

PROPAGATION OF SMALLNESS FOR SOLUTIONS OF GENERALIZED CAUCHY–RIEMANN SYSTEMS

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(Received 25 March 2003)

Abstract Let u be a solution of a generalized Cauchy–Riemann system in \mathbb{R}^n . Suppose that $|u| \leq 1$ in the unit ball and $|u| \leq \varepsilon$ on some closed set E . Classical results say that if E is a set of positive Lebesgue measure, then $|u| \leq C\varepsilon^\alpha$ on any compact subset of the unit ball. In the present work the same estimate is proved provided that E is a subset of a hyperplane and the (capacitary) dimension of E is greater than $n - 2$. The proof gives control of constants C and α .

Keywords: Generalized Cauchy–Riemann system; Riesz capacity; real-analytic functions; quantitative continuation

2000 Mathematics subject classification: Primary 31B35
Secondary 35B35; 35J45

1. Introduction

Given a subset E of the unit ball $B \subset \mathbb{R}^n$ and a class of (vector) functions $\mathcal{A} \subset C(B)$, we say that E is a propagation of smallness set for \mathcal{A} if for any $u \in \mathcal{A}$

$$|u(x)| \leq C \|u\|_B^{1-\alpha} \|u\|_E^\alpha, \quad (1.1)$$

where $\alpha = \alpha(x, E) > 0$ is independent of u and bounded away from 0 on compact subsets of B . We will consider this problem for \mathcal{A} being the space of (real) analytic functions or the space of solutions to some differential equation.

Propagation of smallness was intensively studied for solutions of second-order elliptic equations. In particular, if E is a small ball, then (1.1) is an immediate consequence of the three-balls theorem. Various versions of this theorem for solutions of second-order elliptic equations were proved by Gusarov and Landis [11], and, using another approach, by Brummelhuis [5]. The result due to Korevaar and Meyers [10] shows that when \mathcal{A} is the space of harmonic functions and E is a ball one can take $C = 1$ in (1.1). The case of a set E of positive Lebesgue measure was investigated independently by Nadirashvili and Vessella. It is proved in [15, 20] that if \mathcal{A} is the space of real analytic functions that have analytic extension to a fixed domain in \mathbb{C}^n (for example, \mathcal{A} is a space of solutions to an elliptic equation with analytic coefficients) and $m(E) > 0$, then (1.1) holds. For

arbitrary second-order elliptic equations the question of propagation of smallness from a set of positive measure is open. Interesting inequalities are proved in [16, 21], but they are weaker than (1.1).

In this paper \mathcal{A} is the space of solutions to a generalized Cauchy–Riemann (GCR) system. More precisely, we consider systems of first-order differential equations with constant coefficients that are factors of the Laplacian. The simplest example is the classical Cauchy–Riemann system—for this case, \mathcal{A} is the class of functions analytic in the unit disc. It can be proved that E is a propagation of smallness set for analytic functions if and only if E is non-polar (see §2). The author is grateful to Vladimir Eiderman for pointing out this result. It is related to the fact that the logarithm of the module of an analytic function is subharmonic (and harmonic off the zero set of the function). In other words, we have a description of the space of *modules* of analytic functions. Unfortunately, there is no such description for modules of solutions to GCR systems. It is well known (see [19]) that if u is a solution to a GCR system, then $|u|^p$ is subharmonic for some $p < 1$, where p depends on the system. Being a partial substitution to the subharmonicity of the logarithm, this result is difficult to use in our problem. The difference between analytic functions and solutions to higher-dimensional systems is discussed in the remarkable work of Wolff [22]. It shows that solutions of (even the simplest) generalization of Cauchy–Riemann systems do not share a number of important properties of analytic functions.

We formulate a condition that is sufficient for propagation of smallness for GCR systems. It is clear that zero sets of \mathcal{A} are of dimension at most $n - 2$. We assume that E is a subset of a hyperplane and its (capacitary) dimension is greater than $n - 2$. Then E is a propagation of smallness set for \mathcal{A} and we have estimates on α in (1.1).

Main Result. *Let A be a differential operator with constant coefficients, $A = \sum_1^n A_j \partial_j$, that is a factor of the Laplacian. Let \mathcal{A} be the space of solutions to $Au = 0$, where $u : B \rightarrow \mathbb{R}^m$. Suppose that E is a compact subset of a hyperplane, $E \subset B_{1/2}$, and for some $\delta > 0$ the Riesz capacity $C_{n-2+\delta}(E) = \mathcal{C}$ is positive (for the definition of the capacity see (5.3) below). Then (1.1) holds with some $C = C(n)$ and $\alpha = \alpha(x, n, A, \delta, \mathcal{C})$ bounded away from 0 on compact subsets of B .*

The paper is organized as follows. In §2 we give a simple proof of (1.1) for analytic functions. The proof is based on the definition of capacity via Chebyshev constants and gives an estimate for α in terms of the logarithmic capacity. This result is applied in §3 to obtain propagation of smallness for real-analytic functions in \mathbb{R}^N . The following result of §3 is of independent interest.

