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ON SUBMANIFOLDS WITH TAMED SECOND FUNDAMENTAL FORM

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Abstract. Based on the ideas of Bessa, Jorge and Montenegro (*Comm. Anal. Geom.*, vol. 15, no. 4, 2007, pp. 725–732) we show that a complete submanifold M with tamed second fundamental form in a complete Riemannian manifold N with sectional curvature $K_N \leq \kappa \leq 0$ is proper (compact if N is compact). In addition, if N is Hadamard, then M has finite topology. We also show that the fundamental tone is an obstruction for a Riemannian manifold to be realised as submanifold with tamed second fundamental form of a Hadamard manifold with sectional curvature bounded below.

1. Introduction. Let $\varphi : M \hookrightarrow N$ be an isometric immersion of a complete Riemannian m-manifold M into a complete Riemannian n-manifold N with sectional curvature $K_N \leq \kappa \leq 0$. Fix a point $x_0 \in M$, and let $\rho_M(x) = \mathrm{dist}_M(x_0, x)$ be the distance function on M to x_0. Let $\{C_i\}_{i=1}^{\infty}$ be an exhaustion sequence of M by compact sets with $x_0 \in C_0$. Let $\{a_i\} \subset [0, \infty]$ be a non-increasing sequence of possibly extended numbers defined by

$$a_i = \sup\left\{\frac{S_\kappa}{C_\kappa}(\rho_M(x)) \cdot \|\alpha(x)\|,\ x \in M\backslash C_i\right\},$$

where

$$S_\kappa(t) = \begin{cases} \dfrac{1}{\sqrt{-\kappa}}\sinh(\sqrt{-\kappa}\,t) & \text{if} \quad \kappa < 0, \\ \quad t & \text{if} \quad \kappa = 0; \end{cases} \tag{1}$$

$C_\kappa(t) = S'_\kappa(t)$ and $\|\alpha(x)\|$ is the norm of the second fundamental form at $\varphi(x)$. The number $a(M) = \lim\limits_{i\to\infty} a_i$ does not depend on the exhaustion sequence $\{C_i\}$ nor on the base point x_0.

Definition 1.1. An immersion $\varphi : M \hookrightarrow N$ of a complete Riemannian m-manifold M into an n-manifold N with sectional curvature $K_N \leq \kappa \leq 0$ has tamed second fundamental form if $a(M) < 1$.

In [**4**], Bessa, Jorge and Montenegro showed that a complete submanifold $\varphi : M \hookrightarrow \mathbb{R}^n$ with tamed second fundamental form is proper and has finite topology, where finite

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topology means that M is C^∞-diffeomorphic to a compact smooth manifold $\overline{M}$ with boundary. In this paper we show that the ideas of Bessa, Jorge and Montenegro can be adapted to show that a complete submanifold $M \hookrightarrow N$ with tamed second fundamental form is proper. In addition if N is a Hadamard manifold, then M has finite topology. We prove the following theorem.

THEOREM 1.2. *Let $\varphi : M \hookrightarrow N$ be an isometric immersion of a complete m-manifold M into complete Riemannian n-manifold N with sectional curvature $K_N \leq \kappa \leq 0$. Suppose that M has tamed second fundamental form. Then*

(*a*) *M is compact if N is compact;*
(*b*) *φ is proper if N is non-compact;*
(*c*) *M has finite topology if N is a Hadamard manifold.*

REMARK 1.3. Jorge and Meeks [**10**] showed that any complete m-dimensional submanifold M of $\mathbb{R}^n$ homeomorphic to a compact Riemannian manifold $\overline{M}$, punctured at finite number of points $\{p_1, \ldots, p_r\}$ and having a well-defined normal vector at infinity has $a(M) = 0$. This class of submanifolds includes all the complete minimal surfaces $\varphi\colon M^2 \hookrightarrow \mathbb{R}^n$ with finite total curvature $\int_M |K| < \infty$ studied by Chern and Osserman [**7**, **14**], all the complete surfaces $\varphi\colon M^2 \hookrightarrow \mathbb{R}^n$ with finite total scalar curvature $\int_M |\alpha|^2 dV < \infty$ and non-positive curvature with respect to every normal direction studied by White [**16**] and the m-dimensional minimal submanifolds $\varphi\colon M^m \hookrightarrow \mathbb{R}^n$ with finite total scalar curvature $\int_M |\alpha|^m dV < \infty$ studied by Anderson [**1**]. In [**13**], G. Oliveira Filho proved a version of Anderson's theorem for complete minimal submanifolds of $\mathbb{H}^n$ with finite total curvature $\int_M |\alpha|^m dV < \infty$.

Our second result shows that the fundamental tone $\lambda^*(M)$ can be an obstruction for a Riemannian manifold M to be realised as a submanifold with tamed second fundamental form in a Hadamard manifold with bounded sectional curvature. The fundamental tone of a Riemannian manifold M is given by

$$\lambda^*(M) = \inf\left\{\frac{\int_M |\mathrm{grad} f|^2}{\int_M f^2}, f \in H_0^1(M)\backslash\{0\}\right\}, \tag{2}$$

where $H_0^1(M)$ is the completion of $C_0^\infty(M)$ with respect to the norm $|f|^2 = \int_M f^2 + \int_M |\mathrm{grad} f|^2$. We prove the following theorem.

