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RESEARCH ARTICLE

The zero mass problem for Klein-Gordon equations: quadratic null interactions

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Abstract

We study in $\mathbb{R}^{3+1}$ a system of nonlinearly coupled Klein-Gordon equations under the null condition, with (possibly vanishing) mass varying in the interval $[0, 1]$. Our goal is three-fold, which extends the results in the earlier work of [5, 3]: 1) we want to establish the global well-posedness result to the system that is uniform in terms of the mass parameter (i.e., the smallness of the initial data is independent of the mass parameter); 2) we want to obtain a unified pointwise decay result for the solution to the system, in the sense that the solution decays more like a wave component (independent of the mass parameter) in a certain range of time, while the solution decays as a Klein-Gordon component with a factor depending on the mass parameter in the other part of the time range; 3) the solution to the Klein-Gordon system converges to the solution to the corresponding wave system in a certain sense when the mass parameter goes to 0. In order to achieve these goals, we will rely on both the flat and hyperboloidal foliation of the spacetime and prove a mass-independent L^2–type energy estimate for the Klein-Gordon equations with possibly vanishing mass. In addition, the case of the Klein-Gordon equations with certain restricted large data is discussed.

Contents

1. Introduction

1.1. Motivation and review of the classical results

The study of nonlinear wave equations, nonlinear Klein-Gordon equations and their coupled systems has been an active area of research for decades, and the question 'what kind of quadratic nonlinearities lead to global-in-time solutions?' has attracted special attention from researchers. In addition to the mathematical interest and challenges, such studies are also motivated by the Einstein equations, the Einstein-Klein-Gordon equations, the Yang-Mills equations, the electroweak standard model and many other important models from mathematical physics. Recall that, on the one hand, wave equations in $\mathbb{R}^{3+1}$ with null form nonlinearities were proved to admit global-in-time solutions independently by Klainerman [14] and Christodoulou [2] (and Klainerman identified the class of null forms); see also the generalizations of null condition to the weak null condition in [19, 20] and to the non-resonant condition in [22], for instance. On the other hand, it was shown by Klainerman [12] and Shatah [23] that Klein-Gordon equations with general quadratic nonlinearities in $\mathbb{R}^{3+1}$ admit small solutions. In addition, the global well-posedness results for different types of coupled wave and Klein-Gordon systems, with or without physical models behind, were obtained; see, for instance, [1, 6, 5, 7, 9, 10, 11, 15, 16, 17, 18, 21, 27].

It is well known that (linear) Klein-Gordon components decay $t^{-1/2}$ faster than the (linear) wave components in $\mathbb{R}^{n+1}$ ($n \geq 1$) and that the presence of the mass term allows one to control the L^2–type energy of the Klein-Gordon components by their natural energy. Both of these make it less difficult to study nonlinear Klein-Gordon equations in $\mathbb{R}^{3+1}$. When it comes to the study of nonlinear wave equations, we can utilise the scaling vector field, which makes it easy to apply the Klainerman-Sobolev inequality, and we can rely on the conformal energy estimates to obtain L^2–type energy estimates for wave components (with no derivatives). Thus, we can see from the simple comparisons above that different features help to study pure wave equations and pure Klein-Gordon equations.

Concerning the fact that Klein-Gordon equations become wave equations when the masses are set to be 0, a natural interesting question is that for Klein-Gordon equations with varying mass in $[0, 1]$, what kind of quadratic nonlinearities can ensure small data global existence results that are uniform in terms of the varying mass parameter? Our primary goal is to prove that all kinds of null nonlinearities can uniformly guarantee global existence results for systems of Klein-Gordon equations with mass varying in $[0, 1]$. Such results are known to be valid at the end points 0, 1, which correspond to wave equations and Klein-Gordon equations with fixed mass, respectively. But more is involved if one wants to get global existence results uniform in terms of the mass parameter in $[0, 1]$, because, for instance, we cannot use the scaling vector field, cannot obtain mass-independent L^2–type estimates by the mass term or by the conformal energy estimates, and so on. The main difficulties arise when the mass is close to 0, which is why we refer to this problem as the zero mass problem of the Klein-Gordon equations.

In addition, the study of the Klein-Gordon equations with varying mass (especially when the mass goes to 0) is also motivated by the study of mathematical physics. We briefly recall in [5] that when studying the electroweak standard model, there appear several Klein-Gordon equations with different masses, and physical experiments have verified that some of the masses are extremely small (close to 0 but still positive) compared to others. Thus it is important to obtain results that are uniform in terms of the small masses for Klein-Gordon equations. Since the cases of $m = 0, 1$ have been well studied,

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we will only focus on Klein-Gordon equations with one single mass $m \in [0, 1]$ to capture the most interesting feature instead of coupling with Klein-Gordon equations with masses 0, 1.

1.2. Model of interest

We will consider the following system of coupled Klein-Gordon equations with varying mass $m \in [0, 1]$:

$$-\Box v_i + m^2 v_i = N_i^{jk} Q_0(v_j, v_k) + M_i^{jk\alpha\beta} Q_{\alpha\beta}(v_j, v_k), \tag{1.1}$$

with initial data prescribed on $t = t_0 = 2$

$$\big(v_i, \partial_t v_i\big)(t_0) = \big(v_{i0}, v_{i1}\big). \tag{1.2}$$

In the above, $\Box = \eta^{\alpha\beta}\partial_{\alpha\beta}$ is the wave operator, where $\eta = \mathrm{diag}(-1, 1, 1, 1)$ is the metric of the spacetime, and the Einstein summation convention is adopted. We note that $Q_0(v_j, v_k) = \partial_\alpha v_j \partial^\alpha v_k$ and $Q_{\alpha\beta}(v_j, v_k) = \partial_\alpha v_j \partial_\beta v_k - \partial_\alpha v_k \partial_\beta v_j$ are classical null forms of Klainerman. The indices $i, j, k \in \{1, \cdots, N_0\}$ with N_0 the number of equations (also the number of unknowns), and we use $a, b, c, \cdots \in \{1, 2, 3\}$ and $\alpha, \beta, \gamma, \cdots \{0, 1, 2, 3\}$ to denote the space indices and the spacetime indices, respectively. Throughout the paper, we will also use $A \lesssim B$ to indicate $A \leq CB$, with C a generic constant (independent of the mass parameter m).

At the end points of $m = 0$ and $m = 1$, the small data global existence result (as well as other properties of the solution) for the system in equations (1.1)–(1.2) is well known, and the proofs are conducted depending on the different features of the pure wave equations and the pure Klein-Gordon equations. Here we want to establish the global existence result and explore the properties of the solution for the system in equation (1.1), which are **uniform** in terms of the mass parameter $m \in [0, 1]$, by which we mean the smallness of the initial data is independent of m. In addition, we derive the **unified** pointwise decay result for the equations, by which we mean the pointwise decay result unifies the wave and the Klein-Gordon equations with explicit dependence on the mass parameter m. In addition, it is also interesting to show that the solution to equation (1.1) converges to the corresponding wave system when $m \to 0$. Since some features of the pure wave equations or the pure Klein-Gordon equations cannot be relied on when obtaining the uniform result, the analysis of the proof is more subtle and requires new insights.

1.3. Difficulties and new observations

When studying the Klein-Gordon equations, the most well-known difficulty is that one cannot use the scaling vector field because the scaling vector field does not commute with the Klein-Gordon operators. However, more difficulties arise in studying the Klein-Gordon system with possibly vanishing mass.

First, in order to apply Sobolev–type inequalities to obtain pointwise decay results for the Klein-Gordon components $v = (v_i)$ or to estimate the null forms, we need to bound the L^2–type norm for v_i, which is supposed to be mass-independent, and this is the most difficult part. On the one hand, the presence of the mass term in the Klein-Gordon equation does not seem to help obtain the L^2–type energy estimate. That is because what we can get from the mass term is only

$$m\|v\|_{L^2} \lesssim B, \qquad \text{i.e. } \|v\|_{L^2} \lesssim m^{-1}B,$$

[1]which is mass-dependent, and very unfortunately, the bound for $\|v\|_{L^2}$ blows up when m goes to 0; and in the above, B represents some bound from the energy estimates. On the other hand, the conformal energy estimates allow one to get the L^2–type estimates for wave components (i.e., the cases of $m = 0$),

[1] m^{-1} is interpreted as $+\infty$ when $m = 0$.

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but they can no longer be applied due to the presence of the mass term, which means we cannot obtain L^2–type estimates for v_i using the conformal energy. Second, the solution to the system in equation (1.1) does not decay sufficiently fast. In general, we can expect solutions to Klein-Gordon equations with fixed mass to decay like $t^{-3/2}$ in $\mathbb{R}^{3+1}$, but due to the possibly vanishing mass $m \in [0, 1]$, the best we can expect for the Klein-Gordon components v_i is the (mass-independent) wave decay

$$|v_i| \lesssim t^{-1},$$

and the (mass-dependent) Klein-Gordon decay

$$|v_i| \lesssim m^{-1} t^{-3/2}.$$

In addition, the null form of the type $\partial^\alpha v_j \partial_\alpha v_k$ does not seem to decay sufficiently fast. It is not consistent with the Klein-Gordon equations since we lack the scaling vector field to gain a good factor of t^{-1} from $\partial^\alpha v_j \partial_\alpha v_k$. Last but not least, there are some difficulties in gaining the factor t^{-1} from the null forms $Q_0(v_j, v_k), Q_{\alpha\beta}(v_j, v_k)$ in the highest-order energy estimate, which again is due to the lack of the conformal energy estimate.

