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ON A FUNCTIONAL DIFFERENTIAL EQUATION IN LOCALLY CONVEX SPACES

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The notion of accretiveness for multi-valued nonlinear maps is defined in locally convex spaces and it is used to obtain a locally convex space version of a result of M.G. Crandall and J.A. Nohel.

0. Introduction

The aim of this note is to obtain a locally convex version of a result of Crandall and Nohel [2] about the existence of a unique solution of an initial value problem, where the functions involved have their values in a Banach space. The differential equation in the problem contains a multi-valued map on this Banach space. We shall replace this Banach space with a class of locally convex spaces. To carry out this project, we shall use a method which has been introduced in [6] and developed in [7] and [8]. We begin with a brief account of this method.

1. Γ-completions of locally convex spaces

Let E be a vector space and p be a semi-norm of E . A sequence (x_i) in E is said to be *p-Cauchy* if $p(x_i-x_j) \to 0$ as $i, j \to \infty$. Two p-Cauchy sequences (x_i) and (y_i) are said to be *equivalent* if $p(x_i-y_i) \to 0$ as $i \to \infty$.

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Let (x_i) be a p-Cauchy sequence and $\underline{\underline{x}}$ be the set of all p-Cauchy sequences in E which are equivalent to (x_i) . Such a set $\underline{\underline{x}}$ is called a *p-class* on E . The set of all p-classes on E will be denoted by $E[p]$ and it will be called the *p-completion* of E . It is a vector space when $\alpha\underline{\underline{x}} + \beta\underline{\underline{y}}$ is defined to be the p-class which contains the sequence $(\alpha x_i + \beta y_i)$ for some $(x_i) \in \underline{\underline{x}}$ and $(y_i) \in \underline{\underline{y}}$. The zero element of $E[p]$ is, therefore, the p-class which contains a p-null sequence.

For $\underline{\underline{x}} \in E[p]$, we define

$$p(\underline{\underline{x}}) = \lim_{i\to\infty} p(x_i) \quad \text{for} \quad (x_i) \in \underline{\underline{x}} \text{ .}$$

Then the value $p(\underline{\underline{x}})$ does not depend on the choice of (x_i) from $\underline{\underline{x}}$. It is obvious that p is a norm on $E[p]$ and, with this norm, $E[p]$ is a Banach space.

For each $x \in E$, let $S_p(x)$ be the element of $E[p]$ which contains the p-Cauchy sequence whose terms are identical to x . Then we have

$$p\big(S_p(x)\big) = p(x) \quad \text{for every} \quad x \in E \text{ .}$$

For $\underline{\underline{x}} \in E[p]$ and $(x_i) \in \underline{\underline{x}}$, we have

$$\lim_{i\to\infty} p\big(S_p(x_i)-\underline{\underline{x}}\big) = 0 \text{ ,}$$

which shows that $S_p(E)$ is a dense subset of $E[p]$.

Let E be a locally convex space. A directed set Γ of semi-norms on E which induces the topology of E will be called a *calibration* for E . Then, for each $p \in \Gamma$, we have the p-completion $E[p]$ of E . The family $\{E[p] : p \in \Gamma\}$ of Banach spaces will be called the *Γ-completion* of E . Thus we have a projective system

$$S_p : E \to E[p] \quad \text{for all} \quad p \in \Gamma \text{ .}$$

It is easy to see that the projective topology on E defined by this system coincides with the topology of E .

When $q \geq p$ in Γ , that is, $q(x) \geq p(x)$ for every $x \in E$, we have the natural embedding

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$$T_{q,p} : E[q] \to E[p] \ ,$$

which maps every $\underline{\underline{x}} \in E[q]$ to the p-class which contains elements of $\underline{\underline{x}}$. Obviously, this map is linear,

$$p\left(T_{q,p}(\underline{\underline{x}})\right) \le q(\underline{\underline{x}}) \quad \text{for every} \quad \underline{\underline{x}} \in E[q]$$

and

$$T_{q,p} \circ S_q = S_p \ .$$

Furthermore, it is evident that $T_{q,p}(E[q])$ is a dense subset of $E[p]$.