THEOREM 1.4. *Let $\varphi : M \hookrightarrow N$ be an isometric immersion of a complete m-manifold M with $a(M) < 1$ into a Hadamard n-manifold N with sectional curvature $\mu \leq K_N \leq 0$. Given c, $a(M) < c < 1$, there exists $l = l(m, c) \in \mathbb{Z}_+$ and a positive constant $C = C(m, c, \mu)$ such that*

$$\lambda^*(M) \leq C \cdot \lambda^*(\mathbb{N}^l(\mu)) = C \cdot (l-1)^2 \mu^2/4, \tag{3}$$

where $\mathbb{N}^l(\mu)$ is the l-dimensional simply connected space form of sectional curvature μ.

REMARK 1.5. As corollary of Theorem (1.4) we have that $\lambda^*(M) = 0$ for any submanifold M mentioned in this list above.

Question 1.5. *It is known* [**3**, **5**] *that the fundamental tones of the Nadirashvili bounded minimal surfaces* [**12**] *and the Martin–Morales cylindrically bounded minimal surfaces* [**11**] *are positive. We ask if there is a complete properly immersed (minimal) submanifold of the $\mathbb{R}^n$ with positive fundamental tone $\lambda^* > 0$.*

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2. Preliminaries. Let $\varphi : M \hookrightarrow N$ be an isometric immersion, where M and N are complete Riemannian manifolds. Consider a smooth function $g : N \to \mathbb{R}$ and the composition $f = g \circ \varphi : M \to \mathbb{R}$. Identifying X with $d\varphi(X)$ we have at $q \in M$ and for every $X \in T_qM$ that

$$\langle \text{grad} f, X\rangle = df(X) = dg(X) = \langle \text{grad} g, X\rangle.$$

Hence we write

$$\text{grad} g = \text{grad} f + (\text{grad} g)^{\perp},$$

where $(\text{grad} g)^{\perp}$ is perpendicular to T_qM. Let ∇ and $\bar{\nabla}$ be the Riemannian connections on M and N respectively, and let $\alpha(x)(X, Y)$ and $\text{Hess} f(x)(X, X)$ be respectively the second fundamental form of the immersion φ and the Hessian of f at x with $X, Y \in T_xM$. Using the Gauss equation we have that

$$\text{Hess} f(x)(X, Y) = \text{Hess} g(\varphi(x))(X, Y) + \langle \text{grad} g, \alpha(X, Y)\rangle_{\varphi(x)}. \tag{4}$$

Taking the trace in (4), with respect to an orthonormal basis $\{e_1, ..., e_m\}$ for T_xM, we have that

$$\begin{aligned}\Delta f(x) &= \sum_{i=1}^{m} \text{Hess} f(q)(e_i, e_i)\\ &= \sum_{i=1}^{m} \text{Hess} g(\varphi(x))(e_i, e_i) + \langle \text{grad} g, \sum_{i=1}^{m} \alpha(e_i, e_i)\rangle. \end{aligned}\tag{5}$$

We should mention that formulas (4) and (5) first appeared in [**9**]. If $g = h \circ \rho_N$, where $h : \mathbb{R} \to \mathbb{R}$ is a smooth function and ρ_N is the distance function to a fixed point in N, then equation (4) becomes

$$\text{Hess} f(x)(X, X) = h''(\rho_N)\langle \text{grad}\rho_N, X\rangle^2 + h'(\rho_N)[\text{Hess}\rho_N(X, X) + \langle \text{grad}\rho_N, \alpha(X, X)\rangle]. \tag{6}$$

Another important tool in this paper the Hessian comparison theorem (see [**9**] or [**15**]).

THEOREM 2.1 *Hessian comparison theorem. Let N be a complete Riemannian n-manifold and $y_0, y \in N$. Let $\gamma : [0, \rho_N(y)] \to N$ be a minimising geodesic joining y_0 and y, where ρ_N is the distance function to y_0 on N. Let K_γ be the sectional curvatures of N along γ. Denote by $\mu = \inf K_\gamma$ and $\kappa = \sup K_\gamma$. Then for all $X \in T_yN$, $X \perp \gamma'(\rho_N(y))$ the Hessian of ρ_N at $y = \gamma(\rho_N(y))$, satisfies*

$$\frac{C_\mu}{S_\mu}(\rho_N(y))\|X\|^2 \geq \mathit{Hess}\, \rho_N(y)(X, X) \geq \frac{C_\kappa}{S_\kappa}(\rho_N(y))\|X\|^2, \tag{7}$$

where $\mathit{Hess}\rho_N(y)(\gamma', \gamma') = 0$.

Observation 2.2. If $y \in \text{cut}_N(y_0)$, inequality (7) has to be understood in the following sense:

$$\frac{C_\mu}{S_\mu}(\rho_N(y))\|X\|^2 \geq \lim_{j\to\infty} \text{Hess}\rho_N(y_j)(X_j, X_j) \geq \frac{C_\kappa}{S_\kappa}(\rho_N(y))\|X\|^2.$$

For a sequence $(y_j, X_j) \to (y, X) \in TN$, $y_j \notin \text{cut}_N(y_0)$.

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3. Proof of Theorem 1.2.