In order to tackle the problems caused by the presence of the possibly vanishing mass term, we will rely on the following key observations and insights. First, we will use the hyperboloidal foliation of the spacetime to prove the (uniform) global existence result for the system in equation (1.1), which is developed by Klainerman [12], Hormander [8], LeFloch-Ma [16], and so forth. We will take advantage of the fact that the null forms $(Q_0, Q_{\alpha\beta})$ can be decomposed as sums of products of good components in the hyperboloidal foliation setting (see Lemma 2.4); this is true even for the highest-order energy estimate. As a consequence, we are able to obtain the mass-dependent pointwise decay result

$$\big|v_i(t, x)\big| \lesssim m^{-1} t^{-3/2}.$$

Next, we will move to the usual flat foliation of the spacetime to show the unified pointwise decay result, which is the most difficult part. To achieve this, we will obtain the mass-independent L^2–type norm estimates for the solution $v = (v_i)$ by using tricks from the Fourier analysis. To be more precise, we write the Klein-Gordon equation in the Fourier space and solve the corresponding ordinary differential equation to get the solution in the Fourier space, and then obtain the mass-independent L^2–type norm estimates for the solution $v = (v_i)$ (see Proposition 3.1). However, according to Proposition 3.1, in order to get sufficiently good L^2–type estimates for the solution, we need to gain the extra factor t^{-1} from the null nonlinearities. For the null forms of type $Q_{\alpha\beta}$, we easily have

$$\big|Q_{\alpha\beta}(v_j, v_k)\big| \lesssim \frac{1}{t}\big(|\Gamma v_j||\partial v_k| + |\Gamma v_k||\partial v_j|\big), \qquad \Gamma \in \{\partial_\alpha, \Omega_{ab}, L_a\},$$

and for the higher-order case, the following observation (from [11]) helps

$$Q_{\alpha\beta}(v_j, v_k) = \partial_\alpha(v_j \partial_\beta v_k) - \partial_\beta(\partial_\alpha v_k v_j),$$

thanks to the hidden divergence form of the null nonlinearities $Q_{\alpha\beta}$. For the null forms of type Q_0, at first glance, it does not seem to be possible to gain the factor t^{-1} from the null form Q_0 because we do not have good control over the scaling vector field. But we observe that a nonlinear transformation will help, which transforms the quadratic term Q_0 to the sum of cubic terms and quadratic terms with a good factor m^2 in front. These observations allow us to obtain the mass-independent L^2–type estimates, and hence the mass-independent pointwise decay result for the solution

$$\big|v_i(t, x)\big| \lesssim t^{-1}$$

can be obtained with the aid of the Klainerman-Sobolev inequality in Proposition 2.6. More details follow. We will divide the solution into several parts and analyse each part according to its features. For the parts where we can gain t^{-1} factor from the null form or the nonlinearities are cubic, Proposition 3.1 will be sufficient to obtain the mass-independent pointwise decay. For the part with quadratic nonlinearities with the good factor m^2, we will carefully study the m-dependent relation of the norms of the nonlinearities and try to gain the factor m to cancel the one appearing in the energy.

1.4. Main theorem

Now we provide the statement of the main result.

Theorem 1.1. *Consider the system of Klein-Gordon equations (1.1) with mass $m \in [0, 1]$, and let $N \geq 6$ be an integer. There exists small $\epsilon_0 > 0$, which is* ***independent*** *of the mass parameter m, such that for all $\epsilon < \epsilon_0$ and all compactly supported initial data that are small in the sense that*

$$\|v_{i0}\|_{H^{N+1}} + \|v_{i1}\|_{H^N} \leq \epsilon, \qquad \textit{for all } i, \tag{1.3}$$

then the Cauchy problem in equations (1.1)–(1.2) admits a global-in-time solution $v = (v_i)$. In addition, the unified decay of the solution is obtained

$$\left|v_i(t, x)\right| \lesssim \frac{1}{t + mt^{3/2}}. \tag{1.4}$$

In the proof of Theorem 1.1, we will always assume $m \in (0, 1]$ unless specified since the result for the case of $m = 0$ is well-known and classical. We will also assume the initial data (v_{i0}, v_{i1}) are spatially supported in the unit ball $\{(x, t) : t = t_0 = 2, |x| \leq 1\}$, but the results in the theorem still hold for all of the initial data prescribed at any $t = constant$ with compact support; see the remark in [6] that is below the main theorem there. We note that the compactness assumption implies

$$\|v_{i1}\|_{L^{6/5}(\mathbb{R}^3)} \lesssim \|v_{i1}\|_{L^2(\mathbb{R}^3)},$$

and this will be used when applying Proposition 3.1.

It can be seen from Theorem 1.1 that the global existence result and the pointwise decay result are both consistent with the cases of $m = 0$ and $m = 1$, which are the usual wave equations and the usual Klein-Gordon equations (with fixed mass). It is worth mentioning that the unified decay result in equation (1.4) shows that the solution decays more like a wave component (with no m dependence) as t^{-1} in the time range $t \in [t_0, m^{-2})$, while it decays more like a Klein-Gordon component (with m dependence) as $m^{-1}t^{-3/2}$ in the rest of the time range (if non-empty). In addition to the results contained in Theorem 1.1, we have the following convergence result, which tells us that the solution to the system in equation (1.1) converges to the solution to the corresponding wave system (i.e., the system in equation (1.1) with $m = 0$) in a certain sense. Let

$$v^{(m)}, \qquad m \in [0, 1],$$

denote the solution to the system in equation (1.1) with mass m, and we can now demonstrate the convergence theorem.

Theorem 1.2. *Consider the system in equation (1.1), and let the same assumptions in Theorem 1.1 hold. Then the solution to the system in equation (1.1) with mass m converges to the system in equation (1.1) with $m = 0$, in the sense that (with $0 < \delta \ll 1$)*

$$\left\|\partial\partial^I L^J \left(v^{(m)} - v^{(0)}\right)\right\|_{L^2} + m\left\|\partial^I L^J \left(v^{(m)} - v^{(0)}\right)\right\|_{L^2} \lesssim m^2 t^{1+\delta}, \qquad |I| + |J| \leq N. \tag{1.5}$$

We note that Theorem 1.2 indicates that the solution $v^{(m)}$ tends to $v^{(0)}$ at the rate m^2 when $m \to 0$ on each fixed slice $t = constant$, but the bounds for the energy of the difference $v^{(m)} - v^{(0)}$ blow up as $t \to +\infty$ for each fixed m.

The study of the zero mass problem of the Klein-Gordon equations is a new subject. We recall that such studies have appeared in earlier work [5, 3], where the zero mass problem for Dirac equations (we note that Dirac equations can be transformed into Klein-Gordon equations) and Klein-Gordon equations with divergence form nonlinearities were investigated. In this paper, we extend the earlier work to be able to treat quadratic nonlinearities satisfying the null condition, which is a big step forward. It is also natural to deal with null forms as they frequently appear in the field of wave and Klein-Gordon equations. Of great mathematical interest, the zero mass problem for the Klein-Gordon equations with more types of nonlinearities, as well as in the low dimension cases, is to be studied, which also seems very promising and challenging.

1.5. Further discussions

The goal of this part is to discuss one way to show global existence for nonlinear Klein-Gordon equations with some restricted bounded large data. For general bounded large data, our method does not apply.

For simplicity, we consider the Klein-Gordon equation with $m \in (0, 1]$ (note we exclude $m = 0$ with no harm)

$$\begin{aligned} &-\Box u + m^2 u = \partial_\alpha u \partial^\alpha u, \\ &\big(u, \partial_t u\big)(t=0) = (u_0, u_1) = (u_0, 0). \end{aligned} \tag{1.6}$$

From Theorem 1.1, we know there exists ϵ_0 independent of m, such that for all compactly supported initial data satisfying the smallness condition

$$\|u_0\|_{H^7} < \epsilon_0,$$

the Cauchy problem in equation (1.6) admits a global solution with pointwise decay

$$|u(t,x)| \lesssim \frac{1}{t + mt^{3/2}}.$$

We introduce the new spacetime variables

$$(T, X) = (mt, mx)$$

and define the function

$$U(T, X) = u(m^{-1}T, m^{-1}X) = u(t, x).$$

A simple calculation shows that $U(T, X)$ solves the nonlinear Klein-Gordon equation with fixed mass 1, which reads

$$\begin{aligned} &\partial_T \partial_T U(T,X) - \sum_a \partial_{X^a}\partial_{X^a} U(T,X) + U(T,X) = -\big(\partial_T U(T,X)\big)^2 + \sum_a \big(\partial_{X^a} U(T,X)\big)^2, \\ &\big(U(T,X), \partial_T U(T,X)\big)(T=0) = (U_0(X), 0), \end{aligned} \tag{1.7}$$

in which

$$U_0(X) = u_0(m^{-1}X).$$

We observe that

$$\big\|\partial_X^I U_0\big\|_{L^2(\mathbb{R}^3)} = m^{3/2-|I|}\big\|\partial_x^I u_0\|_{L^2(\mathbb{R}^3)}, \qquad |I| \le 7.$$

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Clearly, for any chosen non-zero small initial data $(u_0, 0)$ for equation (1.6), we can always pick $m \in (0, 1]$ sufficiently small so that the initial data for equation (1.7) is large: that is, satisfying

$$\|U_0\|_{H^7} > C_L$$

for any given large constant $C_L > 1$. This means the nonlinear Klein-Gordon equation (1.7) with fixed mass 1 admits a global solution even if the size of the initial data is large (but restricted).

1.6. Outline

The rest of this paper is organised as follows: In Section 2, we revisit some notations, Sobolev–type inequalities and basic results on the Klein-Gordon equations. Next, we provide the key result of obtaining the mass-independent L^2 norm estimates for solutions to the Klein-Gordon equations with possibly vanishing masses in Section 3. Then we prove the global existence result in Section 4. Finally, the proof for the mass-independent decay result and the proof for the convergence result are illustrated in Section 5 and Section 6, respectively.

2. Preliminaries

2.1. Basic notations

We work in the $(3+1)$ dimensional spacetime with metric $\eta = \mathrm{diag}(-1, 1, 1, 1)$. We write a point $(x^0, x^a) = (t, x^a)$, and the indices are raised or lowered by the metric η. We use

$$\begin{aligned}
\partial_\alpha &= \partial_{x^\alpha}, & \alpha &= 0, 1, 2, 3,\\
\Omega_{ab} &= x_a\partial_b - x_b\partial_a, & a, b &= 1, 2, 3, \text{ and } a < b,\\
L_a &= x_a\partial_t + t\partial_a, & a &= 1, 2, 3
\end{aligned}$$

to denote the vector fields of translation, rotation and Lorentz boosts, respectively. For convenience, we use ∂, Ω, L to represent a general vector field of translation, rotation and Lorentz boost, respectively; and with the notation

$$V = \{\partial_\alpha, \Omega_{ab}, L_a\},$$

Γ is used to represent a general vector field in V.