The following fact will be used frequently. For the proof, we refer to [3], p. 231.

(1.1). *Let E be a locally convex space and Γ be a calibration for E. Then E is complete if and only if the following condition is satisfied: if $\underline{\underline{x}}_p \in E[p]$ for all $p \in \Gamma$ and*

$$T_{q,p}\left(\underline{\underline{x}}_q\right) = \underline{\underline{x}}_p \quad \textit{whenever} \quad q \ge p \quad \textit{in} \quad \Gamma \ ,$$

then there exists $x \in E$ such that $S_p(x) = \underline{\underline{x}}_p$ for all $p \in \Gamma$.

2. Γ-extensions of multi-valued maps

Let E and F be locally convex spaces and let Γ be a calibration for (E, F). In other words, each $p \in \Gamma$ has the E-component p_E and the F-component p_F and

$$\Gamma_E = \{p_E : p \in \Gamma\} \quad \text{and} \quad \Gamma_F = \{p_F : p \in \Gamma\}$$

are calibrations for E and F respectively. We shall denote the embeddings S_{p_E} and S_{p_F} by the same S_p.

Let A be a multi-valued map of E into F, that is, A is a subset of the product $E \times F$. For $p \in \Gamma$ and $[x, y] \in A$, we set

$$S_p([x, y]) = \left[S_p(x), S_p(y)\right] \ .$$

Then

$$S_p(A) \subset E[p] \times F[p] \ ,$$

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and we set

$$A_p = \overline{S_p(A)} ,$$

where the closure is taken in the product $E[p] \times F[p]$ of Banach spaces $E[p]$ and $F[p]$. Hence A_p is always closed and it is easy to see that $A_p = (\overline{A})_p$.

(2.1). $\overline{A} = \bigcap_{p \in \Gamma} S_p^{-1}(A_p)$.

Proof. Since $S_p(A) \subset A_p$, we have

$$S_p(\overline{A}) \subset \overline{S_p(A)} = A_p \quad \text{for all } p \in \Gamma .$$

To prove the converse, assume that there exists $[x, y] \in S_p^{-1}(A_p)$ for all $p \in \Gamma$ such that $[x, y] \notin \overline{A}$. Then, since Γ is directed, there exist $p \in \Gamma$ and $\alpha > 0$ such that

$$\big([x, y]+U_E(p, \alpha) \times U_F(p, \alpha)\big) \cap A = \emptyset ,$$

where $U_E(p, \alpha)$ and $U_F(p, \alpha)$ are open p-balls around zeros with radius α in the spaces E and F respectively. However, for this p, since $S_p([x, y]) \in A_p$, we can choose $[x_i, y_i] \in A$ such that $S_p([x_i, y_i]) \to S_p([x, y])$, which is a contradiction.

As usual, the domain of A is denoted by $D(A)$.

(2.2). *(i)* $\overline{D(A)} = \bigcap_{p \in \Gamma} S_p^{-1}\big(\overline{D(A_p)}\big)$.

(ii) $\overline{D(A_p)} = \overline{S_p(D(A))}$.

Proof. Let $x \in \overline{D(A)}$ and choose a net (x_λ) in $D(A)$ such that $x_\lambda \to x$. Then $S_p(x_\lambda) \in D(A_p)$ and $S_p(x_\lambda) \to S_p(x)$. Hence $S_p(x) \in \overline{D(A_p)}$, which holds for every $p \in \Gamma$. Conversely, assume that $S_p(x) \in \overline{D(A_p)}$ for every $p \in \Gamma$ and $x \notin \overline{D(A)}$. We choose $p \in \Gamma$ and $\alpha > 0$ such that

$$\big(x+U_E(p, \alpha)\big) \cap D(A) = \emptyset .$$

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For this p, since $S_p(x) \in \overline{D(A_p)}$, we can find $\underline{\underline{x}}_n \in D(A_p)$ such that $\underline{\underline{x}}_n \to S_p(x)$. Since there exist $x_n \in D(A)$ such that

$$p\left(S_p\left(x_n\right)-\underline{\underline{x}}_n\right) < 1/n ,$$

we can conclude that $S_p\left(x_n\right) \to S_p(x)$, which is a contradiction. Thus *(i)* was proved. The proof of *(ii)* is similar.