Key Lemma. *Let $A(B_R)$ be the set of real analytic functions f on B_R that admit analytic continuation \tilde{f} to the closed complex ball $B_{R,\mathbb{C}} = \{z \in \mathbb{C}^N : |z| \leq R\}$ such that $|\tilde{f}| \leq 1$ on $B_{R,\mathbb{C}}$. Let $K \subset B_r$ be a compact set of positive Riesz $(N - 1 + \delta)$ -capacity. Then*

$$\sup_{B_r} |f| \leq \sup_K |f|^\alpha,$$

for any $f \in A(B_R)$, where $\alpha = \alpha(N, \delta, r^{1-N-\delta} C_{N-1+\delta}(K))$.

We use this lemma to prove the Main Result in §4, which begins with definitions and examples of GCR systems and factors of the Laplacian. In the last section, notions and results from potential theory are collected.

We use N to denote the dimension of the real (complex) space when working with (real-)analytic functions. Then in §4 we change the notation and use n for the dimension of the space where a GCR system is considered. The reason is that in this section the results on analytic functions in smaller-dimensional spaces are used.

Throughout the paper $c(N), c(n), \dots$ stand for different constants depending only on N, n , etc. The values of these constants changes from line to line and is of no importance for us. By $c_l(N)$, $l = 0, 1, \dots$, we denote constants that are fixed through the paper. We use the standard notation \mathcal{H}^s for the Hausdorff measures and use m to denote the Lebesgue measure.

2. Propagation of smallness for analytic functions

2.1. Formulation of the result

In this section we consider analytic functions of one complex variable. Suppose that f is analytic and bounded by 1 in the ball $B_R = \{|z| < R\}$ and suppose that $|f| \leq \varepsilon$ on a compact subset $K \subset B_r = \{|z| < r\}$, where $r < R$. We consider estimates of the type

$$\sup_{B_r} |f| \leq C\varepsilon^\alpha, \quad \text{for some } \alpha = \alpha(R, r, K) \text{ and an absolute constant } C. \tag{2.1}$$

It is clear that such estimates hold with some α and C if and only if K is a non-polar set. Then one can take $C = 1$ and $\alpha = \max_{B_r} \omega(K, B_R, z)$, where

$$\omega(K, B_R, z) = \inf\{u(z), u \text{ is superharmonic in } B_R, u \geq 0 \text{ on } B_R \text{ and } u \geq 1 \text{ on } K\}$$

is the harmonic measure of K relative to B_R . Our aim is to estimate α in terms of the logarithmic capacity of K . Though the question seems to be classical, we failed to find the reference in classical monographs on potential theory. For this reason, we give an elementary proof of the following statement.

Theorem 2.1. *Suppose that $4r < R < 1$ and f and K are as above. If K is not a polar set, then (2.1) holds with $C = 1$ and*

$$\alpha = \frac{1}{8} \left(\ln \frac{3r}{C_0(K)} \right)^{-1},$$

where $C_0(K)$ is the logarithmic capacity of K (see (5.1) below).

Remark 2.2. As is mentioned in the last section, C_0 is a non-decreasing function on compact sets and $C_0(K) \leq C_0(B_r) = r$, thus the expression above is well defined.

2.2. Proof of Theorem 2.1

We fix m to be the greatest integer such that

$$\varepsilon \left(\frac{3r}{C_0(K)} \right)^m \leq 1. \tag{2.2}$$

Then we have the following inequalities for m :

$$m \leq \frac{-\ln \varepsilon}{\ln(3r) - \ln C_0(K)} \leq \frac{\delta^{-1} \varepsilon^{-\delta}}{(\ln(3r) - \ln C_0(K))}. \tag{2.3}$$

The last inequality is valid for any $\delta > 0$ and is easy to check.

Now, let $z_0, z_1, \dots, z_m \in K$ be such that

$$d_m(K) = \left(\prod_{0 \leq j < k \leq m} |z_j - z_k| \right)^{2/m(m+1)}$$

is the m th transfinite diameter of K (see § 5.2). We consider the Lagrange interpolation polynomial for f with the nodes z_0, z_1, \dots, z_m :

$$P(z) = \sum_{k=0}^m f(z_k) \frac{(z - z_0) \cdots \widehat{(z - z_k)} \cdots (z - z_m)}{(z_k - z_0) \cdots \widehat{(z_k - z_k)} \cdots (z_k - z_m)}.$$

First, we note that for any $z \in K$ the product of the pairwise distances between $z_0, \dots, z_{k-1}, z, z_{k+1}, \dots, z_m$ is less than or equal to the same product for z_0, z_1, \dots, z_m . Hence,

$$(z_k - z_0) \cdots \widehat{(z_k - z_k)} \cdots (z_k - z_m) = \max_{z \in K} (z - z_0) \cdots \widehat{(z - z_k)} \cdots (z - z_m).$$