When it turns to the hyperboloidal foliation of the spacetime of the cone $\mathcal{K} := \{(t, x) : t \geq t_0 = 2, t \geq |x| + 1\}$, we use $\mathcal{H}_s = \{(t, x) : t^2 = |x|^2 + s^2\}$ to denote a hyperboloid at hyperbolic time s with $s \geq s_0 = 2$. We note that throughout we will only consider (unless specified) functions with support in $\mathcal{K}$, since the solution to equation (1.1) is supported in $\mathcal{K}$. We emphasize here that for all points $(t, x) \in \mathcal{K} \bigcap \mathcal{H}_s$ $(s \geq 2)$, the following relations hold:

$$s \leq t \leq s^2, \qquad |x| \leq t. \tag{2.1}$$

In order to adapt to the hyperboloidal foliation of the spacetime, we first recall the semi-hyperboloidal frame introduced in [16], which is defined by

$$\underline{\partial}_0 = \partial_t, \qquad \underline{\partial}_a = \frac{L_a}{t} = \partial_a + \frac{x_a}{t}\partial_t. \tag{2.2}$$

We can also represent the usual partial derivatives ∂_α in terms of the semi-hyperboloidal frame by

$$\partial_0 = \underline{\partial}_0, \qquad \partial_a = -\frac{x_a}{t}\underline{\partial}_0 + \underline{\partial}_a. \tag{2.3}$$

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We denote the Fourier transform of a nice function u by

$$\widehat{u}(\xi) = \int_{\mathbb{R}^3} u(x) e^{-2i\pi x\cdot\xi}\, dx.$$

We recall some properties regarding the Fourier transform, which will be used in the analysis. The partial derivatives are reflected by Fourier multipliers

$$\widehat{\partial_a u}(\xi) = 2\pi i \xi_a \widehat{u}(\xi), \tag{2.4}$$

and the Plancheral identity connects the L^2 norms between the function and its Fourier transform

$$\|u\|_{L^2(\mathbb{R}^3)} = \big\|\widehat{u}\big\|_{L^2(\mathbb{R}^3)}. \tag{2.5}$$

2.2. Estimates for commutators and null forms

Estimates for commutators

We first demonstrate some well-known results regarding the commutators of different vector fields, which can be found in [26, 16].

Lemma 2.1. *Let u be a sufficiently regular function with support $\mathcal{K}$, and denote the commutator by $[\Gamma, \Gamma'] = \Gamma\Gamma' - \Gamma'\Gamma$; then we have*

$$\begin{aligned}
\big|[\partial_\alpha, L_a]u\big| + \big|[\partial_\alpha, \Omega_{ab}]u\big| &\lesssim \sum_\beta \big|\partial_\beta u\big|,\\
\big|[L_c, \Omega_{ab}]u\big| + \big|[L_a, L_b]u\big| &\lesssim \sum_d \big|L_d u\big|,\\
\big|[L_a, (s/t)]u\big| &\lesssim \big|(s/t)u\big|,\\
\big|[L_b L_a, (s/t)]u\big| &\lesssim \big|(s/t)u\big| + \sum_c \big|(s/t)L_c u\big|,\\
\big|[\underline{\partial}_a, L_b]u\big| &\lesssim \sum_c \big|\underline{\partial}_c u\big|.
\end{aligned} \tag{2.6}$$

Next, we recall the following result from [26], which tells us that the null forms acted by a vector field still give us null forms.

Lemma 2.2. *For all nice functions u, w, we have*

$$\begin{aligned}
\partial_\alpha Q_0(u,w) - Q_0(\partial_\alpha u, w) - Q_0(u, \partial_\alpha w) &= 0,\\
\partial_\gamma Q_{\alpha\beta}(u,w) - Q_{\alpha\beta}(\partial_\gamma u, w) - Q_{\alpha\beta}(u, \partial_\gamma w) &= 0,\\
L_a Q_0(u,w) - Q_0(L_a u, w) - Q_0(u, L_a w) &= 0,\\
\big|L_a Q_{\alpha\beta}(u,w) - Q_{\alpha\beta}(L_a u, w) - Q_{\alpha\beta}(u, L_a w)\big| &\le \sum_{\alpha',\beta'} \big|Q_{\alpha'\beta'}(u,w)\big|.
\end{aligned} \tag{2.7}$$

Estimates for null forms

We first recall the classical estimates for null forms of the type $Q_{\alpha\beta}$, which can be found in [26].

Lemma 2.3. *We have for sufficiently regular functions u, w with support in $\mathcal{K} = \{(t,x) : t \ge t_0, t \ge |x| + 1\}$*

$$\big|Q_{\alpha\beta}(u,w)\big| \lesssim \frac{1}{t}\big(|Lu||\partial w| + |\partial u||Lw|\big). \tag{2.8}$$

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In addition, following from [17] of the hyperboloidal setting, we also have the following estimates for all types of null forms.

Lemma 2.4. *It holds for smooth functions u, w with support in $\mathcal{K} = \{(t,x) : t \geq t_0, t \geq |x| + 1\}$ that*

$$\big|Q_0(u,w)\big| + \big|Q_{\alpha\beta}(u,w)\big| \lesssim (s/t)^2 |\partial_t u \partial_t w| + \sum_{a,\alpha} \big(|\underline{\partial}_\alpha u \underline{\partial}_a w| + |\underline{\partial}_\alpha w \underline{\partial}_a u|\big). \tag{2.9}$$

Proof. We revisit the proof for $Q_0(u,w)$ only, from [16], for readers who are not familiar with the hyperboloidal foliation method.

Recall the semi-hyperboloidal frame

$$\partial_t = \underline{\partial}_0, \qquad \partial_a = -\frac{x_a}{t}\underline{\partial}_0 + \underline{\partial}_a,$$

and we express the null form $Q_0(u,w)$ in the semi-hyperboloidal frame to get

$$Q_0(u,w) = -\frac{s^2}{t^2}\underline{\partial}_0 u \underline{\partial}_0 w - \frac{x_a}{t}\big(\underline{\partial}_0 u \underline{\partial}^a w + \underline{\partial}_0 w \underline{\partial}^a u\big) + \underline{\partial}_a u \underline{\partial}^a w.$$

Then the fact $|x| \leq t$ concludes the estimates. □

2.3. *Sobolev–type inequalities*

Klainerman-Sobolev inequality

In order to obtain pointwise decay estimates for the Klein-Gordon components, we need the following Klainerman-Sobolev inequality, which was introduced in [13]. We need the following version of Klainerman-Sobolev inequality because it will not need to rely on the scaling vector field $L_0 = t\partial_t + x^a \partial_a$ (which is not consistent with the Klein-Gordon equations), and this feature is vital in obtaining the mass-independent pointwise decay results for the Klein-Gordon components.

Proposition 2.5. *Assume $u = u(t,x)$ is a sufficiently smooth function that decays sufficiently fast at space infinity for each fixed $t \geq 2$; then for any $t \geq 2$, $x \in \mathbb{R}^3$, we have*

$$|u(t,x)| \lesssim t^{-1} \sup_{0 \leq t' \leq 2t, |I| \leq 3} \big\|\Gamma^I u\big\|_{L^2(\mathbb{R}^3)}, \qquad \Gamma \in V = \{L_a, \partial_\alpha, \Omega_{ab} = x^a \partial_b - x^b \partial_a\}. \tag{2.10}$$

We will use a simplified version of Proposition 2.5 where we do not need to use the rotation vector field because we only need to consider functions supported in $\mathcal{K} = \{(t,x) : t \geq 2, t \geq |x| + 1\}$.

Proposition 2.6. *Assume $u = u(t,x)$ is a sufficiently smooth function with support $\mathcal{K}$; then for any $t \geq 2$, $x \in \mathbb{R}^3$, we have*

$$|u(t,x)| \lesssim t^{-1} \sup_{t_0 \leq t' \leq t_0 + 2t, |I| + |J| \leq 3} \big\|\partial^I L^J u\big\|_{L^2(\mathbb{R}^3)}. \tag{2.11}$$

Proof. The Klainerman-Sobolev inequality in equation (2.11) can be obtained from equation (2.10), the commutator estimates, and the fact that

$$\sum_{a<b} |\Omega_{ab} w| \lesssim \sum_a |L_a w|$$

holds for all nice functions w with support $\mathcal{K}$. □

Sobolev-type inequality on hyperboloids

We now recall a Sobolev-type inequality adapted to the hyperboloids from [16], which allows us to get the (mass-dependent) sup-norm estimates for the Klein-Gordon components.

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Proposition 2.7. *Let $u = u(t,x)$ be a sufficiently nice function with support $\{(t,x) : t \geq |x| + 1\}$; then for all $s \geq 2$, one has*

$$\sup_{\mathcal{H}_s} \left|t^{3/2} u(t,x)\right| \lesssim \sum_{|J| \leq 2} \|L^J u\|_{L^2_f(\mathcal{H}_s)}, \tag{2.12}$$

where the symbol L denotes the Lorentz boosts.

The Sobolev inequality in equation (2.12) combined with the commutator estimates gives us the following inequality:

$$\sup_{\mathcal{H}_s} \left|s t^{1/2} u(t,x)\right| \lesssim \sum_{|J| \leq 2} \|(s/t) L^J u\|_{L^2_f(\mathcal{H}_s)}. \tag{2.13}$$

Hardy inequality on hyperboloids

Proposition 2.8. *Let $u = u(x)$ be a sufficiently smooth function in dimension $d \geq 3$; then it holds ($r = |x|$)*

$$\left\|r^{-1} u\right\|_{L^2(\mathbb{R}^d)} \leq C \sum_a \left\|\partial_a u\right\|_{L^2(\mathbb{R}^d)}. \tag{2.14}$$

The Hardy inequality can also be adapted to the hyperboloidal setting; see, for instance, [16, 17].