3.1. Proof of items (a) and (b). Since $a(M) < 1$, we have that for each $a(M) < c < 1$, there is i such that $a_i \in (a(M), c)$. This means that there exists a geodesic ball $B_M(r_0) \subset M$, with $C_i \subset B_M(r_0)$, centred at x_0 with radius $r_0 > 0$ such that

$$\frac{S_\kappa}{C_\kappa}(\rho_M(x)) \cdot \|\alpha(x)\| \le c < 1, \quad \text{for all } \; x \in M \backslash B_M(r_0). \tag{8}$$

To fix the notation, let $x_0 \in M$, $y_0 = \varphi(x_0)$ and $\rho_M(x) = \mathrm{dist}_M(x_0, x)$ and $\rho_N(y) = \mathrm{dist}_N(y_0, y)$. Suppose first that $\kappa = 0$. Letting $h(t) = t^2$ we have that $f(x) = \rho_N(\varphi(x))^2$. By equation (6) the Hessian of f at $x \in M$ in the direction X is given by

$$\mathrm{Hess}f(x)(X, X) = 2\,[\rho_N \,\mathrm{Hess}\rho_N(X, X) + \rho_N \,\langle \mathrm{grad}\rho_N, \alpha(X, X)\rangle + \langle \mathrm{grad}\rho_N, X\rangle^2](y), \tag{9}$$

where $y = \varphi(x)$. By the Hessian comparison theorem, we have that

$$\mathrm{Hess}\rho_N(y)(X, X) \ge \frac{1}{\rho_N(y)} \|X^\perp\|^2, \tag{10}$$

where $\langle X^\perp, \mathrm{grad}\rho_N\rangle = 0$. Therefore for every $x \in M \backslash B_M(r_0)$,

$$\begin{aligned}
\mathrm{Hess}f(x)(X, X) &= 2[\rho_N \,\mathrm{Hess}\rho_N(X, X) + \langle \mathrm{grad}\rho_N, X\rangle^2 \\
&\quad + \rho_N \langle \mathrm{grad}\rho_N, \alpha(X, X)\rangle](y) \\
&\ge 2\left[\rho_N \,\frac{1}{\rho_N}\|X^\perp\|^2 + \|X^\top\|^2 + \rho_N \langle \mathrm{grad}\rho_N, \alpha(X, X)\rangle\right](y) \\
&\ge 2\,[\|X^\top\|^2 + \|X^\perp\|^2 - \rho_M \,\|\alpha\| \cdot \|X\|^2] \\
&\ge 2(1 - c)\|X\|^2.
\end{aligned} \tag{11}$$

In the third and fourth lines of (11) we have used $\rho_N(\varphi(x)) \le \rho_M(x)$. If $\kappa < 0$, we let $h(t) = \cosh(\sqrt{-\kappa}\, t)$; then $f(x) = \cosh(\sqrt{-\kappa}\, \rho_N)(\varphi(x))$. By equation (6) the Hessian of f is given by

$$\begin{aligned}
\mathrm{Hess}f(x)(X, X) = \big[&-\kappa \cosh(\sqrt{-\kappa}\, \rho_N)\langle \mathrm{grad}\rho_N, X\rangle^2 + \sqrt{-\kappa}\sinh(\sqrt{-\kappa}\, \rho_N) \\
&\times \;\mathrm{Hess}\rho_N(X, X) + \sqrt{-\kappa}\sinh(\sqrt{-\kappa}\, \rho_N)\langle \mathrm{grad}\rho_N, \alpha(X, X)\rangle\big]\,(\varphi(x)).
\end{aligned} \tag{12}$$

By Hessian comparison theorem we have that

$$\mathrm{Hess}\rho_N(y)(X, X) \ge \sqrt{-\kappa}\,\frac{\cosh(\sqrt{-\kappa}\,\rho_N)}{\sinh(\sqrt{-\kappa}\,\rho_N)}\|X^\perp\|^2. \tag{13}$$

Since $a(M) < 1$, we have

$$\|\alpha(x)\| \le c\,\sqrt{-\kappa}\,\frac{\cosh(\sqrt{-\kappa}\,\rho_M)}{\sinh(\sqrt{-\kappa}\,\rho_M)}(x) \le c\,\sqrt{-\kappa}\,\frac{\cosh(\sqrt{-\kappa}\,\rho_N)}{\sinh(\sqrt{-\kappa}\,\rho_N}(\varphi(x)) \tag{14}$$

for every $x \in M \backslash B_M(r_0)$ and some $c \in (0, 1)$. The last inequality follows from the fact that $\rho_N(\varphi(x)) \le \rho_M(x)$ and that the function $\sqrt{-\kappa}\coth(\sqrt{-\kappa}\, t)$ is non-increasing.

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Substituting in equation (12), we obtain

$$\begin{aligned}\text{Hess}f(x)(X,X) &\geq -\kappa\cosh(\sqrt{-\kappa}\,\rho_N)\|X^{\perp}\|^2 - \kappa\cosh(\sqrt{-\kappa}\,\rho_N)\|X^{\top}\|^2 \\ &\quad + \kappa\cdot c\cdot\cosh(\sqrt{-\kappa}\,\rho_N)\|X\|^2 \\ &\geq -\kappa\cdot\cosh(\rho_N)(1-c)\|X\|^2 \\ &\geq -\kappa\cdot(1-c)\cdot\|X\|^2. \end{aligned} \tag{15}$$

Let $\sigma : [0, \rho_M(x)] \to M$ be a minimal geodesic joining x_0 to x. For all $t > r_0$ we have that $(f\circ\sigma)''(t) = \text{Hess}f(\sigma(t))(\sigma', \sigma') \geq 2(1-c)$ if $\kappa = 0$ and $(f\circ\sigma)''(t) \geq -\kappa(1-c)$ if $\kappa < 0$.
For $t \leq r_0$ we have that $(f\circ\sigma)''(t) \geq b = \inf\{\text{Hess}f(x)(\nu,\nu), x \in B_M(r_0), |\nu| = 1\}$. Hence ($\kappa = 0$),

$$\begin{aligned}(f\circ\sigma)'(s) &= (f\circ\sigma)'(0) + \int_0^s (f\circ\sigma)''(\tau)d\tau \\ &\geq (f\circ\sigma)'(0) + \int_0^{r_0} b\,d\tau + \int_{r_0}^s 2(1-c)d\tau \\ &\geq (f\circ\sigma)'(0) + b\,r_0 + 2(1-c)(s-r_0). \end{aligned} \tag{16}$$