Thus the right-hand side of the above inequality is greater than or equal to ρ_m^m , where ρ_m is the m th Chebyshev constant for K (see § 5.2 for the definition). And, using an elementary estimate $\rho_m \geq C_0(K)$ (see (5.2) below), we get

$$(z_k - z_0) \cdots \widehat{(z_k - z_k)} \cdots (z_k - z_m) \geq C_0(K)^m,$$

for any k . Thus, we get an estimate for the interpolation polynomial on B_r :

$$\max_{B_r} |P(z)| \leq (m + 1) \varepsilon \left(\frac{2r}{C_0(K)} \right)^m. \tag{2.4}$$

Using the integral formula for the remainder of the Lagrange interpolation, we get the following inequality for $z \in B_r$:

$$|f(z) - P(z)| = \left| \frac{1}{2\pi i} \int_{|\zeta|=R} \frac{f(\zeta)(z - z_0) \cdots (z - z_m)}{(\zeta - z)(\zeta - z_0) \cdots (\zeta - z_m)} d\zeta \right| \leq \left(\frac{2r}{R - r} \right)^m. \tag{2.5}$$

We now combine estimates (2.4) and (2.5), and use (2.2):

$$\begin{aligned} \max_{B_r} |f| &\leq (m + 1)\varepsilon \left(\frac{2r}{C_0(K)} \right)^m + \left(\frac{2r}{R - r} \right)^m \\ &\leq \left((m + 1)\varepsilon \left(\frac{3r}{C_0(K)} \right)^m + 1 \right) \left(\frac{2}{3} \right)^m \\ &\leq \frac{3}{2}(m + 2)\varepsilon^\beta, \quad \text{where } \beta = \frac{\ln \frac{3}{2}}{\ln(3r) - \ln(C_0(K))}. \end{aligned}$$

Then we apply (2.3) with $\delta = \frac{2}{3}\beta$ to get

$$\max_{B_r} |f| \leq 10\varepsilon^{\beta/3}.$$

So (2.1) is proved with $C = 10$ and $\alpha = \frac{1}{3}\beta$. Now, applying the theorem to functions f^M , where M is any positive integer, we see that if (2.1) is valid with some absolute constant, then it is valid with $C = 1$, and we are done. □

2.3. Corollaries and remarks

Clearly, if K has a positive measure, it is non-polar and we can apply the theorem. Later we need the following result.

Corollary 2.3. *Let f and K be as in the theorem. Suppose that K is a subset of a line and it has a positive linear measure $|K|$. Then estimate (2.1) is satisfied with*

$$\alpha = \frac{1}{8} \left(\ln \frac{12r}{|K|} \right)^{-1}.$$

Proof. Theorem 2.1 combined with the classical inequality $C_0(K) \geq \frac{1}{4}|K|$ (here we use the assumption that K is a subset of a line (see § 5.1)) implies the above estimate. □

If we fix r and R , then α in the theorem can be estimated by the Wiener capacity of E (for the definition see § 5.1).

Corollary 2.4. *Let $r < R < 1$ be some positive numbers and let K be a compact subset of B_r . Then for any function f analytic in B_R ,*

$$\sup_{B_r} |f| \leq \left(\sup_K |f| \right)^\alpha \left(\sup_{B_R} |f| \right)^{1-\alpha}, \quad \text{where } \alpha \geq c(r, R) \text{cap}(K).$$

Proof. First suppose that $4r < R$. Then, applying the theorem, we get the required inequality with

$$\alpha = \frac{1}{8} \left(\ln \frac{3r}{C_0(K)} \right)^{-1} = \frac{1}{8} (\ln 3r + \text{cap}(K)^{-1})^{-1} \geq c(r) \text{cap}(K).$$

If $4r > R$, we set $s = \frac{1}{4}(R - r)$ and take a ball $B(a, s)$ such that $B(a, 4s) \subset B_R$ and $\text{cap}(E) \geq c(r, R) \text{cap}(K)$, where $E = K \cap B(a, s)$. Then, using the theorem, we get

$$\sup_{B(a,s)} |f| \leq \left(\sup_K |f| \right)^\alpha \left(\sup_{B_R} |f| \right)^{1-\alpha}, \quad \text{where } \alpha \geq c(s) \text{cap}(E) \geq c(r, R) \text{cap}(K).$$

We need to replace $B(a, s)$ by B_r . The standard procedure is to take a chain of balls and apply Hadamard's three-circle theorem at each step. Or a continuous version of this 'propagation of smallness' due to Arakelian and Shahgholian [3] can be used. \square

Recently, an estimate between *relative capacity* and *condenser capacity* for compact subsets of \mathbb{C}^N was proved by Siciak [18]. Together with [2, 4, 13], it gives a quantitative relation between the logarithmic capacity (or transfinite diameter) and the condenser capacity. Theorem 2.1 can be obtained from these multidimensional results. In the one-dimensional case we use elementary estimates and interpolation and approximation by polynomials. We end this section by proving an estimate for α from above.