Proposition 2.9. *Assume the function u is sufficiently regular and supported in the region $\mathcal{K}$; then for all $s \geq 2$, one has*

$$\|r^{-1} u\|_{L^2_f(\mathcal{H}_s)} \lesssim \sum_a \|\underline{\partial}_a u\|_{L^2_f(\mathcal{H}_s)}. \tag{2.15}$$

As a consequence, we also have

$$\|t^{-1} u\|_{L^2_f(\mathcal{H}_s)} \lesssim \sum_a \|\underline{\partial}_a u\|_{L^2_f(\mathcal{H}_s)}. \tag{2.16}$$

Sobolev embedding theorem

We recall the following type of Sobolev embedding theorem.

Proposition 2.10. *Let $u = u(x) \in L^{6/5}(\mathbb{R}^3)$; then it holds that*

$$\left\|\frac{u}{\Lambda}\right\|_{L^2(\mathbb{R}^3)} \lesssim \|u\|_{L^{6/5}(\mathbb{R}^3)}, \tag{2.17}$$

in which $\Lambda = \sqrt{-\Delta} = \sqrt{-\partial_a \partial^a}$.

2.4. Energy estimates for Klein-Gordon equations

Given a function $u = u(t,x)$ supported in $\mathcal{K}$, we define its energy $\mathcal{E}_m$, following [16], on a hyperboloid $\mathcal{H}_s$ by

$$\begin{aligned}
\mathcal{E}_m(s,u) &:= \int_{\mathcal{H}_s} \Big((\partial_t u)^2 + \sum_a (\partial_a u)^2 + 2(x^a/t)\partial_t u \partial_a u + m^2 u^2 \Big)\, dx \\
&= \int_{\mathcal{H}_s} \Big(((s/t)\partial_t u)^2 + \sum_a (\underline{\partial}_a u)^2 + m^2 u^2 \Big)\, dx \\
&= \int_{\mathcal{H}_s} \Big((\underline{\partial}_\perp u)^2 + \sum_a ((s/t)\partial_a u)^2 + \sum_{a<b} (t^{-1}\Omega_{ab} u)^2 + m^2 u^2 \Big)\, dx,
\end{aligned} \tag{2.18}$$

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in which $\underline{\partial}_\perp := \partial_t + (x^a/t)\partial_a$ is the orthogonal vector field. The integral $L^2_f(\mathcal{H}_s)$ is defined by

$$\|u\|^2_{L^2_f(\mathcal{H}_s)} := \int_{\mathcal{H}_s} |u|^2\, dx := \int_{\mathbb{R}^3} \big|u(\sqrt{s^2+|x|^2}, x)\big|^2\, dx. \tag{2.19}$$

We note that it holds

$$\big\|(s/t)\partial u\big\|_{L^2_f(\mathcal{H}_s)} + \sum_a \big\|\underline{\partial}_a u\big\|_{L^2_f(\mathcal{H}_s)} \lesssim \mathcal{E}_m(s,u)^{1/2},$$

which will be used frequently.

Next, we demonstrate the energy estimates in the hyperboloidal setting.

Proposition 2.11 (Energy estimates for wave-Klein-Gordon equations). *For $m \geq 0$ and for $s \geq s_0$ (with $s_0 = 2$), it holds that*

$$\mathcal{E}_m(s,u)^{1/2} \leq \mathcal{E}_m(s_0,u)^{1/2} + \int_2^s \big\| -\Box u + m^2 u\big\|_{L^2_f(\mathcal{H}_{s'})}\, ds' \tag{2.20}$$

for all sufficiently regular functions u, which are defined and supported in $\mathcal{K}_{[s_0,s]} = \bigcup_{s_0\leq s'\leq s}\mathcal{H}_{s'}$.

The proof of equation (2.20) can be found in [16, 17].

In comparison with $\mathcal{E}_m$, we use E_m to denote the usual energy on the flat slices $t = constant$, which is expressed as

$$E_m(t,u) = \int_{\mathbb{R}^3} \big|\partial u\big|^2 + m^2u^2\, dx.$$

Similarly, we have the following energy estimate:

$$E_m(t,u)^{1/2} \leq E_m(t_0,u)^{1/2} + \int_2^t \big\| -\Box u + m^2 u\big\|_{L^2(\mathbb{R}^3)}\, dt'. \tag{2.21}$$

3. Mass-independent L^2 norm estimates for Klein-Gordon equations

We will rely on the following key proposition to obtain the mass-independent L^2–type energy estimates for the solution to the system in equation (1.1). A similar result was previously obtained in [3], and we now provide an enhanced version of it. We note that the improvements compared with the results in [3] mainly include the following: 1) the norm for the right-hand side source is taken to be $\|\cdot\|_{L^{6/5}(\mathbb{R}^3)}$ in Proposition 3.1 instead of the previous weighted $\||x|\cdot\|_{L^2(\mathbb{R}^3)}$ of [3], which allows us to benefit from more decay rates in many cases; 2) the norm for the right-hand side source is without r-weight in Proposition 3.1 compared with the result in [3], and this is expected to treat the non-compactly supported case where it is much better to get rid of the r-weight in the region $r \gg t$.

Proposition 3.1. *Consider the wave-Klein-Gordon equation*

$$-\Box u + m^2 u = f, \qquad \big(u, \partial_t u\big)(t_0) = (u_0, u_1),$$

with mass $m \in [0,1]$, and assume

$$\|u_0\|_{L^2(\mathbb{R}^3)} + \|u_1\|_{L^2(\mathbb{R}^3)\cap L^{6/5}(\mathbb{R}^3)} \lesssim C_{t_0}, \quad \|f\|_{L^{6/5}(\mathbb{R}^3)} \leq C_f\, t^{-1+q},$$

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for some numbers C_{t_0} and C_f. Then we have

$$\|u\|_{L^2(\mathbb{R}^3)} \lesssim \begin{cases} C_{t_0} + C_f t^q, & q > 0, \\ C_{t_0} + C_f \log t, & q = 0, \\ C_{t_0} + C_f, & q < 0. \end{cases} \tag{3.1}$$

Proof. We first write the equation of u in the Fourier space (t, ξ)

$$\partial_t \partial_t \widehat{u} + \xi_m^2 \widehat{u} = \widehat{f}$$

and solve the ordinary differential equation to get the solution

$$\widehat{u}(t, \xi) = \cos\big(t\xi_m\big)\widehat{u}_0 + \frac{\sin\big(t\xi_m\big)}{\xi_m}\widehat{u}_1 + \frac{1}{\xi_m}\int_{t_0}^t \sin\big((t-t')\xi_m\big)\widehat{f}(t')\,dt',$$

with the notations defined by

$$\widehat{u}_0 = \widehat{u_0}, \qquad \widehat{u}_1 = \widehat{u_1}, \qquad \xi_m = \sqrt{4\pi^2|\xi|^2 + m^2} \geq |\xi|.$$

Next, we take the L^2 norm in the frequency space to obtain

$$\begin{aligned} &\big\|\widehat{u}(t,\cdot)\big\|_{L^2(\mathbb{R}^3)} \\ &\lesssim \big\|\cos\big(t\xi_m\big)\widehat{u}_0\big\|_{L^2(\mathbb{R}^3)} + \Big\|\frac{\sin\big(t\xi_m\big)}{\xi_m}\widehat{u}_1\Big\|_{L^2(\mathbb{R}^3)} + \Big\|\frac{1}{\xi_m}\int_{t_0}^t \sin\big((t-t')\xi_m\big)\widehat{f}(t')\,dt'\Big\|_{L^2(\mathbb{R}^3)} \\ &\lesssim \big\|\widehat{u}_0\big\|_{L^2(\mathbb{R}^3)} + \Big\|\frac{1}{|\xi|}\widehat{u}_1\Big\|_{L^2(\mathbb{R}^3)} + \int_{t_0}^t \Big\|\frac{1}{|\xi|}\widehat{f}(t')\Big\|_{L^2(\mathbb{R}^3)}\,dt', \end{aligned}$$

which in the physical space reads

$$\big\|u(t,\cdot)\big\|_{L^2(\mathbb{R}^3)} \lesssim \big\|u_0\big\|_{L^2(\mathbb{R}^3)} + \Big\|\frac{u_1}{\Lambda}\Big\|_{L^2(\mathbb{R}^3)} + \int_{t_0}^t \Big\|\frac{f(t')}{\Lambda}\Big\|_{L^2(\mathbb{R}^3)}\,dt',$$

with $\Lambda = \sqrt{-\partial_a \partial^a}$.

Then by the Sobolev embedding theorem in equation (2.17), we admit

$$\big\|u(t,\cdot)\big\|_{L^2(\mathbb{R}^3)} \lesssim \big\|u_0\big\|_{L^2(\mathbb{R}^3)} + \big\|u_1\big\|_{L^{6/5}(\mathbb{R}^3)} + \int_{t_0}^t \big\|f(t')\big\|_{L^{6/5}(\mathbb{R}^3)}\,dt',$$

and the simple result of the integral

$$\int_{t_0}^t t'^{-1+q}\,dt'$$

implies the desired result of equation (3.1). □

We note in the proof that to obtain the L^2–type energy estimates for the solution, we transformed the original equation to the Fourier space, solved the corresponding ordinary differential equation and then conducted the analysis. It is worth mentioning that we find such procedures to bound the L^2–type norms for Klein-Gordon equations (with possibly vanishing mass) can also be applied to pure wave equations, especially in the low dimension where the conformal energy cannot bound the L^2 norm of the solution; see, for instance, [4].

