-1)$ for some hyperplane $H$ in $\mathbb{Q}^3$. Hence $E_2^{-2,2}$ admits $\mathcal{O}_H(-1)$ as a quotient. This is a contradiction. Thus $a = 0$ and $E_2^{-2,2} = 0$. Moreover, we see that $E_2^{-3,2} \cong \mathcal{O}(-2)$. Therefore, $\mathcal{E}$ fits in the following exact sequence:

$$0 \to \mathcal{O}(-2) \to \mathcal{O}^{\oplus r+1} \to \mathcal{E} \to 0.$$

This is Case (8) of Theorem 1.1.

## 18. The case where $\mathcal{E}|_{\mathbb{Q}^2}$ belongs to Case (11) of Theorem 2.3

Suppose that $\mathcal{E}|_{\mathbb{Q}^2}$ fits in the following exact sequence:

$$0 \to \mathcal{O}(-2,-2) \to \mathcal{O}^{\oplus r+1} \to \mathcal{E}|_{\mathbb{Q}^2} \to k(p) \to 0.$$

Then $c_2 h = 7$. It then follows from (6.1) that

$$c_3 \geq 12.$$

https://doi.org/10.1017/S0013091524000919 Published online by Cambridge University Press

We claim here that $h^0(\mathcal{E}(-1)|_{\mathbb{Q}^2}) = 0$. Indeed, if $h^0(\mathcal{E}(-1)|_{\mathbb{Q}^2}) \neq 0$, then

$$c_2 h \leq c_1(\mathcal{E}|_{\mathbb{Q}^2})(c_1(\mathcal{E}|_{\mathbb{Q}^2}) - c_1(\mathcal{O}_{\mathbb{Q}^2}(1,1))) = 4$$

by [12, Lemma 10.1]. This is a contradiction. Hence $h^0(\mathcal{E}(-1)|_{\mathbb{Q}^2}) = 0$. Thus, we have $h^0(\mathcal{E}(-1)) = 0$ by (7.7). It follows from (4.4) that

$$\chi(\mathcal{E}(-1)) = -\frac{11}{2} + \frac{1}{2}c_3.$$

In particular $c_3$ is odd, and thus $c_3 > 12$. Therefore $h^q(\mathcal{E}(-1)) = 0$ for all $q > 0$ by (7.2). This implies that $\chi(\mathcal{E}(-1)) = 0$, which is a contradiction. Therefore, $\mathcal{E}|_{\mathbb{Q}^2}$ cannot belong to Case (11) of Theorem 2.3.

## 19. The case where $\mathcal{E}|_{\mathbb{Q}^2}$ belongs to Case (12) or (13) of Theorem 2.3

Suppose that $\mathcal{E}|_{\mathbb{Q}^2}$ fits in either of the following exact sequences:

$$0 \to \mathcal{O}(-2,-2) \to \mathcal{O}^{\oplus r} \to \mathcal{E}|_{\mathbb{Q}^2} \to \mathcal{O} \to 0;$$

$$0 \to \mathcal{O}(-1,-1)^{\oplus 4} \to \mathcal{O}^{\oplus r} \oplus \mathcal{O}(-1,0)^{\oplus 2} \oplus \mathcal{O}(0,-1)^{\oplus 2} \to \mathcal{E}|_{\mathbb{Q}^2} \to 0.$$

Then $c_2 h = 8$. It then follows from (6.1) that

$$c_3 \geq 16.$$

We claim here that $h^0(\mathcal{E}(-1)|_{\mathbb{Q}^2}) = 0$. Indeed, if $h^0(\mathcal{E}(-1)|_{\mathbb{Q}^2}) \neq 0$, then

$$c_2 h \leq c_1(\mathcal{E}|_{\mathbb{Q}^2})(c_1(\mathcal{E}|_{\mathbb{Q}^2}) - c_1(\mathcal{O}_{\mathbb{Q}^2}(1,1))) = 4$$

by [12, Lemma 10.1]. This is a contradiction. Hence $h^0(\mathcal{E}(-1)|_{\mathbb{Q}^2}) = 0$. Thus, we have $h^0(\mathcal{E}(-1)) = 0$ by (7.7). Note that $h^q(\mathcal{E}|_{\mathbb{Q}^2}) = 0$ for all $q > 0$. Since $h^q(\mathcal{E}) = 0$ for all $q > 0$ by (7.1), this implies that $h^q(\mathcal{E}(-1)) = 0$ for all $q \geq 2$. It follows from (4.4) that

$$0 \geq -h^1(\mathcal{E}(-1)) = \chi(\mathcal{E}(-1)) = -7 + \frac{1}{2}c_3 \geq 1.$$

This is a contradiction. Therefore, $\mathcal{E}|_{\mathbb{Q}^2}$ cannot belong to Case (12) or (13) of Theorem 2.3.

**Acknowledgements.** Deep appreciation goes to the referee for his careful reading the manuscript and kind suggestions and comments. In particular, the referee kindly informed the author of the brilliant short proofs of Lemmas 5.1, 5.3 and 6.1. This work was partially supported by JSPS KAKENHI (C) Grant Number 21K03158.

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