Remark 2.5. Suppose that inequality (2.1) is satisfied with some constant C . Let P_m^* be an extremal polynomial of degree m for K , i.e. the one that gives the m th Chebyshev constant. It is clear that all roots of P_m^* lie in B_r . Thus we have $\max_{B_R} |P_m^*| \leq (R+r)^m$ and $\max_K |P_m^*| = \rho_m(K)^m$. Using (2.1), we get

$$\rho_m^m(B_r) \leq \max_{B_r} |P_m^*| \leq C(R+r)^{m(1-\alpha)} \rho_m(K)^{m\alpha}.$$

Taking the m th root, we obtain the inequality for the limits

$$r \leq (R+r)^{1-\alpha} C_0(K)^\alpha.$$

This implies

$$\alpha \leq \frac{\ln(R+r) - \ln r}{\ln(R+r) - \ln(C_0(K))} \leq \frac{\text{cap}(K)}{\text{cap}(K) \ln r + 1} \ln \left(1 + \frac{R}{r} \right).$$

In particular, $\text{cap}(K) > 0$, and we have an estimate for α from above.

3. Propagation of smallness for real analytic functions in \mathbb{R}^N

3.1. Introductory remarks

We denote by $A(B_R)$ the set of all real analytic functions f on $B_R \subset \mathbb{R}^N$ that have analytic continuation \tilde{f} to the closed complex ball $B_{R,\mathbb{C}} = \{z \in \mathbb{C}^N : |z| \leq R\}$ such that $|\tilde{f}| \leq 1$ on $B_{R,\mathbb{C}}$.

Let K be a compact subset of B_r , we want to obtain an estimate

$$\sup_{B_r} |f| \leq \sup_K |f|^\alpha, \tag{3.1}$$

for any $f \in A(B_R)$. It follows from the remark in the previous section that this inequality holds for some α if and only if K is not a pluri-polar set and α can be estimated in terms of the (pluri-)capacity of K . We will use the fact that K is a subset of \mathbb{R}^n and get a condition on K in real-analytic terms.

Consider the Hausdorff dimension of K . We will use the definition of dimension through the Riesz capacities and refer the reader to §5.3 for the details. It is clear that an $(N-1)$ -dimensional subset $\{x = (x_1, \dots, x_N) \in B_R, x_1 = 0\}$ is not a uniqueness set for

the class $A(B_R)$. On the other hand, if $\dim K > N - 1$, then K is not pluri-polar and (3.1) holds. In fact, for this case $C_s(K) > 0$ for some $s > N - 1$. Then for almost all $b \in \mathbb{S}^{N-1}$ we have

$$\mathcal{H}^{N-1}\{a \in b^\perp : C_{s-N+1}(K \cap l_{a,b}) > 0\} > 0,$$

where $l_{a,b} = \{x = a + tb, t \in \mathbb{R}\}$ (see [14, Theorem 10.8]). Thus $l_{a,b} \cap K$ is not a polar set in the complex plane $L_{a,b} = \{z = a + \lambda b, \lambda \in \mathbb{C}\}$. Suppose that $K \subset \{u = -\infty\}$ for some pluri-subharmonic function u . Then for $b \in \mathbb{S}^{N-1}$ and $a \in b^\perp$ the restriction of u to $L_{a,b}$ is equal to $-\infty$ identically provided that $C_{s-N+1}(K \cap l_{a,b}) > 0$. Thus for almost all $b \in \mathbb{S}^{N-1}$ the restriction of u on b^\perp is equal to $-\infty$ on a set of positive measure. Since u is pluri-subharmonic this implies $u = -\infty$ on b^\perp for almost all $b \in \mathbb{S}^{N-1}$. And, finally, $u = -\infty$ identically.

Our aim is to get a quantitative estimate (3.1) for the case $\dim K > N - 1$. We fix $\delta > 0$ and assume that K has positive Riesz $(N - 1 + \delta)$ -capacity.

3.2. Estimates for functions in $A(B)$

First we prove inequality (3.1) under the assumption that K has positive N -dimensional Lebesgue measure.

