GENERALIZED VECTOR SPACES. I.

The Structure of Finite-Dimensional Spaces

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1. Introduction

During the last fifty years, the concept of the Euclidean space (an n-dimensional coordinate space with a Pythagorean distance) has undergone various profound generalizations.

Hilbert introduced the infinitely-dimensional Euclidean space whose points are infinite sequences of coordinates having from the origin, and thus from each other, finite Pythagorean distances.

Minkowski generalized the Pythagorean distance. Any surface which is symmetric about the origin, o, and intersects every ray issuing from o in exactly one point, is admitted as the "unit sphere" about o, that is, as the set of all points having the distance 1 from o. The distance from o to a point whose coordinates are k times those of a point on the unit sphere, is k. Minkowski chose a congruent unit sphere about every point. He discovered the equivalence of the convexity of these spheres and the triangle inequality for the distance. Finsler introduced spaces which are locally Minkowskian in the same sense in which Riemann spaces are locally Euclidean. With each point, a "tangential" Minkowskian space is associated—the unit sphere varying from point to point. Finsler found that each positively definite problem of the Calculus of Variations gives rise to one of his spaces.

In the finite-dimensional case, Weyl noticed that the definition of points by coordinates could be replaced by the assumption that undefined points can be added, and multiplied by real numbers. Banach, Hahn, and Wiener [1], independently of each other, introduced the following concept. A set of elements, $v, w, \ldots$ (called vectors) is said to be a *vector space* if

(a) the set is a commutative group, the operation being denoted by $+$, the neutral element by o, so that $v + o = o + v = v$;

(b) an associative and doubly distributive multiplication of vectors by real numbers, $\alpha, \beta, \ldots$ is defined, that is to say,

$$\alpha(\beta v) = (\alpha\beta)v, \quad (\alpha + \beta)v = \alpha v + \beta v, \quad \alpha(v + w) = \alpha v + \alpha w;$$

for the multiplication by the numbers 1, -1, and 0 we have

$$1v = v, \quad -1v = -v, \quad 0v = o;$$

(c) with each vector, v, a real number $|v|$ is associated, called the norm of v, which satisfies the following three conditions

(1) $|\alpha v| = |\alpha||v|$ for every v;

(2) $|v + w| \leq |v| + |w|$;

(3) if $v \neq o$, then $|v| > 0$.

Received October 1, 1948.

https://doi.org/10.4153/CJM-1949-010-1 Published online by Cambridge University Press

Much earlier, Fréchet had introduced the most radical generalization of the Euclidean space by assuming only that a number (called distance) be associated with every unordered pair of elements of a set, identical elements having the distance 0, distinct elements a positive distance, while the distance satisfies the triangle inequality. As a price for the generality of these metric spaces we have to accept the possible absence of directions of any kind.

In applying metric methods to the Calculus of Variations we made use of all these generalizations of the concept of space [2]. We studied the minima of line integrals even in a general metric space. Our integrand is a function of the point and (in absence of a direction) of an ordered pair of distinct points. Multiplying the distance by this function we obtain a new distance which we call the *variational distance*. If, in particular, the metric space is a vector space, and the function is positive and endowed with strong continuity properties, then one obtains a Finsler space. If the metric space is Euclidean, then our results generalized Tonelli's existence theorems for the parametric case.

Besides synthesizing the various known concepts, the metric ideas in the Calculus of Variations led to a generalization of the idea of space in a new direction. Minkowski spaces as well as the vector spaces of Banach, Hahn, and Wiener, and even Fréchet's metric spaces, have the following two important features in common: distinct points have distances $\neq 0$; and distances are non-negative. But, on every level of generality, the only source of the lower semi-continuity of the line integral is the local triangle inequality of the variational distance. The two other traditional features (and still more, of course, the symmetry) of the distance appeared to be quite inessential. As a result, one can, in particular, generalize Finsler's concept in such a way that one can associate a generalized Finsler space also with semi-definite and indefinite parametric variational problems in vector spaces.