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As a consequence, combined with the Klainerman-Sobolev inequality in equation (2.10), we also get

$$\|u\|_{L^\infty(\mathbb{R}^3)} \lesssim \begin{cases} C_{t_0} t^{-1} + C_f t^{-1+q}, & q > 0, \\ C_{t_0} t^{-1} + C_f t^{-1} \log t, & q = 0, \\ C_{t_0} t^{-1} + C_f t^{-1}, & q < 0, \end{cases} \tag{3.2}$$

if additional information for higher-order energy is true (with $\Gamma \in \{\partial, \Omega, L\}$)

$$\|\Gamma^I u(t_0)\|_{L^2(\mathbb{R}^3)} + \|\Gamma^I \partial_t u(t_0)\|_{L^2(\mathbb{R}^3) \cap L^{6/5}(\mathbb{R}^3)} \lesssim C_{t_0}, \quad \|\Gamma^I f\|_{L^{6/5}(\mathbb{R}^3)} \le C_f t^{-1+q}, \qquad |I| \le 3.$$

4. Proof for the global existence result

4.1. Initialisation of the bootstrap method

In this section, we aim to prove the uniform global existence result for the system in equation (1.1) via the hyperboloidal foliation method.

As usual, we will rely on the bootstrap method. The local well-posedness result allows us to assume (for all i)

$$\mathcal{E}_m(s, \partial^I L^J v_i)^{1/2} \le C_1 \epsilon, \qquad |I| + |J| \le N, \tag{4.1}$$

for all $s \in [s_0, s_1)$ with $s_1 > s_0$. In equation (4.1), $C_1 \gg 1$ is some large constant to be determined, and $\epsilon > 0$ is the size of the initial data satisfying $C_1 \epsilon \ll \delta \ll 1$, and s_1 is defined by

$$s_1 := \sup\{s : s > s_0,\ \textit{equation } (4.1)\ \textit{holds}\}. \tag{4.2}$$

If $s_1 = +\infty$, then the global existence result is done. So in the following proof, we first assume $s_1 < +\infty$ and then deduce contradictions to assert that s_1 must be $+\infty$.

By recalling the definition of the energy $\mathcal{E}_m$, we easily have the following estimates.

Lemma 4.1. *Assume equation (4.1) holds; then for all $s \in [s_0, s_1)$ and $|I| + |J| \le N$, we have the following estimates:*

$$\begin{aligned} \|(s/t)\partial\partial^I L^J v_i\|_{L^2_f(\mathcal{H}_s)} + m\|\partial^I L^J v_i\|_{L^2_f(\mathcal{H}_s)} + \sum_a \|\underline{\partial}_a \partial^I L^J v_i\|_{L^2_f(\mathcal{H}_s)} &\lesssim C_1 \epsilon, \\ \|(s/t)\partial^I L^J \partial v_i\|_{L^2_f(\mathcal{H}_s)} + \sum_a \|\partial^I L^J \underline{\partial}_a v_i\|_{L^2_f(\mathcal{H}_s)} &\lesssim C_1 \epsilon. \end{aligned} \tag{4.3}$$

Combined with the Sobolev–type inequality on hyperboloids in equation (2.12), the following pointwise estimates are valid.

Lemma 4.2. *For all $|I| + |J| \le N - 2$, we have*

$$|(s/t)\partial\partial^I L^J v_i| + m|\partial^I L^J v_i| + \sum_a |\underline{\partial}_a \partial^I L^J v_i| \lesssim C_1 \epsilon t^{-3/2}. \tag{4.4}$$

In addition to the estimates above, we also introduce estimates obtained by using the Hardy inequality in equations (2.15)–(2.16). They will not be used in the current section but will be used in Section 5.

Lemma 4.3. *The following estimates are valid:*

$$\begin{aligned} \|t^{-1} \partial^I L^J v_i\|_{L^2_f(\mathcal{H}_s)} &\lesssim C_1 \epsilon, \qquad |I| + |J| \le N, \\ |\partial^I L^J v_i| &\lesssim C_1 \epsilon t^{-1/2}, \qquad |I| + |J| \le N - 2. \end{aligned} \tag{4.5}$$

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4.2. Improved energy estimates and global existence result

We now want to show the improved energy estimates for the solution $v = (v_i)$ and conclude the global existence result.

Proposition 4.4 (Improved energy estimates). *Let the assumptions in equation (4.1) be true; then for all $s \in [s_0, s_1)$, it holds that*

$$\mathcal{E}_m(s, \partial^I L^J v_i)^{1/2} \lesssim \epsilon + (C_1\epsilon)^2, \qquad |I| + |J| \le N. \tag{4.6}$$

Proof. When the vector field $\partial^I L^J$ with $|I| + |J| \le N$ acts on the model equations in (1.1), we get

$$-\Box \partial^I L^J v_i + m^2 \partial^I L^J v_i = N_i^{jk} \partial^I L^J Q_0(v_j, v_k) + M_i^{jk\alpha\beta} \partial^I L^J Q_{\alpha\beta}(v_j, v_k).$$

According to the commutator estimates in Lemma 2.2, we can bound the right-hand side as follows:

$$\begin{aligned}
&\big|N_i^{jk} \partial^I L^J Q_0(v_j, v_k) + M_i^{jk\alpha\beta} \partial^I L^J Q_{\alpha\beta}(v_j, v_k)\big| \\
&\lesssim \sum_{\substack{j,k,\alpha,\beta \\ |I_1|+|I_2|\le|I|, |J_1|+|J_2|\le|J|}} \Big(\big|Q_0(\partial^{I_1} L^{J_1} v_j, \partial^{I_2} L^{J_2} v_k)\big| + \big|Q_{\alpha\beta}(\partial^{I_1} L^{J_1} v_j, \partial^{I_2} L^{J_2} v_k)\big|\Big).
\end{aligned}$$

Then the estimates for null forms in the hyperboloidal setting in Lemma 2.4 yield

$$\begin{aligned}
&\big|N_i^{jk} \partial^I L^J Q_0(v_j, v_k) + M_i^{jk\alpha\beta} \partial^I L^J Q_{\alpha\beta}(v_j, v_k)\big| \\
&\lesssim \sum_{\substack{j,k,\alpha,a \\ |I_1|+|I_2|\le|I|, |J_1|+|J_2|\le|J|}} \Big((s/t)^2 \big|\partial_t \partial^{I_1} L^{J_1} v_j \partial_t \partial^{I_2} L^{J_2} v_k\big| \\
&\qquad + \big|\underline{\partial}_\alpha \partial^{I_1} L^{J_1} v_j \underline{\partial}_a \partial^{I_2} L^{J_2} v_k\big| + \big|\underline{\partial}_\alpha \partial^{I_2} L^{J_2} v_k \underline{\partial}_a \partial^{I_1} L^{J_1} v_j\big|\Big).
\end{aligned}$$

We rely on the energy estimates for Klein-Gordon equations on hyperboloids in equation (2.20) to get

$$\begin{aligned}
&\mathcal{E}_m(s, \partial^I L^J v_i)^{1/2} \\
&\le \mathcal{E}_m(s_0, \partial^I L^J v_i)^{1/2} + \int_{s_0}^s \big\|N_i^{jk} \partial^I L^J Q_0(v_j, v_k) + M_i^{jk\alpha\beta} \partial^I L^J Q_{\alpha\beta}(v_j, v_k)\big\|_{L^2_f(\mathcal{H}_{s'})}\, ds' \\
&\lesssim \epsilon + \sum_{\substack{j,k,\alpha,a \\ |I_1|+|I_2|\le|I|, |J_1|+|J_2|\le|J|}} \int_{s_0}^s \Big\|(s/t)^2 \big|\partial_t \partial^{I_1} L^{J_1} v_j \partial_t \partial^{I_2} L^{J_2} v_k\big| \\
&\qquad + \big|\underline{\partial}_\alpha \partial^{I_1} L^{J_1} v_j \underline{\partial}_a \partial^{I_2} L^{J_2} v_k\big| + \big|\underline{\partial}_\alpha \partial^{I_2} L^{J_2} v_k \underline{\partial}_a \partial^{I_1} L^{J_1} v_j\big|\Big\|_{L^2_f(\mathcal{H}_{s'})}\, ds'.
\end{aligned}$$

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We thus have (since[2] $N \geq 6$)

$$\begin{aligned}
&\mathcal{E}_m(s, \partial^I L^J v_i)^{1/2} \\
&\lesssim \epsilon + \sum_{\substack{j,k,\alpha,a \\ |I_1|+|J_1|\leq N, |I_2|+|J_2|\leq N-2}} \int_{s_0}^{s} \Big(\big\|(s'/t)\partial_t \partial^{I_1} L^{J_1} v_j\big\|_{L^2_f(\mathcal{H}_{s'})} \big\|(s'/t)\partial_t \partial^{I_2} L^{J_2} v_k\big\|_{L^\infty_f(\mathcal{H}_{s'})} \\
&\qquad + \big\|(s'/t)\underline{\partial}_\alpha \partial^{I_1} L^{J_1} v_j\big\|_{L^2_f(\mathcal{H}_{s'})} \big\|(t/s')\underline{\partial}_a \partial^{I_2} L^{J_2} v_k\big\|_{L^\infty_f(\mathcal{H}_{s'})} \\
&\qquad + \big\|\underline{\partial}_a \partial^{I_1} L^{J_1} v_j\big\|_{L^2_f(\mathcal{H}_{s'})} \big\|\underline{\partial}_\alpha \partial^{I_2} L^{J_2} v_k\big\|_{L^\infty_f(\mathcal{H}_{s'})} \Big)\, ds' \\
&\lesssim \epsilon + (C_1\epsilon)^2 \int_{s_0}^{s} \big\|t^{-3/2} + t^{-1/2} s'^{-1}\big\|_{L^\infty(\mathcal{H}_{s'})}\, ds' \lesssim \epsilon + (C_1\epsilon)^2.
\end{aligned}$$

Hence the proof is done. □

As a consequence of the improved energy estimates in Proposition 4.4, we conclude the global existence result of the system in equation (1.1).