Now, $\rho_N(\varphi(x_0)) = \text{dist}_N(y_0, y_0) = 0$; then $(f\circ\sigma)'(0) = 0$ and $f(x_0) = 0$; therefore

$$\begin{aligned} f(x) &= \int_0^{\rho_M(x)} (f\circ\sigma)'(s)ds \\ &\geq \int_0^{\rho_M(x)} \{b\,r_0 + 2(1-c)(s-r_0)\}\,ds \\ &\geq b\,r_0\,\rho_M(x) + 2(1-c)\left(\frac{\rho_M^2(x)}{2} - r_0\,\rho_M(x)\right) \\ &\geq (1-c)\,\rho_M^2(x) + (b - 2(1-c))\,r_0\,\rho_M(x). \end{aligned} \tag{17}$$

Thus

$$\rho_N^2(\varphi(x)) \geq (1-c)\,\rho_M^2(x) + (b-2(1-c))r_0\,\rho_M(x) \tag{18}$$

for all $x \in M$. Similarly, for $\kappa < 0$ we obtain that

$$\cosh(\sqrt{-\kappa}\,\rho_N)(\varphi(x)) \geq \sqrt{-\kappa}(1-c)\rho_M^2(x) + (b/\sqrt{-\kappa} - \sqrt{-\kappa}(1-c))r_0\rho_M(x) + 1. \tag{19}$$

If N is compact, the left-hand sides of the inequalities (18) and (19) are bounded above. That implies that M must be compact. In fact, we can find $\mu = \mu(\text{diam}(N), c, \kappa)$ so that $\text{diam}(M) \leq \mu$. Otherwise (if N is complete non-compact) if $\rho_M(x) \to \infty$, then $\rho_N(\varphi) \to \infty$ and φ is proper.

3.2. Proof of item (c). Recall that we have by hypothesis that $\varphi : M \hookrightarrow N$ is a complete m-dimensional submanifold with tamed second fundamental form immersed in complete n-dimensional Hadamard manifold N with $K_N \leq \kappa \leq 0$. We can assume that M is non-compact. Moreover, by item (a), proved in the last subsection, φ is a proper immersion. Let $B_N(r_0)$ be the geodesic ball of N centred at y_0 with radius r_0

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and $S_{r_0} = \partial B_N(r_0)$. Since φ is proper and $a(M) < 1$ we can take r_0 so that

$$\frac{S_\kappa}{C_\kappa}(\rho_M(x))\|\alpha(x)\| \le c < 1, \quad \text{for all } \ x \in M\backslash\varphi^{-1}(B_N(r_0)), \tag{20}$$

and by Sard's theorem (see [**8**], p. 79), r_0 can be chosen so that $\Gamma_{r_0} = \varphi(M)\cap S_{r_0} \neq \emptyset$ is a submanifold of $\dim \Gamma_{r_0} = m-1$. For each $y \in \Gamma_{r_0}$, let us denote by $T_y\Gamma_{r_0} \subset T_y\varphi(M)$ the tangent spaces of Γ_{r_0} and $\varphi(M)$, respectively, at y. Since the dimension $\dim T_y\Gamma_{r_0} = m-1$ and $\dim T_y\varphi(M) = m$, there exist only one unit vector $\nu(y) \in T_y\varphi(M)$ such that $T_y\varphi(M) = T_y\Gamma_{r_0} \oplus [[\nu(y)]]$, with $\langle \nu(y), \text{grad}\rho_N(y)\rangle > 0$. This defines a smooth vector field ν on a neighborhood V of $\varphi^{-1}(\Gamma_{r_0})$. Here $[[\nu(y)]]$ is the vector space generated by $\nu(y)$. Consider the function on $\varphi(V)$ defined by

$$\psi(y) = \langle \nu, \text{grad}\rho_N\rangle(y) = \langle \nu, \text{grad}R\rangle(y) = \nu(y)(R), \ y = \varphi(x). \tag{21}$$

Then $\psi(y) = 0$ if and only if every $x = \varphi^{-1}(y) \in V$ is a critical point of the extrinsic distance function R. Now for each $y \in \Gamma_{r_0}$ fixed, let us consider the solution $\xi(t, y)$ of the following Cauchy problem on $\varphi(M)$:

$$\begin{cases} \xi_t(t, y) = \dfrac{1}{\psi}\, \nu(\xi(t, y)), \\ \xi(0, y) = y. \end{cases} \tag{22}$$

We will prove that along the integral curve $t \mapsto \xi(t, y)$ there are no critical points for $R = \rho_N \circ \varphi$. For this, consider the function $(\psi \circ \xi)(t, y)$ and observe that

$$\begin{aligned} \psi_t &= \xi_t\langle \text{grad}\rho_N, \nu\rangle \\ &= \langle \bar\nabla_{\xi_t}\text{grad}\rho_N, \nu\rangle + \langle \text{grad}\rho_N, \bar\nabla_{\xi_t}\nu\rangle \\ &= \frac{1}{\psi}\langle \bar\nabla_\nu \text{grad}\rho_N, \nu\rangle + \frac{1}{\psi}\langle \text{grad}\rho_N, \nabla_\nu\nu + \alpha(\nu,\nu)\rangle \\ &= \frac{1}{\psi}\text{Hess}\rho_N(\nu,\nu) + \frac{1}{\psi}\,[\langle \text{grad}\rho_N, \nabla_\nu\nu\rangle + \langle \text{grad}\rho_N, \alpha(\nu,\nu)\rangle] \\ &= \frac{1}{\psi}\,[\text{Hess}\rho_N(\nu,\nu) + \langle \text{grad}\rho_N, \nabla_\nu\nu\rangle + \langle \text{grad}\rho_N, \alpha(\nu,\nu)\rangle]\,. \end{aligned} \tag{23}$$