(2.3). *Assume that* $q \geq p$ *in* Γ. *Then, for every* $\underline{\underline{x}}_q \in D\left(A_q\right)$,

(i) $T_{q,p}\underline{\underline{x}}_q \in D\left(A_p\right)$,

(ii) $T_{q,p}A_q\underline{\underline{x}}_q \subset A_pT_{q,p}\underline{\underline{x}}_q$.

Proof. For $\underline{\underline{x}}_q \in D\left(A_q\right)$, assume that $\left[\underline{\underline{x}}_q, \underline{\underline{y}}_q\right] \in A_q$ and choose $\left[x_i, y_i\right] \in A$ such that $S_q\left(\left[x_i, y_i\right]\right) \to \left[\underline{\underline{x}}_q, \underline{\underline{y}}_q\right]$. Then

$$S_p\left(\left[x_i, y_i\right]\right) = T_{q,p} \circ S_q\left(\left[x_i, y_i\right]\right) \to T_{q,p}\left[\underline{\underline{x}}_q, \underline{\underline{y}}_q\right] ,$$

where we used the following notation:

$$T_{q,p}\left[\underline{\underline{x}}_q, \underline{\underline{y}}_q\right] = \left[T_{q,p}\underline{\underline{x}}_q, T_{q,p}\underline{\underline{y}}_q\right] .$$

Thus we have *(i)* and *(ii)*.

3. Surjectivity

Let Γ be a calibration for (E, F) and $A \subset E \times F$ be a multi-valued map. The range of A will be denoted by $R(A)$.

(3.1). *Assume that*

(i) E *is complete,*

(ii) A_p^{-1} *is a single-valued map for every* $p \in \Gamma$,

(iii) $R\left(A_p\right) = F[p]$ *for every* $p \in \Gamma$.

Then $R(\overline{A}) = F$.

Proof. Let $y \in F$. Then, by *(iii)*, there exists $\underline{\underline{x}}_p \in D\left(A_p\right)$ such that $\left[\underline{\underline{x}}_p, S_p(y)\right] \in A_p$ for every $p \in \Gamma$. Assume that $q \geq p$ in Γ. Then, by (2.3),

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$$T_{q,p}A_{q}\underline{\underline{x}}_q \subset A_p T_{q,p}\underline{\underline{x}}_q$$

and, since $S_q(y) \in A_q\underline{\underline{x}}_q$,

$$S_p(y) = T_{q,p} \circ S_q(y) \in A_p T_{q,p}\underline{\underline{x}}_q ,$$

or $[T_{q,p}\underline{\underline{x}}_q, S_p(y)] \in A_p$. Then, by *(ii)*, we have $T_{q,p}\underline{\underline{x}}_q = \underline{\underline{x}}_p$. Therefore, by *(i)* and (1.1), there exists $x \in E$ such that $S_p(x) = \underline{\underline{x}}_p$ for all $p \in \Gamma$. Hence, by (2.1), $[x, y] \in \bar{A}$, or $y \in R(\bar{A})$.

The converse of (3.1) is given by the following.

(3.2). *Assume that*

(i) $R(A) = F$,

(ii) *if* $[x_i, y_i] \in A$ *and* (y_i) *is a p-Cauchy sequence for some* $p \in \Gamma$, *then* (x_i) *is also p-Cauchy.*

Then $R(A_p) = F[p]$.