As a by-product, these studies yielded a generalization of the concept of a vector space. Alt [3] proved the equivalence of the triangle inequality with what he called "projective convexity" of the unit sphere. Pauc [4] and Aronszajn continued this work in many interesting ways and the latter first explicitly formulated the concept of general vector spaces [5] which implicitly was contained in our remarks [6] about what we called "generalized Minkowskian metric." In our spaces we had admitted that distinct points might have the distance 0, and that the distance of two points might be negative. In fact, we mentioned that vectors might have negative norms or the norm 0. We had dropped the symmetry of the distance and the norm. All we had retained was the triangle inequality for distances and norms, and the assumption that by multiplying a vector by a positive number, k, the norm was multiplied by k.

Now we intend to study these generalized vector spaces in a series of papers. The present first paper contains a few remarks about all generalized vector spaces but essentially deals with the structure of generalized vector spaces of finite dimension. We prove that each such space is built up of a subspace all of whose vectors (except o) have a positive norm; a subspace all of whose vectors have the norm 0; and possibly one single line containing a vector with a non-positive norm while the norm of the opposite vector is positive.

In subsequent papers we shall study spaces of infinitely many dimensions, metric properties of our spaces as well as topological aspects of the theory ("triangular topologies"), non-real multipliers, and applications which, besides the Calculus of Variations, comprise the theories of operators and of normed rings.

2. The Main Types of Generalized Vector Spaces

A *generalized vector space* is a set, V, of elements for which addition, and multiplication by real numbers, are defined according to Postulates (a) and (b) while the norm $|v|$ of a vector is a real number satisfying only one and one half of the three Postulates (c), namely,

$$(1^+) \qquad \text{If } a > 0, \text{ then } |av| = a|v|.$$

$$(2) \qquad |v + w| \leq |v| + |w|.$$

We do not postulate the other half of (1), that is

$$(1^-) \qquad \text{If } a < 0, \text{ then } |av| = -a|v|,$$

nor the two important properties of the ordinary vector spaces which are jointly postulated in (3), namely, that $|v| \geq 0$ and that $v \neq o$ implies $|v| \neq 0$.

We shall briefly call a vector, v, *positive, negative* or *null* according to whether $|v|$ is positive, negative or 0. We call the vector v *degenerate* if $v \neq o$ and $|v| = |-v| = 0$.

We call a general vector space, V,

definite if every vector, except o, is positive;

semi-definite if V is not definite but no vector is negative and at least one vector is positive;

indefinite if V contains both positive and negative vectors;

degenerate if V contains at least one degenerate vector;

non-degenerate if V is not degenerate;

totally degenerate if every vector of V is degenerate and thus null.

The vector spaces of Banach, Hahn, and Wiener are the definite vector spaces. If by O^* we mean the set containing only a vector o, then, according to the above terminology, O^* is a definite vector space, and consequently non-degenerate. In fact, O^* is a vector space in the ordinary sense.

That we use the terms definite and semi-definite instead of *positively* definite and *positively* semi-definite will not lead to ambiguities since we shall see in Section 4 that no space is negatively definite or negatively semi-definite. There would be only negatively definite and negatively semi-definite spaces if we postulated 1^+) in conjunction with the triangle contra-inequality

$$|v + w| \geq |v| + |w|.$$

https://doi.org/10.4153/CJM-1949-010-1 Published online by Cambridge University Press

3. Vector-Algebraic Preliminaries

A subset, V', of V is called a *subspace* if for every two vectors, v and w, of V' and for every number α, the vectors $v + w$ and αv belong to V'. The set consisting of o alone is the subspace O^*.