Proof of the global existence result. We choose C_1 large and ϵ small such that

$$C\big(\epsilon + (C_1\epsilon)^2\big) \leq \frac{1}{2} C_1 \epsilon,$$

in which C is the hidden constant in equation (4.6), which thus leads us to the improved estimates

$$\mathcal{E}_m(s, \partial^I L^J v_i)^{1/2} \leq \frac{1}{2} C_1 \epsilon, \qquad |I| + |J| \leq N.$$

If $s_1 > s_0$ is some finite number, then the improved estimates above imply that we can extend the solution $v = (v_i)$ to a strictly larger (hyperbolic) time interval, which contradicts the definition of s_1 in equation (4.2). Hence s_1 must be $+\infty$, which implies the global existence of the solution $v = (v_i)$ to the system in equation (1.1). □

5. Proof for the uniform pointwise decay result

5.1. Decomposition and nonlinear transformation

Our task in this section is to show the unified pointwise decay estimate in equation (1.4)

$$\big|v_i(t,x)\big| \lesssim \frac{1}{t + m t^{3/2}},$$

which corresponds to the usual wave decay and Klein-Gordon decay with $m = 0, 1$, respectively. This is the bulk of the paper.

Recall the system in equation (1.1):

$$\begin{aligned}
&-\Box v_i + m^2 v_i = N_i^{jk} Q_0(v_j, v_k) + M_i^{jk\alpha\beta} Q_{\alpha\beta}(v_j, v_k), \\
&\big(v_i, \partial_t v_i\big)(t_0) = \big(v_{i0}, v_{i1}\big).
\end{aligned}$$

In order to arrive at equation (1.4), it suffices to show

$$\big|v_i(t,x)\big| \lesssim t^{-1}, \tag{5.1}$$

[2] Actually, $N \geq 4$ is enough to ensure the global existence result.

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which is because we already obtain

$$\left|v_i(t,x)\right| \lesssim m^{-1}t^{-3/2}$$

in Section 4. To achieve equation (5.1), we first do a nonlinear transformation from v_i to $V_i = v_i + N_i^{jk} v_j v_k$ and then decompose V_i into pieces

$$V_i = V_{c,i} + V_{m,i} + V_{n,i}. \tag{5.2}$$

For clarity, we note that we use $V_{c,i}$ to denote the decomposition with cubic nonlinearities, use $V_{m,i}$ to denote the decomposition with nonlinearities with m-dependent factors and use $V_{n,i}$ to denote the decomposition with null nonlinearities of the type $Q_{\alpha\beta}$. The functions V's are solutions to the following (linear) Klein-Gordon equations:

$$\begin{aligned} -\Box V_i + m^2 V_i &= M_i^{jk\alpha\beta} Q_{\alpha\beta}(v_j, v_k) - m^2 N_i^{jk} v_j v_k \\ &\quad + N_i^{jk} v_k \big(N_j^{mn} Q_0(v_m, v_n) + M_j^{mn\alpha\beta} Q_{\alpha\beta}(v_m, v_n)\big) \\ &\quad + N_i^{jk} v_j \big(N_k^{mn} Q_0(v_m, v_n) + M_k^{mn\alpha\beta} Q_{\alpha\beta}(v_m, v_n)\big), \\ \big(V_i, \partial_t V_i\big)(t_0) &= \big(V_{i0}, V_{i1}\big) := \big(v_{i0} + N_i^{jk} v_{j0} v_{k0}, v_{i1} + N_i^{jk}(v_{j0} v_{k1} + v_{j1} v_{k0})\big), \end{aligned} \tag{5.3}$$

$$\begin{aligned} -\Box V_{c,i} + m^2 V_{c,i} &= N_i^{jk} v_k \big(N_j^{mn} Q_0(v_m, v_n) + M_j^{mn\alpha\beta} Q_{\alpha\beta}(v_m, v_n)\big) \\ &\quad + N_i^{jk} v_j \big(N_k^{mn} Q_0(v_m, v_n) + M_k^{mn\alpha\beta} Q_{\alpha\beta}(v_m, v_n)\big), \\ \big(V_{c,i}, \partial_t V_{c,i}\big)(t_0) &= \big(V_{i0}, V_{i1}\big), \end{aligned} \tag{5.4}$$

$$\begin{aligned} -\Box V_{m,i} + m^2 V_{m,i} &= -m^2 N_i^{jk} v_j v_k, \\ \big(V_{m,i}, \partial_t V_{m,i}\big)(t_0) &= (0,0), \end{aligned} \tag{5.5}$$

as well as

$$\begin{aligned} -\Box V_{n,i} + m^2 V_{n,i} &= M_i^{jk\alpha\beta} Q_{\alpha\beta}(v_j, v_k), \\ \big(V_{n,i}, \partial_t V_{n,i}\big)(t_0) &= (0,0). \end{aligned} \tag{5.6}$$

In addition to the decomposition above, we find it helps to utilise the divergence structure of the null forms of the form $Q_{\alpha\beta}$ if we further decompose the $V_{n,i}$ part as

$$V_{n,i} = V_{n,i}^5 + \partial_\gamma V_{n,i}^\gamma. \tag{5.7}$$

We use $V_{n,i}^5, V_{n,i}^\gamma$ to denote the decomposition with 0 nonlinearities and divergent nonlinearities without ∂_γ, respectively. Similarly, $V_{n,i}^5, V_{n,i}^\gamma$ are solutions to the following (linear) Klein-Gordon equations:

$$\begin{aligned} -\Box V_{n,i}^5 + m^2 V_{n,i}^5 &= 0, \\ \big(V_{n,i}^5, \partial_t V_{n,i}^5\big)(t_0) &= (0,0), \end{aligned} \tag{5.8}$$

$$\begin{aligned} -\Box V_{n,i}^\gamma + m^2 V_{n,i}^\gamma &= M_i^{jk\gamma\beta} v_j \partial_\beta v_k - M_i^{jk\alpha\gamma} v_j \partial_\alpha v_k, \\ \big(V_{n,i}^\gamma, \partial_t V_{n,i}^\gamma\big)(t_0) &= (0,0). \end{aligned} \tag{5.9}$$

5.2. The mass-independent L^2 norm estimates

Recall that our goal is to obtain the following mass-independent L^2 estimates for functions Vs, and we will rely on the bootstrap method one more time to achieve it.

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Proposition 5.1. *For all $|I| + |J| \le N - 1$ (and for each i), we have*

$$\big\|\partial^I L^J \big(V_{c,i}, V_{m,i}, V_{n,i}, V^5_{n,i}\big)\big\|_{L^2(\mathbb{R}^3)} + t^{-\delta}\big\|\partial^I L^J \partial V^\gamma_{n,i}\big\|_{L^2(\mathbb{R}^3)} \le C_3\epsilon, \tag{5.10}$$

and for all $|I| + |J| = N$, we have

$$\big\|\partial^I L^J \big(V_{c,i}, V_{m,i}, V^5_{n,i}\big)\big\|_{L^2(\mathbb{R}^3)} + t^{-\delta}\big\|\partial^I L^J \partial V^\gamma_{n,i}\big\|_{L^2(\mathbb{R}^3)} \le C_3\epsilon, \tag{5.11}$$

in which $C_3 \ge C_2 \ge C_1$ are some constants to be determined.

We have the following results, which are a consequence of the pointwise decay estimate

$$\big|\partial^I L^J \partial v, \partial\partial^I L^J v\big| \lesssim C_1\epsilon t^{-1}, \qquad |I| + |J| \le N - 2,$$

obtained in Section 4.

Lemma 5.2. *We have for all $|I| + |J| \le N - 2$ that*

$$E_m(t, \partial^I L^J v)^{1/2} \lesssim C_1\epsilon t^{\delta/2}. \tag{5.12}$$

Since we are proving energy estimates for linear equations, we know the solutions already exist. We first prove the energy estimates for the low-order cases. By the continuity of the energy, we assume the following bounds are valid for all $t \in [t_0, t_1)$:

$$\begin{aligned} \big\|\partial^I L^J \big(V_{c,i}, V_{m,i}, V_{n,i}, V^5_{n,i}\big)\big\|_{L^2(\mathbb{R}^3)} + t^{-\delta}\big\|\partial^I L^J \partial V^\gamma_{n,i}\big\|_{L^2(\mathbb{R}^3)} &\le C_2\epsilon, \qquad |I| + |J| \le N - 3,\\ \big\|\partial^I L^J \big(V_{c,i}, V_{m,i}, V^5_{n,i}\big)\big\|_{L^2(\mathbb{R}^3)} + t^{-\delta}\big\|\partial^I L^J \partial V^\gamma_{n,i}\big\|_{L^2(\mathbb{R}^3)} &\le C_2\epsilon, \qquad |I| + |J| \le N - 2. \end{aligned} \tag{5.13}$$

Similar to the definition of s_1 in Section 4, t_1 is defined by

$$t_1 := \sup\{t : t > t_0,\ \textit{equation } (5.13)\ \textit{holds}\}. \tag{5.14}$$

A direct result from the bootstrap assumption in equation (5.13) is the following.