Proof. Let $\underline{\underline{y}} \in F[p]$ and $(y_i) \in \underline{\underline{y}}$. Then, by *(i)*, we can choose $x_i \in D(A)$ such that $[x_i, y_i] \in A$. By *(ii)*, (x_i) is p-Cauchy. Hence $(x_i) \in \underline{\underline{x}}$ for some $\underline{\underline{x}} \in E[p]$. Then it is the definition of A_p that $[\underline{\underline{x}}, \underline{\underline{y}}] \in A_p$.

4. Γ-contractions

Let Γ be a calibration for E . Then a map f of a subset $D(f)$ of E into E is said to be a Γ-*contraction* if

$$p\big(f(x)-f(y)\big) \le p(x-y)$$

for all $p \in \Gamma$ and $x, y \in D(f)$. When f is a linear map, it is a Γ-contraction if and only if $p\big(f(x)\big) \le p(x)$ for all $p \in \Gamma$ and $x \in D(f)$. In this case, the following theorem of Moore [4] is of fundamental importance.

(4.1). *Let* E *be a locally convex space and* S *be an algebraic semigroup of continuous linear maps of* E *into* E . *Then* S *is equicontinuous if and only if there is a calibration* Γ *for* E *such that* S

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consists of Γ*-contractions.*

From this theorem we can immediately obtain a sufficient condition for a nonlinear map f to be a Γ-contraction for some Γ. We recall that f is said to be Gâteaux-differentiable on $D(f)$ if, for each $a \in D(f)$ and $x \in E$, the limit

$$\lim_{\varepsilon \to 0} \varepsilon^{-1}\left(f(a+\varepsilon x)-f(a)\right) = f'(a)(x)$$

exists and $f'(a)$ is a continuous linear map on E.

(4.2). *Let* f *be a Gâteaux-differentiable map on a convex subset* $D(f)$ *into* E. *If the set* $\{f'(x) : x \in D(f)\}$ *is contained in an equicontinuous semigroup, there exists a calibration* Γ *for* E *such that* f *is a* Γ*-contraction.*

The proof is an immediate consequence of the mean-value theorem (see [5], p. 15).

When f is a Γ-contraction and $p \in \Gamma$, $\left(f\left(x_i\right)\right)$ is a p-Cauchy sequence whenever $\left(x_i\right)$ is a p-Cauchy sequence. Hence, for every $\underline{\underline{x}} \in \overline{S_p\left(D(f)\right)}$, we can set

$$f_p(\underline{\underline{x}}) = \lim_{i \to \infty} S_p\left(f\left(x_i\right)\right) .$$

Then f_p is a contraction of $\overline{S_p\left(D(f)\right)}$ into $E[p]$ and

$$f_p \circ S_p = S_p \circ f .$$

We shall use the following fact later.

(4.3). *Let* f *be a* Γ*-contraction and* $p \in \Gamma$. *If* $x_i \in D(f)$ *and* $S_p\left(x_i\right) \to \underline{\underline{x}}$ *for some* $\underline{\underline{x}} \in E[p]$, *then* $S_p\left(f\left(x_i\right)\right) \to f_p(\underline{\underline{x}})$.

5. Γ-accretive maps

Let Γ be a calibration for E. We shall also denote by Γ the calibration for (E, E) with the identical components.

A map $A \subset E \times E$ is said to be Γ-*accretive* if, for every $\lambda > 0$, $(1+\lambda A)^{-1}$ is a single-valued Γ-contraction. If, furthermore,

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$R(1+\lambda A) = E$, then A is said to be *m-*Γ*-accretive*.

(5.1). *For any map* $A \subset E \times E$ *and* $\lambda > 0$,

(i) $(1+\lambda A)_p = 1 + \lambda A_p$ *for all* $p \in \Gamma$,

(ii) $\left((1+\lambda A)^{-1}\right)_p = \left(1+\lambda A_p\right)^{-1}$ *for all* $p \in \Gamma$.