If S is a subset of V, then we denote by $V(S)$ the subspace of V consisting of all vectors $\alpha_1 v_1 + \alpha_2 v_2 + \ldots + \alpha_n v_n$ where n is any integer, the α_i are numbers, the v_i vectors of S. In particular, if S consists of only one vector $v \neq o$, then $V(S)$ is called the *v-line* and denoted by $[v]$. In the usual way, we mean by $V' + V''$, the *join* of two subspaces V' and V'', the subspace $V(S)$ where S is the set of all vectors belonging to V' and/or V''; by $V' . V''$, the *intersection* of V' and V'', the subspace of all vectors belonging to both V' and V''. If $V' \neq O^* \neq V''$ and $V' . V'' = O^*$, then V' and V'' are called *independent*.

Lemma. *If V' is a subspace of V, then there exists a subspace, V'', of V such that $V' + V'' = V$ and $V' . V'' = O^*$.*

If $V' = V$, then $V'' = O^*$. If $V' \neq V$, then there exists a vector, v_1, which does not belong to V'. In this case, let Ω be any ordinal number about which we make the following assumption: with every ordinal number $\omega < \Omega$ a vector, v_ω, has been associated in such a way that if S_ω is the set of all vectors $v_1, \ldots, v_\omega$, then

(1) the set S_ω does not contain any finite subset of dependent vectors;

(2) $V(S_\omega) . V' = O^*$.

We call T_Ω the set of all vectors v_ω such that $\omega < \Omega$. Then two cases are possible. Either $V = V(T_\Omega)$ in which case we set $V'' = V(T_\Omega)$ and our proposition holds. Or V contains vectors not belonging to the join $V' + V(T_\Omega)$. In this case, we call one of these vectors v_Ω, and denote the set of all vectors $v_1, \ldots, v_\Omega$ by S_Ω. Then we have associated a vector v_ω with every ordinal number $\omega \leq \Omega$ in such a way that conditions (1) and (2) are satisfied. There exists an ordinal number Ω such that the first case prevails. If V is n-dimensional, this follows by induction, and $\Omega \leq n$. If V is infinitely dimensional, the conclusion is valid by transfinite induction.

If $v \neq o$, then we call the set of the vectors αv for all $\alpha > 0$, the open v-ray or, briefly, since we shall not consider rays which include o, the *v-ray*. We call the $(-v)$-ray the *opposite* ray. The v-line consists of o, the v-ray, and the opposite ray.

If v and w are independent vectors (that is, vectors $\neq o$ neither lying on the line of the other), then we call the set of the vectors $\alpha v + \beta w$ for all real numbers α, β, the *v, w-plane*. We further call the set of the vectors $\alpha v + \beta w$ such that $\alpha \geq 0$, $\beta \geq 0$ ($\alpha > 0$, $\beta > 0$) the *closed* (*open*) *v, w-quadrant*. We denote these quadrants by $[v, w]$ and (v, w), respectively. We can also introduce the half-open quadrants $[v, w)$ and $(v, w]$.

The set of all vectors which are opposite to those of the open, the closed, the half-open first v, w-quadrants are called the open, the closed, the half-open *third v, w-quadrants*, respectively. We denote these sets by $)v, w($, $]v,w[$,

$]v,w($, $)v,w[$, respectively. Clearly, the first and third quadrants are associated with the unordered vector pair, v, w. With the ordered pair v, w we can also associate the closed *second v, w-quadrant*, that is, the set $]v,w]$ of the vectors $-\alpha v + \beta w$ such that $\alpha \geq 0$, $\beta \geq 0$. Similarly we define $)v,w)$, $]v,w)$, $)v,w]$, and the *fourth v,w-quadrants*. One readily proves

REMARK 1. *If v, w, x are pairwise independent vectors and x belongs to (v,w), then a vector y belongs to (v,w) if and only if y either belongs to (v,x) or to (x,w) or to the x-ray.*

REMARK 2. *If v,w,x are pairwise independent and x belongs to $)v,w($, then every vector of the v,w-plane belongs to (v,w) or to (v,x) or to (w,x) or to the rays of one of the vectors v,w,x.*

4. Corollaries of the Assumptions About the Norm

We shall deduce immediate consequences of the assumptions (1^+) and (2) about the norm in a generalized vector space.