Lemma 5.3. *For all $t \in [t_0, t_1)$, we have*

$$\begin{aligned} \big\|\partial^I L^J v\big\|_{L^2(\mathbb{R}^3)} &\lesssim C_2\epsilon, \qquad |I| + |J| \le N - 3,\\ \big\|\partial^I L^J v\big\|_{L^2(\mathbb{R}^3)} &\lesssim C_2\epsilon t^{\delta}, \qquad |I| + |J| = N - 2. \end{aligned} \tag{5.15}$$

Proof. For all $|I| + |J| \le N - 3$, we first have

$$\big\|\partial^I L^J V\big\|_{L^2(\mathbb{R}^3)} \lesssim C_2\epsilon,$$

which simply follows from the relations

$$V_i = V_{c,i} + V_{m,i} + V_{n,i}, \qquad V_{n,i} = V^5_{n,i} + \partial_\gamma V^\gamma_{n,i},$$

and the commutator estimate in Lemma 2.1

$$\big|[\partial, L]u\big| \lesssim |\partial u|.$$

Next, we recall it holds that

$$v_i = V_i - N_i^{jk} v_j v_k.$$

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Thus we get

$$\begin{aligned}
&\sum_{|I|+|J|\le N-3} \big\|\partial^I L^J v_i\big\|_{L^2(\mathbb{R}^3)} \\
&\lesssim \sum_{|I|+|J|\le N-3} \big\|\partial^I L^J V_i\big\|_{L^2(\mathbb{R}^3)} + \sum_{|I|+|J|\le N-3} \big\|\partial^I L^J v_i\big\|_{L^2(\mathbb{R}^3)} \sum_{|I|+|J|\le N-3} \big\|\partial^I L^J v_i\big\|_{L^\infty(\mathbb{R}^3)},
\end{aligned}$$

and the pointwise estimate obtained in Lemma 4.3 yields

$$\sum_{|I|+|J|\le N-3} \big\|\partial^I L^J v_i\big\|_{L^2(\mathbb{R}^3)} \lesssim \sum_{|I|+|J|\le N-3} \big\|\partial^I L^J V_i\big\|_{L^2(\mathbb{R}^3)} + C_1\epsilon t^{-1/2} \sum_{|I|+|J|\le N-3} \big\|\partial^I L^J v_i\big\|_{L^2(\mathbb{R}^3)},$$

and the smallness of $C_1\epsilon$ allows us finally to obtain

$$\sum_{|I|+|J|\le N-3} \big\|\partial^I L^J v_i\big\|_{L^2(\mathbb{R}^3)} \lesssim \sum_{|I|+|J|\le N-3} \big\|\partial^I L^J V_i\big\|_{L^2(\mathbb{R}^3)}.$$

The bound for the case of $|I| + |J| = N - 2$ can be derived in the same way; hence the proof is complete. □

We are going to derive the improved estimates under the bootstrap assumption of equation (5.13), and we first provide the improved estimates for low-order cases.

Proposition 5.4. *Let the estimate in equation (5.13) be true; then for any $t \in [t_0, t_1)$, we have*

$$\begin{aligned}
\big\|\partial^I L^J \big(V_{c,i}, V_{m,i}, V_{n,i}, V^5_{n,i}\big)\big\|_{L^2(\mathbb{R}^3)} + t^{-\delta}\big\|\partial^I L^J \partial V^\gamma_{n,i}\big\|_{L^2(\mathbb{R}^3)} &\lesssim \epsilon + (C_2\epsilon)^2, \qquad |I| + |J| \le N - 3,\\
\big\|\partial^I L^J \big(V_{c,i}, V_{m,i}, V^5_{n,i}\big)\big\|_{L^2(\mathbb{R}^3)} + t^{-\delta}\big\|\partial^I L^J \partial V^\gamma_{n,i}\big\|_{L^2(\mathbb{R}^3)} &\lesssim \epsilon + (C_2\epsilon)^2, \qquad |I| + |J| \le N - 2.
\end{aligned} \tag{5.16}$$

Proof. In terms of the features of each term, we estimate them one by one.

Estimate for $\big\|\partial^I L^J V_{m,i}\big\|_{L^2(\mathbb{R}^3)}$. We start with the easy one, and the energy estimate for the $\partial^I L^J V_{m,i}$ equation gives

$$E_m(t, \partial^I L^J V_{m,i})^{1/2} \le E_m(t_0, \partial^I L^J V_{m,i})^{1/2} + \int_{t_0}^t \big\|\partial^I L^J F_{V_{m,i}}\big\|_{L^2(\mathbb{R}^3)}\, dt',$$

with

$$F_{V_{m,i}} = -m^2 N_i^{jk} v_j v_k.$$

Recall the m-dependent pointwise estimate

$$m\big|\partial^I L^J v\big| \lesssim C_1\epsilon t^{-3/2},$$

and we get

$$\begin{aligned}
E_m(t, \partial^I L^J V_{m,i})^{1/2} &\lesssim m^2\epsilon + m\int_{t_0}^t \sum_{|I|+|J|\le N-2} \big\|\partial^I L^J v\big\|_{L^2(\mathbb{R}^3)} \sum_{|I|+|J|\le N-2} \big\|m\partial^I L^J v\big\|_{L^\infty(\mathbb{R}^3)}\, dt'\\
&\lesssim m^2\epsilon + mC_1\epsilon C_2\epsilon \int_{t_0}^t t'^{-3/2+\delta}\, dt' \lesssim mC_1\epsilon C_2\epsilon.
\end{aligned}$$

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Hence the definition of the energy E_m implies that

$$\|\partial^I L^J V_{m,i}\|_{L^2(\mathbb{R}^3)} \lesssim C_1\epsilon C_2\epsilon.$$

Estimates for $\|\partial^I L^J V_{c,i}\|_{L^2(\mathbb{R}^3)}, \|\partial^I L^J V_{n,i}\|_{L^2(\mathbb{R}^3)}, \|\partial^I L^J V^5_{n,i}\|_{L^2(\mathbb{R}^3)}$. Since the procedure is the same when estimating these three solutions, we gather the proof here.

For the equations with fast-decay nonlinearities, which is the situation now, we rely on Proposition 3.1 to obtain the m-independent L^2 norm bounds. Thus it suffices to estimate $\|\text{nonlinearities}\|_{L^{6/5}(\mathbb{R}^3)}$.

We find for estimating $\|\partial^I L^J V_{c,i}\|_{L^2(\mathbb{R}^3)}$ it suffices to show that

$$\begin{aligned}
&\sum_{|I|+|J|\le N-2} \Big\|\partial^I L^J \Big(N_i^{jk} v_k \big(N_j^{mn} Q_0(v_m,v_n) + M_j^{mn\alpha\beta} Q_{\alpha\beta}(v_m,v_n)\big) \\
&\qquad\qquad + N_i^{jk} v_j \big(N_k^{mn} Q_0(v_m,v_n) + M_k^{mn\alpha\beta} Q_{\alpha\beta}(v_m,v_n)\big)\Big)\Big\|_{L^{6/5}(\mathbb{R}^3)} \\
&\lesssim \sum_{|I|+|J|\le N-2} \|\partial^I L^J v\|_{L^2(\mathbb{R}^3)} \sum_{|I|+|J|\le N-2} \|\partial^I L^J \partial v\|^2_{L^6(\mathbb{R}^3)} \\
&\lesssim C_2\epsilon t^\delta \sum_{|I|+|J|\le N-2} \|\partial^I L^J \partial v\|^{2/3}_{L^2} \sum_{|I|+|J|\le N-2} \|\partial^I L^J \partial v\|^{4/3}_{L^\infty} \\
&\lesssim C_2\epsilon (C_1\epsilon)^2 t^{-4/3+2\delta},
\end{aligned}$$

in which we used the estimate in equation (5.12).

Similarly, in order to estimate $\|\partial^I L^J V_{n,i}\|_{L^2(\mathbb{R}^3)}$, we need to demonstrate that

$$\begin{aligned}
&\sum_{|I|+|J|\le N-3} \|\partial^I L^J \big(M_i^{jk\alpha\beta} Q_{\alpha\beta}(v_j,v_k)\big)\|_{L^{6/5}(\mathbb{R}^3)} \\
&\lesssim t^{-1} \sum_{|I|+|J|\le N-2} \|\partial^I L^J v\|_{L^2(\mathbb{R}^3)} \sum_{|I|+|J|\le N-3} \|\partial\partial^I L^J v\|_{L^3(\mathbb{R}^3)} \\
&\lesssim C_2\epsilon t^{-1+\delta} \sum_{|I|+|J|\le N-3} \|\partial\partial^I L^J v\|^{2/3}_{L^2(\mathbb{R}^3)} \sum_{|I|+|J|\le N-3} \|\partial\partial^I L^J v\|^{1/3}_{L^\infty(\mathbb{R}^3)} \\
&\lesssim C_2\epsilon C_1\epsilon t^{-4/3+\delta}.
\end{aligned}$$

The estimate for $\|\partial^I L^J V^5_{n,i}\|_{L^2(\mathbb{R}^3)}$ with $|I|+|J| \le N-2$ is trivial according to Proposition 3.1, since the equation is homogeneous.

Estimate for $\|\partial^I L^J \partial V^\gamma_{n,i}\|_{L^2(\mathbb{R}^3)}$. We utilise the energy estimate for the equation of $\partial^I L^J \partial V^\gamma_{n,i}$ with $|I| + |J| \le N-2$ to get

$$\begin{aligned}
E_m(t, \partial^I L^J V^\gamma_{n,i})^{1/2} &\le \int_{t_0}^t \|\partial^I L^J \big(M_i^{jk\gamma\beta} v_j \partial_\beta v_k - M_i^{jk\alpha\gamma} v_j \partial_\alpha v_k\big)\|_{L^2(\mathbb{R}^3)}\, dt' \\
&\lesssim \int_{t_0}^t \sum_{|I|+|J|\le N-2} \|\partial^I L^J v\|_{L^2(\mathbb{R}^3)} \sum_{|I|+|J|\le N-2} \|\partial\partial^I L^J v\|_{L^\infty(\mathbb{R}^3)}\, dt' \\
&\lesssim \int_{t_0}^t C_2\epsilon t'^\delta C_1\epsilon t'^{-1}\, dt' \lesssim C_2\epsilon C_1\epsilon t^\delta.
\end{aligned}$$

By the definition of E_m and the commutator estimates, we deduce

$$\|\partial^I L^J \partial V^\gamma_{n,i}\|_{L^2(\mathbb{R}^3)} \lesssim C_2\epsilon C_1\epsilon t^\delta, \qquad |I| + |J| \le N-2.$$

□

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According to the refined bounds in Proposition 5.4, we easily know that the estimates in equation (5.13) are true for all $t \in [t_0, +\infty)$ after carefully choosing C_2 large enough and ϵ sufficiently small (we might shrink the choice of ϵ in Proof 4.2 if needed, and nothing else is affected).

The Klainerman-Sobolev inequality in equation (2.11) together with equation (5.13) provides the following mass-independent results

$$\begin{aligned} \left\|\partial^I L^J v\right\|_{L^2(\mathbb{R}^3)} &\lesssim C_2\epsilon, \qquad |I| + |J| \leq N - 3, \\ \left\|\partial^I L^J v\right\|_{L^2(\mathbb{R}^3)} &\lesssim C_2\epsilon t^{\delta}, \qquad |I| + |J| = N - 2, \\ \left\|\partial^I L^J v\right\|_{L^\infty(\mathbb{R}^3)} &\lesssim C_2\epsilon t^{-1}, \qquad |I| + |J| \leq N - 6, \end{aligned} \tag{5.17}$$

which are valid for all $t \in [t_0, +\infty)$.