Lemma 3.1. *Suppose that $K \subset B_r$, $4r < R < 1$, and $m = m(K) > 0$. Then (3.1) is satisfied for any $f \in A(B_R)$ with $\alpha = \alpha(r, N, m)$.*

Proof. We fix a point $a \in B_r$ and consider all lines passing through a . We can find at least one line l such that $|K \cap l| > c(N)r^{1-N}m$. In fact,

$$m = m(K) = \int \chi_K(x) \, dm(x) = \int_{\mathbb{S}^{N-1}} \int_0^{2r} t^{N-1} \chi_K(a + tx') \, dt \, ds(x'),$$

and there exists $x' \in \mathbb{S}^{N-1}$ such that

$$\begin{aligned} |K \cap \{a + sx'\}| &= \int_0^{2r} \chi_K(a + tx') \, dt \\ &\geq (2r)^{1-N} \int_0^{2r} t^{N-1} \chi_K(a + tx') \, dt \geq c(N)r^{1-N}m. \end{aligned}$$

We take l to be $\{a + sx' : s \in \mathbb{R}\}$. Now let $2\tilde{r} = |B_r \cap l| < 2r$. We apply Corollary 2.3 replacing r, R and K by $\tilde{r}, \sqrt{R^2 - r^2 + \tilde{r}^2}$ and $K \cap l$. Denote by \tilde{f} the extension of $f|_l$ to the complex ball of radius $\sqrt{R^2 - r^2 + \tilde{r}^2}$, then we have

$$|f(a)| \leq \sup_{B_{\tilde{r}}} |\tilde{f}| \leq \sup_{K \cap l} |f|^\alpha,$$

where

$$\alpha \geq \frac{1}{8} \left(\ln \frac{c(N)r^N}{m(K)} \right)^{-1}.$$

This completes the proof. □

We finish this section by proving the Key Lemma formulated in §1.

Lemma 3.2. *Let $K \subset B_r$, $4r < R < 1$, and let $\mathcal{C}(K) = C_{N-1+\delta}(K) > 0$. Then (3.1) holds with*

$$\alpha = \alpha(N, \delta, r^{1-N-\delta}\mathcal{C}(K)).$$

Proof. Applying Lemma 5.3 to K we can find the line l such that

$$\mathcal{H}^{N-1}\{a \in l^\perp : C_\delta(K \cap l_a) \geq k(N)r^{1-N}\mathcal{C}(K)\} \geq c_1(N)r^{-\delta}\mathcal{C}(K).$$

We denote by P_l the orthogonal projection of \mathbb{R}^N onto l^\perp and by L the set of $a \in l^\perp$ such that $C_\delta(K \cap l_a) \geq k(N)r^{1-N}\mathcal{C}(K)$. Then, we apply Theorem 2.1 to each set $K \cap l_a$ and use a simple estimate $C_0(K \cap l_a) \geq (C_\delta(K \cap l_a))^{1/\delta}$ (see (5.4) below). We get

$$|f(x)| \leq \sup_K |f|^\alpha \quad \text{for any } x \in B_r \cap P_l^{-1}(L), \quad (3.2)$$

where

$$\alpha \geq \frac{1}{8\delta} \left(\ln \frac{c(N)r^{N-1+\delta}}{\mathcal{C}(K)} \right)^{-1}.$$

Now we consider the hyperplanes $\{l^\perp + h\}$, where $h \in l$, $|h|^2 < c_3(N)r^{-N+3-\delta}\mathcal{C}(K)$. Each of them intersects $P_l^{-1}(L) \cap B_r$ and, provided that $c_3(N)$ is small enough, we can estimate the measure of $E_h = (l^\perp + h) \cap B_r \cap P_l^{-1}(L)$,

$$\mathcal{H}^{N-1}(E_h) \geq c_4(N)r^{-\delta}\mathcal{C}(K).$$

On E_h estimate (3.2) is valid and, applying Lemma 3.1 to each $(N-1)$ -dimensional set E_h , we get

$$|f(x)| \leq \sup_K |f|^\alpha, \quad \text{whenever } \text{dist}(x, l^\perp) \leq \sqrt{c_3(N)r^{-N+3-\delta}\mathcal{C}(K)},$$

where

$$\alpha \geq \frac{1}{64\delta} \left(\ln \frac{c(N)r^{N-1+\delta}}{\mathcal{C}(K)} \right)^{-2}.$$

Finally,

$$m\left\{x \in B_r : \text{dist}(x, l^\perp) \leq c(N)\sqrt{r^{-N+3-\delta}\mathcal{C}(K)}\right\} \geq c(N)\sqrt{r^{N+1-\delta}\mathcal{C}(K)}$$

and, using Lemma 3.1 once again, we have

$$\sup_{B_r} |f| \leq \sup_K |f|^\alpha,$$

where

$$\alpha \geq \frac{1}{256\delta} \left(\ln \frac{c(N)r^{N-1+\delta}}{\mathcal{C}(K)} \right)^{-3}.$$

This completes the proof of the theorem. We obtained an explicit estimate for α , but it looks too complicated to be used. \square

4. Prolongation of smallness for factors of the Laplacian

4.1. Definition and examples

Let A be a first-order linear differential operator with constant coefficients defined on vector functions, $A = \sum_{j=1}^n A_j \partial_j$, where A_j are constant matrices. The system of equations

$$(AF =) \sum_{j=1}^n A_j \frac{\partial F}{\partial x_j} = 0 \tag{4.1}$$

is called a generalized Cauchy–Riemann (GCR) system if for any solution F to this system, $F : \mathbb{R}^n \rightarrow \mathbb{R}^m$, $F = (f_1, \dots, f_m)$, all its components f_1, f_2, \dots, f_m are harmonic (see [19]). (We suppose that F is well defined and satisfies (4.1) in an open subset of \mathbb{R}^n .)