Proof. If $[\underline{\underline{x}}, \underline{\underline{y}}] \in (1+\lambda A)_p$, we can choose $[x_i, y_i] \in 1+\lambda A$ such that $S_p([x_i, y_i]) \to [\underline{\underline{x}}, \underline{\underline{y}}]$. Then $\left[x_i, \lambda^{-1}(y_i-x_i)\right] \in A$ and

$$S_p\left(\left[x_i, \lambda^{-1}(y_i-x_i)\right]\right) \to [\underline{\underline{x}}, \lambda^{-1}(\underline{\underline{y}}-\underline{\underline{x}})] .$$

Hence $[\underline{\underline{x}}, \lambda^{-1}(\underline{\underline{y}}-\underline{\underline{x}})] \in A_p$ which implies $[\underline{\underline{x}}, \underline{\underline{y}}] \in 1+\lambda A_p$. The converse and *(ii)* can be proved similarly.

We are going to express the Γ-accretiveness of A in terms of the accretiveness of the family $\{A_p : p \in \Gamma\}$ of maps on Banach spaces.

(5.2). A *is* Γ*-accretive if and only if every* A_p *is accretive.*

Proof. Assume that A is Γ-accretive and $p \in \Gamma$. Then, for $J_\lambda = (1+\lambda A)^{-1}$, we have

$$p\left(J_\lambda x-J_\lambda y\right) \le p(x-y) \quad \text{for} \quad x, y \in R(1+\lambda A) .$$

Let $\underline{\underline{x}}, \underline{\underline{y}} \in R\left(1+\lambda A_p\right)$. Then, by (5.1) *(i)*, there exist $x_i, y_i \in R(1+\lambda A)$ such that

$$S_p\left(x_i\right) \to \underline{\underline{x}} \quad \text{and} \quad S_p\left(y_i\right) \to \underline{\underline{y}} .$$

Then, by (4.3), we have

$$S_p\left(J_\lambda x_i\right) \to \left(J_\lambda\right)_p(\underline{\underline{x}}) \quad \text{and} \quad S_p\left(J_\lambda y_i\right) \to \left(J_\lambda\right)_p(\underline{\underline{y}}) .$$

Hence

$$p\left(\left(J_\lambda\right)_p(\underline{\underline{x}})-\left(J_\lambda\right)_p(\underline{\underline{y}})\right) = \lim_{i\to\infty} p\left(J_\lambda x_i-J_\lambda y_i\right)$$
$$\le \lim_{i\to\infty} p\left(x_i-y_i\right) = p(\underline{\underline{x}}-\underline{\underline{y}}) .$$

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Since $(J_\lambda)_p = (J_p)_\lambda$ by (5.1) *(ii)*, we have proved that A_p is accretive. Conversely, assume that every A_p is accretive and $x, y \in R(1+\lambda A)$. Then, for every $p \in \Gamma$, $S_p(x), S_p(y) \in R(1+\lambda A_p)$. Therefore,

$$p\left((J_p)_\lambda S_p(x)-(J_p)_\lambda S_p(y)\right) \le p(x-y) .$$

However, we always have $(J_p)_\lambda \circ S_p = S_p \circ J$. Hence

$$p(J_\lambda x-J_\lambda y) \le p(x-y) ,$$

which shows that A is Γ-accretive.

(5.3). *(i) If A is m-Γ-accretive, every A_p is m-accretive.*

(ii) If E is complete, A is closed and every A_p is m-accretive, then A is m-Γ-accretive.

Proof. *(i)* By the assumption, $R(1+\lambda A) = E$. To prove that the condition (3.2) *(ii)* is satisfied for $1 + \lambda A$, assume that $[x_i, y_i] \in 1+\lambda A$ and (y_i) is p-Cauchy. Then, since $x_i = J_\lambda y_i$ and J_λ is a Γ-contraction, (x_i) is also p-Cauchy. Hence, by (3.2), we have $R(1+\lambda A_p) = E[p]$.