Thus

$$\psi_t\psi = \text{Hess}\rho_N(\nu,\nu) + \langle \text{grad}\rho_N, \nabla_\nu\nu\rangle + \langle \text{grad}\rho_N, \alpha(\nu,\nu)\rangle. \tag{24}$$

Since $\langle \nu, \nu\rangle = 1$, we have at once that $\langle \nabla_\nu\nu, \nu\rangle = 0$. As $\nabla_\nu\nu \in T_xM$, we have that

$$\langle \text{grad}\rho_N, \nabla_\nu\nu\rangle = \langle \text{grad}R, \nabla_\nu\nu\rangle.$$

By equation (21), we can write $\text{grad}R(x) = \psi(\varphi(x))\cdot\nu(\varphi(x))$, since $\text{grad}R(x) \perp T_{\varphi(x)}\Gamma_{\rho_N(y)}$, $(\Gamma_{\rho_N(y)} = \varphi(M)\cap\partial B_N(\rho_N(y)))$. Then

$$\langle \text{grad}\rho_N, \nabla_\nu\nu\rangle = \langle \text{grad}R, \nabla_\nu\nu\rangle = \psi\langle \nu, \nabla_\nu\nu\rangle = 0.$$

Writing

$$\nu(y) = \cos\beta(y)\,\text{grad}\rho_N + \sin\beta(y)\,\omega \tag{25}$$

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and

$$\text{grad}\rho_N(y) = \cos\beta\ \nu(y) + \sin\beta\ \nu^*, \tag{26}$$

where $\langle \omega, \text{grad}\rho_N\rangle = 0$ and $\langle \nu, \nu^*\rangle = 0$, equation (24) becomes

$$\psi_t\psi = \sin^2\beta\ \text{Hess}\rho_N(\omega,\omega) + \sin\beta\ \langle \nu^*, \alpha(\nu,\nu)\rangle. \tag{27}$$

From (25) we have that $\psi(y) = \cos\beta(y)$,

$$\psi_t\psi = \sqrt{1-\psi^2}\sqrt{1-\psi^2}\text{Hess}\rho_N(\omega,\omega) + \sqrt{1-\psi^2}\langle \nu^*, \alpha(\nu,\nu)\rangle. \tag{28}$$

Hence

$$\frac{\psi_t\psi}{\sqrt{1-\psi^2}} = \sqrt{1-\psi^2}\text{Hess}\rho_N(\omega,\omega) + \langle \nu^*, \alpha(\nu,\nu)\rangle. \tag{29}$$

Thus we arrive at the following differential equation:

$$-(\sqrt{1-\psi^2})_t = \sqrt{1-\psi^2}\ \text{Hess}\rho_N(\omega,\omega) + \langle \nu^*, \alpha(\nu,\nu)\rangle. \tag{30}$$

The Hessian comparison theorem implies that

$$\text{Hess}\rho_N(\omega,\omega) \geq \frac{C_\kappa}{S_\kappa}(\rho_N(\xi(t,y))). \tag{31}$$

Substituting it in equation (30) the following inequality is obtained:

$$-(\sqrt{1-\psi^2})_t \geq \sqrt{1-\psi^2}\ \frac{C_\kappa}{S_\kappa}(\rho_N(\xi(t,y)))\ + \langle \nu^*, \alpha(\nu,\nu)\rangle. \tag{32}$$

Denoting by $R(t,y)$ the restriction of $R = \rho_N \circ \varphi$ to $\varphi^{-1}(\xi(t,y))$ we have

$$R(t,y) = R(\varphi^{-1}(\xi(t,y))) = \rho_N(\xi(t,y)).$$

On the other hand we have that

$$R_t = \left\langle \text{grad}R, \frac{1}{\psi}\nu \right\rangle = \left\langle \psi\nu, \frac{1}{\psi}\nu \right\rangle = 1; \tag{33}$$

then

$$R(t,y) = t + r_0. \tag{34}$$

Writing $\frac{C_k}{S_k}(\rho_N(\xi(t,y))) = \frac{C_\kappa}{S_\kappa}(t+r_0)$ in (32) we have

$$-(\sqrt{1-\psi^2})_t \geq \sqrt{1-\psi^2}\ \frac{C_\kappa}{S_\kappa}(t+r_0) + \langle \nu^*, \alpha(\nu,\nu)\rangle. \tag{35}$$

Multiplying (35) by $S_\kappa(t+r_0)$, the following is obtained:

$$-\left[S_\kappa(t+r_0)(\sqrt{1-\psi^2})_t + C_\kappa(t+r_0)\sqrt{1-\psi^2}\right] \geq S_k(t+r_0)\langle \nu^*, \alpha(\nu,\nu)\rangle.$$

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The last inequality can be written as

$$\left[S_\kappa(t+r_0)\sqrt{1-\psi^2}\right]_t \leq -S_\kappa(t+r_0)\langle \nu^*, \alpha(\nu,\nu)\rangle. \tag{36}$$

Integrating (36) from 0 to t the resulting inequality is as follows:

$$S_\kappa(t+r_0)\sin\beta(\xi(t,y)) \leq S_\kappa(r_0)\sin\beta(y) + \int_0^t -S_k(s+r_0)\langle \nu^*, \alpha(\nu,\nu)\rangle ds.$$