Definition 4.1. A GCR system (4.1) is said to be a *factor of the Laplacian* if there exists a first-order differential operator $B = \sum_{j=1}^n B_j \partial_j$ with constant coefficients such that $BA = \Delta$.

Let us remark that $BA = \Delta$ means that

$$B_j A_j = I_m, \quad j = 1, \dots, n, \quad \text{and} \quad B_j A_k + B_k A_j = 0, \quad 1 \leq j < k \leq n,$$

where I_m is the identity matrix.

Example 4.2. The simplest GCR system in \mathbb{R}^n is

$$\sum_{j=1}^n \frac{\partial f_j}{\partial x_j} = 0, \quad \frac{\partial f_j}{\partial x_k} = \frac{\partial f_k}{\partial x_j}, \quad 1 \leq j < k \leq n. \tag{4.2}$$

Clearly, it is a factor of the Laplacian. A solution to this system is locally the gradient of a harmonic function. In $\mathbb{R}^2 = \mathbb{C}$, (4.2) is the classical Cauchy–Riemann system for $(f_1, -f_2)$.

Example 4.3. We consider (non-homogeneous) differential forms in \mathbb{R}^n . Let d be the exterior differential operator and δ the adjoint operator (see, for example, [7]). Then the system

$$(d + \delta)\omega = 0$$

can be written as a system of first-order differential equations with constant coefficients and it is a factor of the Laplacian as $\Delta\omega = -(d\delta + \delta d)\omega = -(d + \delta)(d + \delta)\omega$. When $\omega = \sum_{j=1}^N f_j dx^j$ is a 1-form, the system above coincides with (4.2).

Further examples and a description of rotationally invariant factors of the Laplacian can be found in [6] (see also [8] and references therein).

4.2. Prolongation to an $(n - 1)$ -dimensional ball

Now we are ready to prove the Main Result formulated in § 1. First we shall apply the results of § 3. In order to do this we need the following classical lemma. It can be found in textbooks on harmonic functions (see, for example, [9, Chapter 1.5]).

Lemma 4.4. *Suppose that h is a harmonic function in the ball $B_R \subset \mathbb{R}^n$ and $|h| \leq 1$ on B_R , $r = \frac{1}{3}R$. Then h has an analytic continuation \tilde{h} to $B_{r,\mathbb{C}}$ that is bounded by $c(n)$ and, for $z \in B_{r,\mathbb{C}}$,*

$$|\tilde{h}(z) - T_k \tilde{h}(z)| \leq c(n)\rho^k,$$

where $T_k \tilde{h}$ is the Taylor polynomial for \tilde{h} and $\rho < 1$.

Let $F = (f_1, \dots, f_m)$ be a solution to a GCR system $AF = 0$ in $B_R \subset \mathbb{R}^n$ and $\sup_{B_R} |F| \leq 1$. And let K be a subset of the hyperplane $\{x_n = 0\}$ with $\mathcal{C}(K) = C_{n-2+\delta}(K) > 0$ for some $\delta > 0$. To prove the main result we should estimate $\sup_{B_t} |F|$ (where $t < R$) in terms of $\sup_K |F|$. We may assume that $K \subset B_r$, where $r < \frac{1}{12}R$. The first step is to estimate $|F|$ on $B_r \cap \{x_n = 0\}$.

Each component f_j of F is a harmonic function. Applying Lemma 4.4, we see that f_j can be extended to a bounded analytic function in $B_{R/3,\mathbb{C}}$. Let $B'_r = B_r \cap \{x_n = 0\}$. Theorem 3.2 implies

$$\sup_{B'_r} |F| \leq c(n) \left(\sup_K |F| \right)^\alpha, \tag{4.3}$$

where $\alpha = \alpha(r, n, \mathcal{C}(K), \delta)$.

4.3. From hyperplane to a cylinder

Suppose that $|F| \leq 1$ on B_R and $|F| \leq \varepsilon$ on K , where K is as above. Then by (4.3) we have $|F| \leq c(n)\varepsilon^\alpha$ on B'_r , where $12r < R$. We shall extend this estimate to a ball in \mathbb{R}^n .

