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Countable Amenable Identity Excluding Groups

Wojciech Jaworski

Abstract. A discrete group G is called *identity excluding* if the only irreducible unitary representation of G which weakly contains the 1-dimensional identity representation is the 1-dimensional identity representation itself. Given a unitary representation π of G and a probability measure μ on G, let P_μ denote the μ-average $\int \pi(g)\mu(dg)$. The goal of this article is twofold: (1) to study the asymptotic behaviour of the powers P_μ^n, and (2) to provide a characterization of countable amenable identity excluding groups. We prove that for every adapted probability measure μ on an identity excluding group and every unitary representation π there exists and orthogonal projection E_μ onto a π-invariant subspace such that $s\text{-}\lim_{n\to\infty}\big(P_\mu^n - \pi(a)^n E_\mu\big) = 0$ for every $a \in \operatorname{supp}\mu$. This also remains true for suitably defined identity excluding locally compact groups. We show that the class of countable amenable identity excluding groups coincides with the class of FC-hypercentral groups; in the finitely generated case this is precisely the class of groups of polynomial growth. We also establish that every adapted random walk on a countable amenable identity excluding group is ergodic.

1 Introduction

Let G be a locally compact (Hausdorff) group and μ a regular probability measure on G. We will always assume that μ is adapted, *i.e.*, not supported on a proper closed subgroup of G. Given a continuous representation π of G by isometries in a Banach space $\mathfrak{H}$ let P_μ denote the μ-average of π, *i.e.*, the contraction

$$P_\mu = \int_G \pi(g)\mu(dg).$$

The asymptotic behaviour of the powers P_μ^n when $n \to \infty$ has been of considerable interest in ergodic theory, probability, and harmonic analysis, see, *e.g.*, [3–6, 13, 14, 16, 17, 24, 25].

The asymptotic behaviour depends on the spectral properties of the contraction P_μ. These in turn are strongly influenced by the properties of the group G itself. For example, it is well known that when π is the left regular representation in $L^2(G)$, then the spectral radius of P_μ is 1 when G is amenable and is strictly smaller than 1 when G is not amenable. When μ is absolutely continuous and G has Kahzdan Property T, then for every continuous unitary representation all unimodular elements of the spectrum of P_μ are eigenvalues. When G is nilpotent and μ is any aperiodic probability measure on G, *i.e.*, an adapted measure which is not supported on a coset of a proper closed normal subgroup, then the spectral radius of P_μ is strictly smaller than 1 for every nontrivial continuous irreducible unitary representation [17].

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The latter result for nilpotent groups remains, in fact, valid for a wider class of groups and, as observed by Jones, Rosenblatt, and Tempelman [14], is a consequence of the following property of continuous irreducible unitary representations of G: If a continuous irreducible representation π of G admits a dense subgroup $D \subseteq G$ such that with D given discrete topology the restriction of π to D weakly contains the 1-dimensional identity representation, then π itself is the identity representation. Locally compact groups whose continuous unitary representations possess this property were named in [13] *identity excluding groups*. The fact that a group is identity excluding ensures a regular asymptotic behaviour of the powers P_μ^n for all aperiodic measures μ and all continuous unitary representations: the sequence P_μ^n converges in the strong operator topology to the orthogonal projection onto the subspace of the fixed points of the representation [17].

As we show in the sequel, continuous irreducible unitary representations of locally compact nilpotent groups enjoy a property that is stronger than identity excluding: If a continuous irreducible unitary representation π of G admits a dense subgroup $D \subseteq G$ such that with D given discrete topology the restriction of π to D weakly contains a 1-dimensional representation of D, then π itself is 1-dimensional. General locally compact groups with this property could be called *strongly identity excluding*. The fact that G is strongly identity excluding implies that for every continuous irreducible unitary representation of dimension greater than 1 the spectral radius of P_μ is strictly smaller than 1 not only for aperiodic μ, but for arbitrary adapted μ. This results in the following regular asymptotic behaviour of the powers P_μ^n for arbitrary adapted measures and arbitrary continuous unitary representations. For a given μ let N_μ denote the smallest closed normal subgroup of G such that μ is carried on a coset of N_μ. The result is that for any $a \in G$ with $\mu(aN_\mu) = 1$, the sequence $\pi(a)^{-n}P_\mu^n$ converges in the strong operator topology to the orthogonal projection onto the subspace $\mathfrak{N}_\mu = \{x \in \mathfrak{H}\,;\, \pi(g)x = x \text{ for every } g \in N_\mu\}$.