If in (1^+) we set $\alpha = 2$ and $v = o$, then since $2o = o$ we conclude

$$(1^\circ) \qquad |o| = 0.$$

If in (2) we set $v = v' + v''$ and $w = v''$, we obtain

$$|v'| = |v' + v'' - v''| \leq |v' + v''| + |-v''|,$$

thus

$$|v' + v''| \geq |v'| - |-v''|.$$

Similarly, $|v' + v''| \geq |v''| - |-v'|$. Hence

$$(2a) \qquad \text{Max}\,[|v| - |-w|, |w| - |-v|] \leq |v + w| \leq |v| + |w|.$$

If in $(2a)$ we set $w = -v$, then by (1°) we have $0 \leq |v| + |-v|$ and thus

$$|v| \geq -|-v| \quad \text{and} \quad |-v| \geq -|v|.$$

In particular, we can formulate the following

LEMMA. *The opposite of a negative vector is positive. The opposite of a null vector is non-negative.*

As a corollary of this lemma we see that *no space is negatively definite or negatively semi-definite.*

In absence of a general concept of limit, we can prove only two restricted continuity properties of the norm.

ADDITIVE CONTINUITY. *For every $\epsilon > 0$ there exists a $\delta > 0$, namely $\delta = \epsilon$, such that for every vector v*

from $|w| < \delta$ and $|-w| < \delta$ it follows that $|v| - \delta < |v + w| < |v| + \delta$.

This is an immediate consequence of $2a$).

FINITE-DIMENSIONAL CONTINUITY. *For every $\epsilon > 0$ every integer n, and every n-tuple of vectors $w_1, w_2, \ldots, w_n$, there exists a $\delta > 0$ (depending upon ϵ, $w_1, \ldots, w_n$) such that from*

$$|\delta_1| < \delta, |\delta_2| < \delta, \ldots, |\delta_n| < \delta$$

for every vector v it follows that

$$|v| - \epsilon \leq |v + \delta_1 w_1 + \delta_2 w_2 + \ldots + \delta_n w_n| \leq |v| + \epsilon.$$

Setting $\delta_1 w_1 + \ldots + \delta_n w_n = w$ we see that both $|w|$ and $|-w|$ are $\leq n \text{ Max} |\delta_i| \cdot \text{Max} [|w_i|, |-w_i|]$.

Thus $\delta = \frac{\epsilon}{n} \text{ Max} [|w_i|, |-w_i|]$ satisfies the requirement.

5. Lemmas

Lemma 1. *If v and w are independent non-positive vectors, then every vector of $[v, w]$, i.e., the closed first v, w-quadrant, is non-positive.*

For if $\alpha \geq 0$ and $\beta \geq 0$, then

$$|\alpha v + \beta w| \leq |\alpha v| + |\beta w| = \alpha |v| + \beta |w|.$$

The last expression is ≤ 0 if $|v| \leq 0$ and $|w| \leq 0$.

The last expression is < 0 if $\alpha > 0$, $\beta > 0$ and at least one of the vectors v and w is negative. We thus have proved

Lemma 2. *If of two independent vectors, v and w, one is non-positive and the other negative, then every vector of (v,w), i.e. the open first v,w-quadrant, is negative.*

Lemma 3. *If v and w are independent null vectors, then either every vector of (v, w) is null or every vector of (v,w) is negative.*

By Lemma 1, every vector of (v,w) is non-positive. Either every vector of (v, w) is null or there exists a negative vector, x, of (v,w). In the latter case, by Lemma 2, every vector of (v,x) and of (x,w) is negative. Since by Remark 1 of Section 3, every vector of (v,w) belongs either to (v,x) or to (x,w) or to the x-ray, every vector of (v,w) is negative.