Thus

$$\sin\beta(\xi(t,y)) \leq \frac{S_\kappa(r_0)}{S_k(t+r_0)}\sin\beta(y) + \frac{1}{S_\kappa(t+r_0)}\int_0^t S_\kappa(s+r_0)(-\langle \nu^*, \alpha(\nu,\nu)\rangle)ds. \tag{37}$$

Since $a(M) < 1$,

$$\begin{aligned}-\langle \nu^*, \alpha(\nu,\nu)\rangle(\xi(s,y)) &\leq \|\alpha(\xi(s,y))\| \leq c\frac{C_\kappa}{S_\kappa}(\rho_M(\xi(s,y)))\\ &\leq c\frac{C_\kappa}{S_\kappa}(\rho_N(\xi(s,y))) = c\frac{C_\kappa}{S_\kappa}(s+r_0)\end{aligned}$$

for every $s \geq 0$. Substituting in (37), we have

$$\begin{aligned}\sin\beta(\xi(t,y)) &\leq \frac{S_\kappa(r_0)}{S_\kappa(t+r_0)}\sin\beta(y) + \frac{c}{S_\kappa(t+r_0)}\int_0^t C_\kappa(s+r_0)ds\\ &= \frac{S_\kappa(r_0)}{S_\kappa(t+r_0)}\sin\beta(y) + \frac{c}{S_\kappa(t+r_0)}(S_\kappa(t+r_0) - S_\kappa(r_0))\\ &= \frac{S_\kappa(r_0)}{S_\kappa(t+r_0)}(\sin\beta(y) - c) + c < 1 \end{aligned}\tag{38}$$

for all $t \geq 0$. Therefore, along the integral curve $t \mapsto \xi(t,y)$, there are no critical points for the function $R(x) = \rho_N(\varphi(x))$ outside the geodesic ball $B_N(r_0)$. The flow ξ_t maps $\partial B_N(r_0)$ diffeomorphically into $\partial B_N(r_0+t)$, for all $t \geq 0$. This shows that M has finite topology (see also [**6**]). This concludes the proof of Theorem 1.2. For the sake of clarity we show that $\frac{S_\kappa(r_0)}{S_\kappa(t+r_0)}(\sin\beta(y)-c)+c < 1$. Let $h(t) = \frac{S_\kappa(r_0)}{S_\kappa(t+r_0)}(\sin\beta(y)-c)+c$. We have that $h(0) = \sin\beta < 1$ and $h'(t) = -\frac{C_\kappa(t+r_0)S_\kappa(r_0)}{S_\kappa^2(t+r_0))}(\sin\beta - c)$. If $\sin\beta \geq c$, then $h'(t) \leq 0$ and $h(t) \leq h(0)$. If $\sin\beta < c$, suppose by contradiction that there exists a $T > 0$ such that $h(T) > 1$. This implies that $0 > S_\kappa(r_0)(\sin\beta - c) > (1-c)S_\kappa(T+r_0) > 0$.

4. Proof of Theorem 1.4. The first ingredient for the proof of Theorem 1.4 is the well-known Barta's theorem [**2**] stated here for the sake of completeness.

THEOREM 4.1 (Barta). *Let Ω be a bounded open of a Riemannian manifold with piecewise smooth boundary. Let $f \in C^2(\Omega) \cap C^0(\bar{\Omega})$ with $f|\Omega > 0$ and $f|\partial\Omega = 0$. The first Dirichlet eigenvalue $\lambda_1(\Omega)$ has the following bounds:*

$$\sup_\Omega\left(-\frac{\Delta f}{f}\right) \geq \lambda_1(\Omega) \geq \inf_\Omega\left(\frac{-\Delta f}{f}\right), \tag{39}$$

with equality in (4) *if and only if f is the first eigenfunction of Ω.*

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Let $\varphi : M \hookrightarrow N$ be an isometric immersion with tamed second fundamental form of a complete m-manifold M into a Hadamard n-manifold N with sectional curvature $\mu \leq K_N \leq 0$. Let $x_0 \in M$, $y_0 = \varphi(x_0) \in N$, and let $\rho_N(y) = \mathrm{dist}_N(y_0, y)$ be the distance function on N and $\rho_N \circ \varphi$ the extrinsic distance on M. By the proof of Theorem (1.2) there is an $r_0 > 0$ such that there is no critical point $x \in M \setminus \varphi^{-1}(B_N(r_0))$ for $\rho_N \circ \varphi$, where $B_N(r_0)$ is the geodesic ball in N centred at y_0 with radius r_0. Let $R > r_0$, and let $\Omega \subset \varphi^{-1}(B_N(R))$ be a connected component. Since φ is proper we have that Ω is bounded with boundary $\partial\Omega$ that we may suppose to be piecewise smooth. Let $v : B_{\mathbb{N}^l(\mu)}(R) \to \mathbb{R}$ be a positive first eigenfunction of the geodesic ball of radius R in the l-dimensional simply connected space form $\mathbb{N}^l(\mu)$ of constant sectional curvature μ, where l is to be determined. The function v is radial, i.e. $v(x) = v(|x|)$, and satisfies the following differential equation:

$$v''(t) + (l-1)\,\frac{C_\mu}{S_\mu}(t)\,v'(t) + \lambda_1(B_{\mathbb{N}^l(\mu)}(R))v(t) = 0, \ \ \forall\, t \in [0, R], \tag{40}$$