We consider analytic continuation \tilde{F} of (all components of) F that is obtained by the series expansion and diverges in $B_{r,\mathbb{C}}$. By Lemma 4.4, we have $\sup_{B'_{t,\mathbb{C}}} |\tilde{F}| \leq c(n)\varepsilon^\alpha$ for $t < \frac{1}{3}r$. Thus we can estimate the partial derivatives of F with respect to the first $n - 1$ variables. Letting $\lambda = (\lambda_1, \dots, \lambda_{n-1}, 0)$, we have

$$|\mathcal{D}^\lambda F| \leq c(n)|\lambda|! \left(\frac{2}{t}\right)^{|\lambda|} \varepsilon^\alpha \quad \text{on } B'_s, \quad s < \frac{1}{2}t.$$

To get estimates for the partial derivatives with respect to x_n , note that F (with all its partial derivatives) satisfies a GCR system and all components of F are harmonic functions. Thereby we have

$$|\partial_n F| = |B_n A_n \partial_n F| = \left| B_n \left(- \sum_1^{n-1} A_j \partial_j F \right) \right| \leq c(A, n) \frac{2}{t} \varepsilon^\alpha \quad \text{on } B'_s.$$

Furthermore, the following estimates hold on B'_s :

$$|\partial_n^{2k} F| = \left| \left(- \sum_1^{n-1} \partial_j^2 \right)^k F \right| \leq c(n)n^k(2k)! \left(\frac{2}{t}\right)^{2k} \varepsilon^\alpha$$

and

$$|\partial_n^{2k+1} F| = \left| B_n \left(- \sum_1^{n-1} A_j \partial_j \left(- \sum_1^{n-1} \partial_j^2 \right)^k \right) F \right| \leq c(n, A)n^k(2k+1)! \left(\frac{2}{t}\right)^{2k+1} \varepsilon^\alpha.$$

Finally, for $x = (x', x_n)$ with $|x'| < s$ and $x_n < t/4\sqrt{n}$, we have

$$|F(x', x_n)| = \left| \sum_0^\infty \frac{1}{l!} \partial_n^l F(x', 0) x_n^l \right| \leq c(n, A) \varepsilon^\alpha.$$

In particular, the last estimate is valid in the ball of radius $r/12\sqrt{n}$. Once again, using the three-balls theorem, we can replace the ball by any other that is compactly supported in B_R and we are done.

5. Some definitions and results from potential theory

In this section results from potential theory that were used in the work are collected. Our main sources are [1, 12, 14].

5.1. The Wiener and logarithmic capacities on the complex plane

We follow Landkof’s book [12, Chapter II, no. 15] to define the capacity of subsets of (the unit disc in) the complex plane. Let E be a compact subset of the unit disc. Then the Wiener capacity of E is

$$\text{cap}(E) = \left(\inf \left\{ I(\mu) = \int_E \int_E \ln \frac{1}{|x - y|} d\mu(x) d\mu(y), \text{supp}(\mu) \subset E, \mu(E) = 1 \right\} \right)^{-1},$$

where the infimum is taken over all Radon measures μ .

A compact subset E of the unit disc is called polar if there exists a subharmonic function u such that $u|_E = -\infty$; E is a polar set if and only if $\text{cap}(E) = 0$. We shall also use the logarithmic capacity of E defined by

$$C_0(E) = \exp(-\text{cap}(E)^{-1}). \tag{5.1}$$

It is easy to see that C_0 is a non-negative monotone non-decreasing function of compact sets and the logarithmic capacity of a ball is given by $C_0(B_r) = r$ (see [12, Chapter II, nos 16, 17]). Another simple inequality we use is if K is a compact subset of a line, then $C_0(K) \geq \frac{1}{4}|K|$, where $|K|$ is the linear measure of K .

5.2. Chebyshev constants and transfinite diameter

Denote by \mathcal{P}_m the space of all polynomials of degree m whose leading coefficient equals one. Then Chebyshev’s constants for E are given by

$$\rho_m(E) = \min_{\mathcal{P}_m} \max_E |P_m(z)|^{1/m}.$$

Now, the m th transfinite diameter of E is

$$d_m(E) = \max_{z_j \in E} \left(\prod_{0 \leq j < k \leq m} |z_j - z_k| \right)^{2/m(m+1)}.$$

Then limits of both sequences $\{\rho_m\}$ and $\{d_m\}$ exist and are equal to the logarithmic capacity of E (see [1, Part I, § 7]):

$$\lim_{m \rightarrow \infty} \rho_m(E) = \lim_{m \rightarrow \infty} d_m(E) = C_0(E).$$

The following simple inequality was used:

$$\rho_m(E) \geq C_0(E). \quad (5.2)$$

In order to prove it let us take $P_m^* \in \mathcal{P}_m$ such that $\rho_m(E) = \max_E |P_m^*|^{1/m}$. Then for any positive integer n we have

$$\rho_m(E) = \max_E |(P_m^*)^n|^{1/mn} \geq \rho_{mn}(E).$$

As we know that $\lim_{m \rightarrow \infty} \rho_m(E)$ exists, we get (5.2) immediately.