Lemma 4. *If v and w are independent and v is degenerate, then every vector of the open half-plane $(v,w] + [w, -v)$ of the v, w-plane has the same sign as w.*

If w or any other vector of $(v,w] + [w, -v)$ is negative, then by Lemma 2 every vector in both quadrants is negative. If w is null, then by Lemma 1 every vector of $[v,w]$ and every vector of $[-v,w]$ is non-positive. By Lemma 3 none of these vectors is negative. Similarly, if any vector w' of the open half-plane is null, all vectors are null. Consequently, if w is positive, then every vector of the open half-plane is positive.

An obvious consequence of Lemma 4 is the following

Corollary. *If v and w are independent degenerate vectors, then every vector of the v, w-plane is degenerate.*

6. The Degenerate Part of Generalized Vector Spaces

The set, V_d, of all degenerate vectors of the vector space V is a subspace of V which we shall call *the degenerate part* of V. For if v is a degenerate vector, then obviously αv is degenerate for every α; and if v and w are degenerate, then by the Corollary of Lemma 4, the vector $v + w$ is degenerate.

Theorem 1. *Every generalized vector space, V, is the sum of a uniquely determined totally degenerate subspace, V_d, and a non-degenerate subspace V'. The latter can be chosen in such a way that V' and V_d are independent unless V is totally degenerate or non-degenerate. In the former case we have $V' = O^*$, in the latter case, $V_d = O^*$.*

Let V_d be the degenerate part of the vector space V. By the Lemma of Section 3, there exists a subspace V' such that $V = V_d + V'$ and $V_d \cdot V' = O^*$. The subspace V' is non-degenerate since V and V_d have only the vector o in common, and V_d contains all degenerate vectors.

7. The Non-Positive Part of a Vector Space

We shall call a subset, C, of a vector space a *cone* if

(*a*) C contains o and at least one vector besides o;

(*b*) for every vector, v, of C, except o, the v-ray is a subset of C.

We shall call the cone *convex* if

(*c*) for every two independent vectors, v and w, of C every vector of the first quadrant (v,w) belongs to C.

We shall call the cone *proper* if

(*d*) C does not contain two opposite vectors.

We shall refer to proper convex cones briefly as *cones*. We shall call the cone C *open* if every vector of C, except o, is an interior element of C. Here we define interior elements without reference to spherical neighbourhoods in the following way. The vector w is an *interior* element of the subset S of the vector space V if for every vector, v, of V there exists a positive number α (depending upon v) such that for every ϵ between 0 and α the vector $w + \alpha v$ belongs to S. We call a cone, C, *closed* if the set of all vectors not belonging to C is open. By the *boundary* of an open cone, C, we mean the set of all vectors w which do not belong to C while for some vector, v, of the vector space and every sufficiently small positive number ϵ the vector $w + \epsilon v$ does belong to C.

In terms of these concepts we can formulate the following

Theorem 2. *In a non-degenerate vectorspace, V, which is not definite, the set of all non-positive vectors is a closed cone. If V is indefinite, then the set consisting of o and all negative vectors is an open cone with the set of all null vectors as its boundary.*

Let V be a non-degenerate vector space which is not definite, that is to say, contains a non-positive vector $v \neq o$. Then the set, C, of all non-positive vectors is a cone since: (*a*) o is non-positive and, by assumption, V contains at least one vector $\neq o$; (*b*) every positive multiple of a non-positive vector is non-positive; (*c*) the convexity condition is satisfied by virtue of Lemma 1 of Section 5; condition (*d*) is satisfied because, by assumption, V is non-degenerate. The cone C is closed since, by virtue of the continuity of the norm, the set of all positive vectors is open.

Now let V be indefinite, that is, contain a negative vector. Then the set consisting of o and all negative vectors satisfies: conditions (*a*) and (*b*); moreover, by virtue of Lemma 2 of Section 5, condition (*c*); condition (*d*) since the opposite of a negative vector is positive. C is open by virtue of the continuity of the norm. Each null vector, v, belongs to the boundary of the open cone. For if x is a negative vector, then, for every positive α which is < 1, the vector $v + \alpha(x - v)$ is negative. Hence v belongs to the boundary of the cone.