(ii) We prove that the conditions *(i)*, *(ii)* and *(iii)* in (3.1) are satisfied for $1 + \lambda A$. However *(i)* is a part of our assumptions and *(ii)* is included in the definition of A_p being accretive. Finally, *(iii)* is satisfied by $(1+\lambda A)_p$. Thus we have $R(\overline{1+\lambda A}) = E$. However, $\overline{1 + \lambda A} = 1 + \lambda\overline{A}$ and, since A is closed, we arrive at $R(1+\lambda A) = E$.

The following facts are immediate consequences of the definition of m-Γ-accretiveness and the fact that m-Γ-accretive maps are "maximal" Γ-accretive.

(5.4). *(i) A m-Γ-accretive subset is closed in $E \times E$.*

(ii) If A is m-Γ-accretive and $x \in D(A)$, then Ax is closed.

6. Some function spaces

Let E be a locally convex space equipped with a calibration Γ.

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For a positive number T , we shall denote the closed interval $[0, T]$ by I , and let $C(I, E)$ be the set of all continuous functions of I into E . Then, for each $p \in \Gamma$ and $u \in C(I, E)$, $p \circ u$ is a real-valued continuous function defined on the compact subset I . Therefore, we can set

$$p_t(u) = \sup\{p(u(s)) : 0 \le s \le t\}$$

for $t \in I$, and we set $p^\infty(u) = p_T^\infty(u)$ for $u \in C(I, E)$. Then the set

$$\Gamma^\infty = \{p^\infty : p \in \Gamma\}$$

defines a locally convex topology on $C(I, E)$ and it is complete if E is complete.

Let A be a multi-valued map on E and we denote by $C(I, \overline{D(A)})$ the set of all continuous functions of I into $\overline{D(A)}$. Then it is a closed subset of $C(I, E)$.

We shall denote the p-completion of $C(I, E)$ by $C(I, E)[p]$. Then we have the natural embedding

$$S_p : C(I, E) \to C(I, E)[p] .$$

We now set

$$C(I, \overline{D(A)})[p] = \overline{S_p(C(I, \overline{D(A)}))} .$$

(6.1). *If* $\overline{D(A)}$ *is convex,* $C(I, \overline{D(A)})[p] = C(I, \overline{D(A_p)})$.

Proof. By (2.2) *(ii)*, we have

$$S_p(C(I, \overline{D(A)})) \subset C(I, \overline{D(A_p)})$$

and $C(I, \overline{D(A_p)})$ is closed. Hence we only need to show that $C(I, \overline{D(A_p)}) \subset C(I, \overline{D(A)})[p]$. Now let $\underline{\underline{v}} \in C(I, \overline{D(A_p)})$ and we set

$$\underline{\underline{v}}_n(t) = \frac{kT-nt}{kT} \underline{\underline{a}}_{k-1} + \frac{nt}{kT} \underline{\underline{a}}_k \quad \text{if} \quad \frac{(k-1)T}{n} \le t \le \frac{kT}{n}$$

for $k = 1, 2, \ldots, n$, where

$$\underline{\underline{a}}_k = \underline{\underline{v}}(kT/n) \quad \text{for} \quad k = 0, 1, \ldots, n .$$

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Then, since $\overline{D(A_p)}$ is convex, $\underline{\underline{v}}_n \in C(I, \overline{D(A_p)})$ and $\lim_{n\to\infty} \underline{\underline{v}}_n = \underline{\underline{v}}$ in the Banach space $C(I, E[p])$. Since $\underline{\underline{a}}_k \in \overline{D(A_p)}$, we can choose $a_{k,i} \in D(A)$ such that $S_p(a_{k,i}) \to \underline{\underline{a}}_k$ for $k = 0, 1, \ldots, n$, and we define functions $v_{n,i} : I \to E$ by

$$v_{n,i}(t) = \frac{kT-nt}{kT} a_{k-1,i} + \frac{nt}{kT} a_{k,i} \quad \text{if} \quad \frac{(k-1)T}{n} \le t \le \frac{kT}{n} .$$