Next, to proceed to prove the bounds of high-order energy in Proposition 5.1, we make new bootstrap assumptions for $t \in [t_0, t_2)$ (and for all i):

$$\begin{aligned} \left\|\partial^I L^J \left(V_{c,i}, V_{m,i}, V_{n,i}, V^5_{n,i}\right)\right\|_{L^2(\mathbb{R}^3)} + t^{-\delta}\left\|\partial^I L^J \partial V^\gamma_{n,i}\right\|_{L^2(\mathbb{R}^3)} &\leq C_3\epsilon, \qquad |I| + |J| \leq N - 1, \\ \left\|\partial^I L^J \left(V_{c,i}, V_{m,i}, V^5_{n,i}\right)\right\|_{L^2(\mathbb{R}^3)} + t^{-\delta}\left\|\partial^I L^J \partial V^\gamma_{n,i}\right\|_{L^2(\mathbb{R}^3)} &\leq C_3\epsilon, \qquad |I| + |J| \leq N. \end{aligned} \tag{5.18}$$

Similar to the definition of t_1, t_2 is defined by

$$t_2 := \sup\{t : t > t_0,\ \textit{equation (5.18) holds}\}. \tag{5.19}$$

Recall that $C_3 \geq C_2$ is to be determined.

We have the following refined estimates for high-order cases.

Proposition 5.5. *Assuming the estimate in equation (5.18) is true, then for all $t \in [t_0, t_2)$, we have*

$$\begin{aligned} \left\|\partial^I L^J \left(V_{c,i}, V_{m,i}, V_{n,i}, V^5_{n,i}\right)\right\|_{L^2(\mathbb{R}^3)} + t^{-\delta}\left\|\partial^I L^J \partial V^\gamma_{n,i}\right\|_{L^2(\mathbb{R}^3)} &\lesssim \epsilon + (C_3\epsilon)^2, \qquad |I| + |J| \leq N - 1, \\ \left\|\partial^I L^J \left(V_{c,i}, V_{m,i}, V^5_{n,i}\right)\right\|_{L^2(\mathbb{R}^3)} + t^{-\delta}\left\|\partial^I L^J \partial V^\gamma_{n,i}\right\|_{L^2(\mathbb{R}^3)} &\lesssim \epsilon + (C_3\epsilon)^2, \qquad |I| + |J| \leq N. \end{aligned} \tag{5.20}$$

The proof for Proposition 5.4 also applies to Proposition 5.5, so we omit it. Also, similarly, we can choose C_3 large enough, and ϵ sufficiently small (we shrink it further if needed), so that we can improve the estimates in equation (5.18) with a factor $1/2$ in front of the original bounds. And this indicates that the estimates in equation (5.18) are valid for all $t \in [t_0, +\infty)$.

The proof for Proposition 5.1 follows from the established estimates in equations (5.13) and (5.18), which are valid for all $t \in [t_0, +\infty)$.

5.3. The mass-independent wave decay for the solution

With the estimates built in Proposition 5.1, we have the following results for the original solution $v = (v_i)$.

Lemma 5.6. *For all $t \in [t_0, +\infty)$, we have*

$$\begin{aligned} \left\|\partial^I L^J v\right\|_{L^2(\mathbb{R}^3)} &\lesssim C_3\epsilon, \qquad |I| + |J| \leq N - 1, \\ \left\|\partial^I L^J v\right\|_{L^2(\mathbb{R}^3)} &\lesssim C_3\epsilon t^{\delta}, \qquad |I| + |J| = N. \end{aligned} \tag{5.21}$$

The proof for Lemma 5.6 follows from the proof for Lemma 5.3.

Next, we apply the Klainerman-Sobolev inequality in equation (2.11) to arrive at the mass-independent pointwise decay results.

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Proposition 5.7. *It holds that*

$$\begin{aligned} \|\partial^I L^J v\|_{L^\infty(\mathbb{R}^3)} &\lesssim C_3 \epsilon t^{-1}, \qquad |I|+|J| \le N-4, \\ \|\partial^I L^J v\|_{L^\infty(\mathbb{R}^3)} &\lesssim C_3 \epsilon t^{-1+\delta}, \qquad |I|+|J| = N-3. \end{aligned} \tag{5.22}$$

We now establish equation (1.4) in Theorem 1.1.

Proof of equation (1.4). From Proposition 5.7, we obtain the mass-independent pointwise decay result

$$|v_i| \lesssim t^{-1},$$

while the estimates in Lemma 4.2 give us the mass-dependent Klein-Gordon decay

$$|v_i| \lesssim m^{-1} t^{-3/2}, \qquad m \in (0,1].$$

Combining these two kinds of decay bounds, we are led to

$$|v_i| \lesssim \frac{1}{t + m t^{3/2}}, \qquad m \in (0,1]. \tag{5.23}$$

But we see equation (5.23) is obviously true for the case of $m = 0$ (because it is just the case of wave equations), and hence the proof for equation (1.4) is complete. □

6. Proof for the covergence result

With the global existence result and the unified pointwise decay result prepared in the last two sections, we now want to build the convergence result when the mass parameter m goes to 0.

Proof of Theorem 1.2. We take the difference between the equation of $v_i^{(m)}$ and the equation of $v_i^{(0)}$ to have

$$\begin{aligned} &-\Box\big(v_i^{(m)} - v_i^{(0)}\big) + m^2\big(v_i^{(m)} - v_i^{(0)}\big) \\ &= -m^2 v_i^{(0)} + N_i^{jk} Q_0\big(v_j^{(m)} - v_j^{(0)}, v_k^{(m)}\big) + N_i^{jk} Q_0\big(v_j^{(0)}, v_k^{(m)} - v_k^{(0)}\big) \\ &\quad + M_i^{jk\alpha\beta} Q_{\alpha\beta}\big(v_j^{(m)} - v_j^{(0)}, v_k^{(m)}\big) + M_i^{jk\alpha\beta} Q_{\alpha\beta}\big(v_j^{(0)}, v_k^{(m)} - v_k^{(0)}\big) =: F^{(m)}, \end{aligned}$$

with zero initial data

$$\big(v_i^{(m)} - v_i^{(0)}, \partial_t v_i^{(m)} - \partial_t v_i^{(0)}\big)(t_0) = (0,0).$$

The vector field $\partial^I L^J$ with $|I|+|J| \le N-2$ acts on the equation $v_i^{(m)} - v_i^{(0)}$, and then the energy estimates give

$$E_m\big(t, \partial^I L^J v_i^{(m)} - \partial^I L^J v_i^{(0)}\big)^{1/2} \le E_m\big(t_0, \partial^I L^J v_i^{(m)} - \partial^I L^J v_i^{(0)}\big)^{1/2} + \int_{t_0}^t \|\partial^I L^J F^{(m)}\|_{L^2(\mathbb{R}^3)}\, dt',$$

and by the fact $|\partial \partial^I L^J v| \lesssim C_1 \epsilon t^{-1}$, we further have

$$\begin{aligned} &E_m\big(t, \partial^I L^J v_i^{(m)} - \partial^I L^J v_i^{(0)}\big)^{1/2} \\ &\lesssim m^2 C_1 \epsilon t + C_1 \epsilon \sum_{|I|+|J| \le N-2} \int_{t_0}^t t'^{-1} E_m\big(t', \partial^I L^J v_i^{(m)} - \partial^I L^J v_i^{(0)}\big)^{1/2}\, dt'. \end{aligned}$$

In succession, we apply Gronwall's inequality to obtain

$$\sum_{|I|+|J|\le N-2} E_m\big(t, \partial^I L^J v_i^{(m)} - \partial^I L^J v_i^{(0)}\big)^{1/2} \lesssim m^2 C_1 \epsilon t^{1+CC_1\epsilon}, \tag{6.1}$$

with C a generic constant.

Thus by choosing ϵ sufficiently small such that $CC_1\epsilon \le \delta/2$, we complete the proof for the cases of $|I| + |J| \le N - 2$.

Based on the estimates obtained and the Klainerman-Sobolev inequality in equation (2.11), we obtain

$$\sum_{|I|+|J|\le N-5} \big\|\partial\partial^I L^J v_i^{(m)} - \partial\partial^I L^J v_i^{(0)}\big\|_{L^\infty(\mathbb{R}^3)} \lesssim m^2 C_1 \epsilon t^{\delta/2};$$

then for $|I| + |J| \le N$ with $N \ge 6$, we can bound

$$\int_{t_0}^t \big\|\partial^I L^J F^{(m)}\big\|_{L^2(\mathbb{R}^3)}\, dx \lesssim m^2 C_1 \epsilon t^{1+\delta/2} + C_1\epsilon \sum_{|I|+|J|\le N} \int_{t_0}^t t'^{-1} E_m\big(t', \partial^I L^J v_i^{(m)} - \partial^I L^J v_i^{(0)}\big)^{1/2}\, dt',$$

which further yields

$$\begin{aligned}&\sum_{|I|+|J|\le N} E_m\big(t, \partial^I L^J v_i^{(m)} - \partial^I L^J v_i^{(0)}\big)^{1/2} \\ &\quad\lesssim m^2 C_1 \epsilon t^{1+\delta/2} + C_1\epsilon \sum_{|I|+|J|\le N} \int_{t_0}^t t'^{-1} E_m\big(t', \partial^I L^J v_i^{(m)} - \partial^I L^J v_i^{(0)}\big)^{1/2}\, dt'.\end{aligned}$$

Again, Gronwall's inequality deduces that (by letting ϵ sufficiently small)

$$\sum_{|I|+|J|\le N} E_m\big(t, \partial^I L^J v_i^{(m)} - \partial^I L^J v_i^{(0)}\big)^{1/2} \lesssim m^2 C_1 \epsilon t^{1+\delta}. \tag{6.2}$$

Now the proof is complete. □

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