8. The Decomposition of Finite-Dimensional Spaces

Lemma. *If V^* is a finite-dimensional definite subspace and P a plane which is independent of V^* and such that $W = V^* + P$ is non-degenerate, then W contains a vector, w', such that $V^* + [w']$ is definite.* If $V^* = O^*$, the Lemma contends that *every non-degenerate plane contains a definite line.*

From the definiteness of V^* we deduce the following

Remark A. *If z is a non-positive vector of W, then for every vector v^* of V^* and every number $\alpha > 0$, the vector $v^* - \alpha z$ is positive.*

For if $y = v^* - \alpha z$ were non-positive, then, since W is non-degenerate, y and z would be independent, and by Lemma 1, every vector of the first quadrant (y,z) would be non-positive whereas $v^* = y + \alpha z$ is positive.

From the finite dimensionality of V^* (and thus of W) we deduce a Remark B concerning a property of a particular subset, S, of W. *Only this single consequence* of the assumption that V^* has a finite dimension, say $k - 2$, will be used in the proof. If $x_1, \ldots, x_k$ are independent vectors of W, let S denote the set of all vectors $\alpha_1 x_1 + \ldots + \alpha_k x_k$ for which $\Sigma \alpha_i^2 = 1$.

Remark B. *Every sequence $s_1, s_2, \ldots$ of vectors of S contains a subsequence $s_{i_1}, s_{i_2}, \ldots$ for which a vector, s, of S exists such that*

$$\lim_{n \to \infty} |s_{i_n}| = |s|.$$

If, in particular, the kth components of the vectors $s_1, s_2, \ldots$ converge to 0, then s can be so chosen that its last component is 0. If the $(k-1)$th components of all vectors $s_1, s_2, \ldots$ are positive, then s can be so chosen that the $(k-1)$th component of s is non-negative.

By virtue of the compactness of the sphere $\Sigma \alpha_i^2 = 1$ in the k-dimensional Cartesian space of the $(\alpha_1, \ldots, \alpha_k)$, the s_{i_n} can be so selected that if

$$s - s_{i_n} = \alpha_{n,1} x_1 + \ldots + \alpha_{n,k} x_k,$$

then as $n \to \infty$

$$\lim \alpha_{n,1} = \lim \alpha_{n,2} = \ldots = \lim \alpha_{n,k} = 0.$$

Hence Remark B is a consequence of the finite-dimensional continuity of the norm.

If every vector of W is positive, then the proposition of the Lemma holds. We thus can assume that W contains a non-positive vector, v. Its opposite, the vector $-v$, is positive. By Remark A, for every v^* of V^* and every $\alpha > 0$, the vector $v^* - \alpha v$ is positive. W contains a vector, w, such that $[w]$ and $V^* + [v]$ are independent. Now we prove:

There exists a $\beta \geq 0$ with the following property P. If $\alpha > \beta$, then for every v^* of V^*, the vector $v^* - \alpha v + w$ is positive.

We assume that no number $\beta \geq 0$ has the property P and derive a contradiction. By the assumption, for every n there exists a number $\alpha_n > n$ and a vector, v^*_n of V^* such that

$$y_n = v^*_n - \alpha_n v + w$$

is non-positive. Let $x_1, \ldots, x_{k-2}$ be independent vectors of V^* such that

$$v^*_n = \alpha_{n,1} x_1 + \ldots + \alpha_{n,k-2} x_{k-2}.$$

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We set

$$x_{k-1} = -v \text{ and } a_{n,k-1} = a_n,$$
$$x_k = w \text{ and } a_{n,k} = 1,$$
$$\left[\sum_i (a_{n,i})^2\right]^{\frac{1}{2}} = \gamma_n > 0$$