5.3. The Riesz capacities in \mathbb{R}^n and the capacitary dimension

Definition 5.1. Let $s > 0$. The Riesz s -capacity of a set $A \subset \mathbb{R}^N$ is defined by

$$C_s(A) = \sup\{I_s(\mu)^{-1} : \mu(A) = 1\}, \quad (5.3)$$

where the supremum is taken over all Radon measures μ compactly supported on A , and

$$I_s(\mu) = \iint |x - y|^{-s} d\mu(x) d\mu(y).$$

Similarly, the logarithmic capacity of A is defined by

$$C_0(A) = \sup\left\{\exp\left(-\iint \ln \frac{1}{|x - y|} d\mu(x) d\mu(y)\right) : \mu(A) = 1\right\}.$$

It is easy to check that $C_s(B_r) = c(s, n)r^s$ if $0 < s < n$.

By the mean inequality, we have

$$I_s(\mu) \geq \exp\left(s \iint \ln \frac{1}{|x - y|} d\mu(x) d\mu(y)\right) = \exp(sI(\mu)).$$

Thus,

$$C_s(A) \leq C_0^s(A), \quad (5.4)$$

where C_0 is the logarithmic capacity.

Definition 5.2. Let A be a subset of \mathbb{R}^N . The capacitary dimension of A is defined by

$$\dim_c(A) = \sup\{s : C_s(A) > 0\}.$$

The remarkable fact is that the capacitary dimension coincides with the Hausdorff one (see [14, Chapter 8]).

5.4. Slicing

Suppose that $A \subset \mathbb{R}^N$ is such that $\mathcal{C}(A) = C_{N-1+\delta}(A) > 0$ for some positive δ . We consider intersections of A and one-dimensional lines. Consider all lines passing through the origin, and for any such line l and any $a \in l^\perp$ define $l_a = l + a$. The following inequality is used to reduce the multidimensional case to the one-dimensional one (see [14, § 10.9] and [17]):

$$\iint_{l^\perp} C_\delta(A \cap l_a) \, d\mathcal{H}^{N-1}(a) \, d\gamma_{N,1}(l) \geq c_0(N)C_{N-1+\delta}(A), \tag{5.5}$$

where $\gamma_{N,1}$ is the measure on the manifold of all lines (in \mathbb{R}^N) passing through the origin which is induced by the surface measure on \mathbb{S}^{N-1} . This inequality implies the following lemma.

Lemma 5.3. *There exist $k = k(N)$ such that for any $A \subset B_r$*

$$\mathcal{H}^{N-1}\{a \in l^\perp : C_\delta(A \cap l_a) \geq kr^{1-N}\mathcal{C}(A)\} \geq c_1(N)r^{-\delta}\mathcal{C}(A)$$

is valid for at least one line l .

Proof. According to (5.5), there exists l such that

$$\int_{l^\perp} C_\delta(A \cap l_a) \, d\mathcal{H}^{N-1}(a) \geq c_0(N)\mathcal{C}(A).$$

By the hypothesis $A \subset B_r$, $C_\delta(A \cap l_a) \leq Cr^\delta$ and

$$\mathcal{H}^{N-1}\{a \in l^\perp : A \cap l_a \neq \emptyset\} \leq c_2(N)r^{N-1}.$$

For any $k > 0$, we define

$$E_k = \{a \in l^\perp : C_\delta(A \cap l_a) \geq kr^{1-N}\mathcal{C}(A)\} \quad \text{and} \quad H_k = \mathcal{H}^{N-1}(E_k).$$

Then we have

$$\begin{aligned} c_0(N)\mathcal{C}(A) &\leq \int_{l^\perp} C_\delta(A \cap l_a) \, d\mathcal{H}^{N-1}(a) \\ &= \int_{E_k} C_\delta(A \cap l_a) \, d\mathcal{H}^{N-1}(a) + \int_{l^\perp \setminus E_k} C_\delta(A \cap l_a) \, d\mathcal{H}^{N-1}(a) \\ &\leq Cr^\delta H_k + kr^{1-N}\mathcal{C}(A)(c_2(N)r^{N-1} - H_k) \\ &= (Cr^\delta - kr^{1-N}\mathcal{C}(A))H_k + c_2(N)k\mathcal{C}(A). \end{aligned}$$

Taking k small enough, we may assume that $Cr^\delta - kr^{1-N}\mathcal{C}(A) > c(N)r^\delta > 0$ (since $A \subset B_r$) and $c_2(N)k < c_0(N)$. Then we have $H_k \geq c_1(N)r^{-\delta}\mathcal{C}(A)$ and the lemma is proved. □

Acknowledgements. The author was supported by a Norwegian Research Foundation postdoctoral fellowship. I thank Vladimir Eiderman, Victor Havin and Mikhail Sodin for useful discussions on the subject of the article. Thanks also go to Alexander Rashkovskii and Norman Levenberg, who pointed out the references on pluri-potential theory to me. I am very grateful to the referee, whose remarks and suggestions improved the final version of the article.

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