Then, since $\overline{D(A)}$ is convex, $v_{n,i} \in C(I, \overline{D(A)})$ and it is easy to see that

$$p^{\infty}(S_p \circ v_{n,i} - \underline{\underline{v}}_n) \to 0 \quad \text{as} \quad i \to \infty .$$

This means that $(v_{n,i})$ is a p-Cauchy sequence in $C(I, \overline{D(A)})$ and $(v_{n,i}) \in \underline{\underline{v}}_n$. Hence $\underline{\underline{v}}_n \in C(I, \overline{D(A)})[p]$. Since $C(I, \overline{D(A)})[p]$ is closed, $\underline{\underline{v}} \in C(I, \overline{D(A)})[p]$.

We note that, when A is an m-accretive map on a Banach space which, together with its dual, is uniformly convex, then $\overline{D(A)}$ is conves. (See Barbu [1], p. 77, Proposition 3.6.) We shall call a calibration Γ *dually uniformly convex* if, for every $p \in \Gamma$, $E[p]$ and its dual are uniformly convex. Obviously $L^p_{\text{loc}}(R)$ admits such a calibration if $1 < p < \infty$. It is also known that every nuclear space admits a Hilbert calibration which is obviously dually uniformly convex.

Let, as before, E be a locally convex space equipped with a calibration Γ, and let $L^1(I, E)$ be the set of all integrable functions of I into E. Then, for each $p \in \Gamma$ and $u \in L^1(I, E)$, we can set

$$p^1_t(u) = \int_0^t p(u(s))ds \quad \text{for} \quad t \in I ,$$

and we also set $p^1(u) = p^1_T(u)$. Then the set

$$\Gamma^1 = \{p^1 : p \in \Gamma\}$$

defines a locally convex topology on $L^1(I, E)$ and it is complete if E is complete. In the same manner as in the case of $C(I, \overline{D(A)})$, we can prove that

$$L^1(I, E)[p] = L^1(I, E[p]) \quad \text{for every} \quad p \in \Gamma .$$

In the proof of $L^1(I, E[p]) \subset L^1(I, E)[p]$ we can approximate elements of $L^1(I, E[p])$ by step functions which are, as can be proved easily, contained in $L^1(I, E)[p]$.

7. A result of Crandall and Nohel

Let E be a Banach space and $A \subset E \times E$ be an m-accretive map. For $a \in \overline{D(A)}$, let us consider the following initial value problem:

$$(*) \qquad \begin{cases} \frac{d}{dt} u(t) + Au(t) \ni Gu(t) \quad (t \in I) , \\ u(0) = a , \end{cases}$$

where

$$G : C\left(I, \overline{D(A)}\right) \to L^1(I, E) .$$

It has been proved in [2] that the problem (*) has a unique solution in $C\left(I, \overline{D(A)}\right)$ if there exists $\gamma \in L^1(I, R)$ such that

$$(**) \qquad p_t^1\left(G(u)-G(v)\right) \le \int_0^t \gamma(s) p_s^\infty(u-v) ds$$

for all $u, v \in C\left(I, \overline{D(A)}\right)$ and $t \in I$, where p denotes the norm of E .

(7.1). *Let* E *be a complete locally convex space equipped with a dually uniformly convex calibration* Γ . *If* $A \subset E \times E$ *is an* m-Γ-*accretive map and the map* G *satisfies the condition* (**) *for every* $p \in \Gamma$, *then, for each* $a \in \overline{D(A)}$, *the initial value problem* (*) *has a unique solution in* $C\left(I, \overline{D(A)}\right)$.