Since $a_n > n$ we have $\lim \gamma_n = \infty$. If we set

$$s_n = \frac{1}{\gamma_n} y_n ,$$

then by Remark B there exists a subsequence $s_{i_1}, s_{i_2}, \ldots$ and a vector s of S such that

$$\lim | s_{i_n} | = | s |.$$

Since $\lim \gamma_n = \infty$ and $a_{n,k} = 1$ and $a_{n,k-1} > 0$ for every n, we see that s can be chosen as a vector of the form $v^* - av$ where v^* is in V^* and $a \geq 0$. Thus s is positive. This is a contradiction since the s_n, and, in particular, the s_{i_n} are non-positive. (The y_n are non-positive, the $\gamma_n > 0$).

Having established the existence of a $\beta \geq 0$ with the property P, we call β_0 the g.l.b. of all $\beta \geq 0$ with the property P. Two cases are possible.

First Case. $\beta_0 > 0$. Then from the definition of β_0 it follows[1] that there exists a non-positive vector (and thus clearly a null vector) $w_0 = v^*_0 - \beta_0 v + w$ where v^*_0 belongs to V^*. Now if for a vector v^* of V^* and $\kappa > 0$ the vector $w'_\kappa = v^* - (\kappa - \beta_0)v - w$ were non-positive, then since $V^* + P$ is non-degenerate, the vectors w_0 and w' would be independent, and all vectors of the first quadrant (w_0, w') would be non-positive. But this is not the case since $w_0 + w'$, that is, $(v^*_0 + v^*) - \kappa v$ is positive. Hence all vectors w'_κ are positive. Since w_0 is non-positive, by Remark A, for every vector v^* of V^* and every $a > 0$, the vector

$$a\left(\frac{1}{a} v^* - v^*_0\right) + a\beta_0 v - aw$$

is positive. Hence for every v^*_1 of V^*, the vector $v^*_1 + \beta_0 v - w$ is positive. In particular, $- v^*_0 + \beta_0 v - w$ is positive. Since its opposite, the vector w_0, is non-positive, by Remark A all vectors $v^* - aw_0$ or, which is equivalent, all vectors

$$v^* + a(\beta_0 v - w) \text{ for } v^* \text{ in } V^* \text{ and } a > 0$$

are positive. From this fact it follows [exactly as from the positivity of all vectors $v^* - av$ we derived the existence of a $\beta \geq 0$ with the property P] that there exists a $\gamma \geq 0$ such that for each v^* of V^* and every $a > \gamma$ the vector

$$v_a = v^* + a(\beta_0 v - w) + w$$

is positive. No matter how we determine $a > \gamma$, for every $\pi > 0$ the vectors

$$\pi[v^* + a(\beta_0 v - w) + w]$$

[1] If we use an argument similar to the one which yielded the existence of s.

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or, which is equivalent, the vectors

$$v^* + \pi[\alpha(\beta_0 v - w) + w]$$

are positive for every v^* of V^*. Now if $\alpha > 1$, then for every v^* of V^* the vector

$$v^* - \alpha(\beta_0 v - w) - w = (\alpha - 1)\left[\frac{1}{\alpha - 1} v^* + w - \frac{\alpha}{\alpha - 1}\beta_0 v\right]$$

is positive by definition of β_0. Hence every vector

$$v^* - \pi[\alpha(\beta_0 v - w) + w]$$

is positive. Thus if we set, for instance,

$$w' = (2 + \gamma)\,(\beta_0 v - w) + w$$

then $v^* + \alpha w'$ is positive for every v^* in V^* and every α, that is to say, the subspace $V^* + [w']$ is definite.

Second Case. $\beta_0 = 0$. Then we first let w and v play the roles of $-v$ and w, respectively, and arrive at a γ_0 such that for $\alpha > \gamma_0$

$$v^* + \alpha w$$

is positive while one vector

$$w_1 = v^* + \gamma_0 w$$

is non-positive (thus obviously null). We let now this vector play the role of w_0 and exactly as before arrive at a vector w'' such that $V^* + [w'']$ is definite. This completes the proof of the Lemma.