with initial data $v(0) = 1$, $v'(0) = 0$. Moreover, $v'(t) < 0$ for all $t \in (0, R]$; S_μ and C_μ are defined in (1) and $\lambda_1(B_{\mathbb{N}^l(\mu)}(R))$ is the first Dirichlet eigenvalue of the geodesic ball $B_{\mathbb{N}^l(\mu)}(R) \subset \mathbb{N}^l(\mu)$ with radius R. Define $\tilde{v} : B_N(R) \to \mathbb{R}$ by $\tilde{v}(y) = v \circ \rho_N(y)$ and $f : \Omega \to \mathbb{R}$ by $f(x) = \tilde{v} \circ \varphi(x)$. By Barta's theorem we have $\lambda_1(\Omega) \leq \sup_\Omega(-\triangle f/f)$. The Laplacian $\triangle f$ at a point $x \in M$ is given by

$$\begin{aligned}\triangle_M f(x) &= \left[\sum_{i=1}^m \mathrm{Hess}\,\tilde{v}(e_i, e_i) + \langle \mathrm{grad}\tilde{v}, \vec{H}\rangle\right](\varphi(x))\\ &= \sum_{i=1}^m \left[v''(\rho_N)\langle \mathrm{grad}\rho_N, e_i\rangle^2 + v'(\rho_N)\,\mathrm{Hess}\,\rho_N(e_i, e_i)\right] + v'(\rho)\langle \mathrm{grad}\rho_N, \vec{H}\rangle,\end{aligned}$$

where $\mathrm{Hess}\,\tilde{v}$ is the Hessian of $\tilde{v}$ in the metric of N and $\{e_i\}_{i=1}^m$ is an orthonormal basis for T_xM at which we made the identification $\varphi_* e_i = e_i$. We are going to give an upper bound for $(-\triangle f/f)$ on $\varphi^{-1}(B_N(R))$. Let $x \in \varphi^{-1}(B_N(R))$, and choose an orthonormal basis $\{e_1, ..., e_m\}$ for T_xM such that $\{e_2, \ldots, e_m\}$ are tangent to the distance sphere $\partial B_N(r(x))$ of radius $r(x) = \rho_N(\varphi(x))$ and $e_1 = \langle e_1, \mathrm{grad}_N\bar{\rho}\rangle \mathrm{grad}_N\bar{\rho} + \langle e_1, \partial/\partial\theta\rangle\partial/\partial\theta$, where $|\partial/\partial\theta| = 1$, $\partial/\partial\theta \perp \mathrm{grad}_N\bar{\rho}$. To simplify the notation set $t = \rho_N(\varphi(x))$, $\triangle_M = \triangle$. Then

$$\begin{aligned}\triangle f(x) &= \sum_{i=1}^m \left[v''(t)\langle \mathrm{grad}\rho_N, e_i\rangle^2 + v'(t)\,\mathrm{Hess}\,\rho_N(e_i, e_i)\right] + v'(t)\langle \mathrm{grad}\rho_N, \vec{H}\rangle\\ &= v''(t)\langle \mathrm{grad}\rho_N, e_1\rangle^2 + v'(t)\langle e_1, \partial/\partial\theta\rangle^2\,\mathrm{Hess}\,\rho_N(\partial/\partial\theta, \partial/\partial\theta)\\ &\quad + \sum_{i=2}^m v'(t)\,\mathrm{Hess}\,\rho_N(e_i, e_i) + v'(t)\langle \mathrm{grad}\rho_N, \vec{H}\rangle.\end{aligned} \tag{41}$$

Thus from (41)

$$\begin{aligned}-\frac{\triangle f}{f}(x) &= -\frac{v''}{v}(t)\langle \mathrm{grad}\rho_N, e_1\rangle^2 - \frac{v'}{v}(t)\langle e_1, \partial/\partial\theta\rangle^2\,\mathrm{Hess}\,\rho_N(\partial/\partial\theta, \partial/\partial\theta)\\ &\quad - \sum_{i=2}^m \frac{v'}{v}(t)\,\mathrm{Hess}\,\rho_N(e_i, e_i) - \frac{v'}{v}(t)\langle \mathrm{grad}\rho_N, \vec{H}\rangle.\end{aligned} \tag{42}$$

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Equation (40) says that

$$-\frac{v''}{v}(t) = (l-1)\frac{C_\mu}{S_\mu}\frac{v'}{v}(t) + \lambda_1(B_{\mathbb{N}^l(\mu)}(R)).$$

By the Hessian comparison theorem and the fact $v'/v \leq 0$ we have from equation (42) the following inequality:

$$\begin{aligned} -\frac{\Delta f}{f}(x) \leq{}& \lambda_1(B_{\mathbb{N}^l(\mu)}(R))][1 - \langle e_1, \partial/\partial\theta\rangle^2] \\ &- \frac{C_\mu}{S_\mu}(t)\frac{v'}{v}(t)\left[m - l + l\langle e_1, \partial/\partial\theta\rangle^2 + \frac{S_\mu}{C_\mu}\|\vec{H}\|\right]. \end{aligned} \tag{43}$$

On the other hand the mean curvature vector $\vec{H}$ at $\varphi(x)$ has the norm

$$\|\vec{H}\|(\varphi(x)) \leq \|\alpha\|(\varphi(x)) \leq c \cdot (C_\kappa/S_\kappa)(\rho_M(x)) \leq c \cdot (C_\kappa/S_\kappa)(\rho_N(\varphi(x))).$$