Proof. The condition (**) implies

$$p^1\left(G(u)-G(v)\right) \le \left(\int_0^T \gamma(s) ds\right) p^\infty(u-v)$$

for $u, v \in C\left(I, \overline{D(A)}\right)$. Hence $\left(G\left(u_i\right)\right)$ is p^1-Cauchy whenever $\left(u_i\right)$ is p^∞-Cauchy. Therefore, by (6.1), we can define a map

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$$G_p : C\left(I, \overline{D(A_p)}\right) \to L^1(I, E[p])$$

such that

$$G_p\left(S_p \circ u\right) = S_p \circ G(u) \quad \text{for every} \quad u \in C\left(I, \overline{D(A)}\right) ,$$

and, if $q \geq p$ in Γ ,

$$T_{q,p}\left(G_q\left(\underline{\underline{u}}_q\right)(t)\right) = G_p\left(T_{q,p} \circ \underline{\underline{u}}_q\right)(t)$$

for all $\underline{\underline{u}}_q \in C\left(I, \overline{D(A_p)}\right)$ and $t \in I$. Furthermore, G_p satisfies the condition (**). Hence, for each $p \in \Gamma$, the initial value problem

$$\begin{cases} \frac{d}{dt}\,\underline{\underline{u}}(t) + A_p\underline{\underline{u}}(t) \ni G_p(\underline{\underline{u}})(t) & (t \in I) \\ \underline{\underline{u}}(0) = S_p(a) \end{cases}$$

has a unique solution $\underline{\underline{u}}_p$ in $C\left(I, \overline{D(A_p)}\right)$. Then, if $q \geq p$ in Γ ,

$$T_{q,p}\underline{\underline{u}}_q(0) = T_{q,p}\left(S_q(a)\right) = S_p(a) ,$$

and

$$T_{q,p}\left(\frac{d}{dt}\,\underline{\underline{u}}_q(t)\right) + T_{q,p}\left(A_q\underline{\underline{u}}_q(t)\right) \ni T_{q,p}\left(G_q\left(\underline{\underline{u}}_q\right)(t)\right) ,$$

which, by (2.3), implies

$$\frac{d}{dt}\,T_{q,p}\underline{\underline{u}}_q(t) + A_pT_{q,p}\underline{\underline{u}}_q(t) \ni G_p\left(T_{q,p} \circ \underline{\underline{u}}_q\right)(t) \quad (t \in I) .$$

Hence, by the unicity of the solution,

$$T_{q,p}\underline{\underline{u}}_q(t) = \underline{\underline{u}}_p(t) \quad \text{for all} \quad t \in I .$$

Since E is complete, we can apply (1.1) to find $u(t) \in E$ such that

$$\underline{\underline{u}}_p(t) = S_p\left(u(t)\right) \quad \text{for all} \quad p \in \Gamma \quad \text{and} \quad t \in I .$$

Then $u(t)$ is continuous with respect to t and, by (2.2),

$$u(t) \in \bigcap_{p\in\Gamma} S_p^{-1}\left(\overline{D(A_p)}\right) = \overline{D(A)}$$

Hence $u \in C\left(I, \overline{D(A)}\right)$ and, furthermore,

$$S_p\left(\frac{d}{dt}\,u(t)+Au(t)\right) \ni S_p\left(G(u)(t)\right) \quad \text{for all} \quad t \in I \quad \text{and} \quad p \in \Gamma .$$

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Then, by the lemma which will be proved below, we have

$$\frac{d}{dt} u(t) + Au(t) \ni G(u)(t) \quad \text{for all} \quad t \in I .$$

The lemma referred to in the above is the following.

(7.2). *Assume that* B *is a closed subset of* E *and*

$$S_p(x) \in S_p(B) \quad \textit{for every} \quad p \in \Gamma .$$

Then $x \in B$.

Proof. For each $p \in \Gamma$ we choose $b_p \in B$ such that $S_p(x) = S_p(b_p)$, which means that $p(x-b_p) = 0$. Then the net (b_p) converges to x . Since B is closed, we have $x \in B$.

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