Now let V be a finite-dimensional non-degenerate vector space. If V is a plane, then, by the Lemma, V contains a definite line. By induction we build an increasing chain of definite subspaces which, by the Lemma, can be continued as long as there exists a plane which is independent of the subspace. This is the case until we have reached a definite subspace, V', whose dimension is that of V minus 1. If V itself is not definite we can represent V as the sum of V' and the line of any vector not contained in V'. As such a vector we may use any non-positive vector v'. The opposite of v' is positive. We thus have proved the following

Theorem 3. *Every finite-dimensional non-degenerate space which is not definite is the sum of a definite subspace and the v'-line of any non-positive vector v' which is $\neq o$.*

Since every indefinite vector space obviously contains a nullvector besides o, and in a non-degenerate space the line of such a vector is semi-definite, we obtain the following

Corollary. *Each non-degenerate finite-dimensional vector space which is non definite is the sum of a definite subspace and a single semi-definite line.*

Combining Theorems 1 and 3 we can say

Each finite-dimensional vector space, V, can be represented as the sum of the following parts:

(1) *a uniquely determined totally degenerate subspace (which is O^* if and only if V is non-degenerate);*

(2) *a definite subspace (which may be O^*); to which, if and only if V con-*

tains a non-positive vector whose opposite is positive, we have to add a ***third*** *sub-space, namely,*

(3) *a single line for which we may choose the v-line of any* ***non-positive*** *vector.*

9. A Survey of All Finite-Dimensional Vector Spaces

The space O^*, and only this space, may be considered as 0-dimensional. There are four types of 1-dimensional spaces ("lines"): definite, semi-definite and indefinite lines (which are non-degenerate) and totally degenerate lines. By induction we see that there are the following types of n-dimensional spaces:

I. *Non-Degenerate Spaces.*

1. *Definite Spaces.*
2. *Semi-Definite Spaces.* We shall classify them by calling the dimension of the closed cone of null vectors of a space, its *degree.*
3. *Indefinite Spaces* with an n-dimensional cone of negative vectors whose boundary consists of the null vectors.

II. *Degenerate Spaces* which are the sum of a non-degenerate space of a dimension $< n$ (which we shall call the *rank* of the space) and a totally degenerate space.

In subsequent papers we shall refer to the definite, semi-definite and indefinite spaces as *elliptic*, *parabolic*, and *hyperbolic* spaces, respectively. The parabolic spaces of minimum degree, 1, will be called *properly* parabolic. We shall see that in a finite-dimensional vector space every closed cone, for a properly chosen definition of the norm, is the cone of the null vectors of a parabolic space; and that every open cone may be the cone of the negative vectors of an indefinite space. Hence every hyperbolic space can be made parabolic of maximum degree by redefining the norm of each negative vector as 0, while every parabolic space of maximum degree can be made hyperbolic by associating with the interior null vectors proper negative norms.

References

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[2]Cf., in particular, "Die metrische Methode in der Variationsrechnung," *Ergebn. mathem. Kolloquiums*, vol. 8 (1937), 1-32; two notes in the *Proc. Nat. Acad. Sci.*, vol. 23 (1937), 246 and vol. 25 (1939), 474; and the lecture "Analysis and Metric Geometry" in The Rice Institute Pamphlet, vol. 27 (1940), 1-40.

[3]*Ergebn. mathem. Kolloquiums*, vol. 8 (1937), 32.

[4]Cf., in particular, Pauc's résumé in the pamphlet "Les méthodes directes en Calcul des Variations," (Paris, Herman, 1941).

[5]*Rend. della Acc. Naz. Linc.*, vol. 26 (1937).

[6]*Ergebn. math. Kolloquiums*, vol. 8 (1937), 25 and Alt, loc. cit.[3]

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