We have that for any given $a(M) < c < 1$ there exist $r_0 = r_0(c) > 0$ such that there is no critical point $x \in M \setminus \varphi^{-1}(B_N(r_0))$ for $\rho_N \circ \varphi$. A critical point x satisfies $\langle e_1, \partial/\partial\theta\rangle(\varphi(x)) = 1$ (see equation (25), where $\langle e_1, \partial/\partial\theta\rangle(\varphi(x)) = \sin\beta(\varphi(x))$). Inequality (38) shows that for any $x \in M \setminus \varphi^{-1}(B_N(r_0))$ we have ($\kappa = 0$ in our case)

$$\begin{aligned} \langle e_1, \partial/\partial\theta\rangle(\varphi(x)) &\leq \frac{r_0}{\rho_N(\varphi(x)) + r_0}\left(\sup_{z\in\varphi^{-1}(\partial B_N(r_0))} \sin\beta(\varphi(z))) - c\right) + c \\ &\leq \frac{r_0}{r_0 + r_0}(1-c) + c \\ &= \frac{1+c}{2}. \end{aligned} \tag{44}$$

We have then from (43) and (44) the following inequality:

$$-\frac{\Delta f}{f}(x) \leq \lambda_1(B_{\mathbb{N}^l(\mu)}(R)) - \frac{C_\mu}{S_\mu}(t)\frac{v'}{v}(t)\left[m - l + \frac{l}{4}(1+c)^2 + c\right].$$

Choose the least $l \in \mathbb{Z}_+$ such that $m - l + l(1+c)^2/4 + c \leq 0$. With this choice of l we have for all $x \in \varphi^{-1}(B_N(R) \setminus B_N(r_0))$ that

$$-\frac{\Delta f}{f}(x) \leq \lambda_1(B_{\mathbb{N}^l(\mu)}(R)). \tag{45}$$

Now let $x \in \varphi^{-1}(B_N(r_0))$. Since $1 - \langle e_1, \partial/\partial\theta\rangle^2 \leq 1$ and $-l + l\langle e_1, \partial/\partial\theta\rangle^2 \leq 0$ we obtain from (43) the following inequality ($t = \rho_N(\varphi(x))$):

$$-\frac{\Delta f}{f}(x) \leq \lambda_1(B_{\mathbb{N}^l(\mu)}(R))] - \frac{C_\mu}{S_\mu}(t)\frac{v'}{v}(t)\left[m + \frac{S_\mu}{C_\mu}\|\vec{H}\|\right]. \tag{46}$$

We need the following technical lemma.

LEMMA 4.2. *Let v be the function satisfying* (40). *Then* $-v'(t)/t \leq \lambda_1(B_{\mathbb{N}^l(\mu)}(R))$ *for all* $t \in [0, R]$.

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Proof. Consider the function $h : [0, R] \to \mathbb{R}$ given by $h(t) = \lambda \cdot t + v'(t)$, $\lambda = \lambda_1(B_{\mathbb{N}^l(\mu)}(R))$. We know that $v(0) = 1$, $v'(0) = 0$ and $v'(t) \le 0$; besides v satisfies equation (40). Observe that

$$0 = v''(t) + (l-1)v' + \lambda v \le v'' + \lambda.$$

Thus $v'' \ge -\lambda$ and $h'(t) = \lambda + v'' \ge 0$. Since $h(0) = 0$ we have $h(t) = \lambda t + v'(t) \ge 0$. This proves the lemma. □

Since v is a non-increasing positive function we have $v(t) \ge v(r_0)$. Applying Lemma (4.2) we obtain

$$-\frac{\triangle f}{f}(x) \le \lambda_1(B_{\mathbb{N}^l(\mu)}(R)) + \frac{t \cdot C_\mu(t)}{S_\mu(t)}\left(-\frac{v'(t)}{t}\right) \cdot \frac{1}{v(r_0)}[m+c] \tag{47}$$

$$\le \lambda_1(B_{\mathbb{N}^l(\mu)}(R))\left[1 + r_0\frac{C_\mu}{S_\mu}(r_0) \cdot \frac{1}{v(r_0)}[m+c]\right]. \tag{48}$$

Thus for all $x \in \varphi^{-1}(B_N(R))$ we have

$$-(\triangle f/f)(x) \le \max\left\{1, \left[1 + r_0\frac{C_\mu}{S_\mu}(r_0) \cdot \frac{1}{v(r_0)}[m+c]\right]\right\} \cdot \lambda_1(B_{\mathbb{N}^l(\mu)}(R))$$
$$= \left[1 + r_0\frac{C_\mu}{S_\mu}(r_0) \cdot \frac{1}{v(r_0)}[m+c]\right] \cdot \lambda_1(B_{\mathbb{N}^l(\mu)}(R)).$$

Then by Barta's theorem

$$\lambda_1(\Omega) \le \left[1 + r_0\frac{C_\mu}{S_\mu}(r_0) \cdot \frac{1}{v(r_0)}[m+c]\right] \cdot \lambda_1(B_{\mathbb{N}^l(\mu)}(R)).$$

Observe that $C = \left[1 + r_0\frac{C_\mu}{S_\mu}(r_0) \cdot \frac{1}{v(r_0)}[m+c]\right]$ does not depend on R. So letting $R \to \infty$ we have $\lambda^*(M) \le C\lambda^*(\mathbb{N}^l(\mu))$.

COROLLARY 4.3 (From the proof). *Given c, $a(M) < c < 1$, there exists $r_0 = r_0(c) > 0$, $l = l(m, c) \in \mathbb{Z}_+$ and $C = C(m, \mu, c) > 0$ such that for any $R > r_0$ and any connected component Ω of $\varphi^{-1}(B_N(r))$, then*

$$\lambda^*(\Omega) \le C \cdot \lambda_1(B_{\mathbb{N}^l(\mu)}(R)).$$

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