The concepts of an identity excluding group and a strongly identity excluding group are, generally speaking, difficult to deal with, and no characterization of such groups is known at present. However, it can be easily seen that groups for which every continuous irreducible unitary representation is finite dimensional are strongly identity excluding. Thus, in particular, compact groups and locally compact abelian groups are strongly identity excluding. As we already mentioned, locally compact nilpotent groups are strongly identity excluding. However, there exist examples of countable solvable groups as well as connected solvable Lie groups which are not identity excluding [14, Example 3.10], [13, Example 3.13]. According to [20], compactly generated totally disconnected locally compact groups of polynomial growth are identity excluding.

It is not difficult to see that for discrete groups the property of being strongly identity excluding and that of being identity excluding are equivalent. Moreover, for discrete groups identity excluding is clearly a weakening of the Kahzdan Property T. Hence, many nonamenable groups are (strongly) identity excluding. However, by a result of Yoshizawa [26], the free group on 2 generators is not identity excluding.

While we do not know whether there exist nonamenable discrete identity excluding groups which do not have Property T, this is evidently so in the amenable case. In this article we provide a rather simple characterization of countable amenable identity excluding groups and discuss some of their properties. It turns out that this class of groups coincides with the class of groups known as FC-hypercentral [22, Chapter 4.3]. It follows, in particular, that finitely generated amenable identity excluding groups are, precisely, the finitely generated groups of polynomial growth. We also show that for a countable amenable identity excluding group not only the asymptotic behaviour of the powers P_μ^n is highly regular but the convolution powers μ^n themselves behave very regularly: the random walk of law μ is always ergodic.

2 The Spectral Radius of P_μ

Throughout Sections 2 and 3, G will denote a locally compact (Hausdorff) group, μ a regular adapted probability measure on G, π a continuous unitary representation in a Hilbert space $\mathfrak{H}$, and P_μ the μ-average of π. $\rho(P_\mu)$ and $\sigma(P_\mu)$ will stand for the spectral radius and the spectrum of P_μ, respectively. We note that adaptedness and regularity imply σ-compactness of G, so G will be always assumed σ-compact.

According to a result of Jones, Rosenblatt, and Tempelman [14, Theorem 3.6], recast by Lin and Wittmann [17, Theorem 2.3], if μ is aperiodic and $\rho(P_\mu) = 1$, then there exists a dense subgroup $D \subseteq G$ and a sequence x_n of unit vectors in $\mathfrak{H}$ such that $\lim_{n\to\infty} \|\pi(g)x_n - x_n\| = 0$ for every $g \in D$, *i.e.*, the 1-dimensional identity representation of D is weakly contained in the restriction of π to D. We begin by generalizing this result to arbitrary adapted measures μ.

Given a subset A of a group let $\operatorname{gp}(A)$ denote the subgroup generated by A and $\operatorname{ngp}(A)$ the smallest normal subgroup of $\operatorname{gp}(A)$ containing A in one of its cosets.

Lemma 2.1 $\operatorname{ngp}(A) = \operatorname{gp}\big(\bigcup_{n=1}^\infty (A^{-n}A^n \cup A^nA^{-n})\big)$ *and* $\operatorname{gp}(A)/\operatorname{ngp}(A)$ *is a cyclic group.*

Proof See the proof of Proposition 1.1 in [6]. ■

Theorem 2.2 *If $\sigma(P_\mu)$ contains a unimodular element α, then there exists a sequence of unit vectors $x_n \in \mathfrak{H}$, a σ-compact subgroup $D \subseteq G$, and a character χ on D such that:*

(i) $\mu(D) = 1$;
(ii) $\lim_{n\to\infty} \|\pi(g)x_n - \chi(g)x_n\| = 0$ *for every* $g \in D$;
(iii) $\operatorname{Ker}\chi$ *is a σ-compact subgroup whose closure contains N_μ, $D/\operatorname{Ker}\chi$ is cyclic and admits a generator $a\operatorname{Ker}\chi$ such that $\mu(a\operatorname{Ker}\chi) = 1$ and $\chi(a) = \alpha$.*

Proof Since α is on the boundary of $\sigma(P_\mu)$, there exists a sequence of unit vectors $y_n \in \mathfrak{H}$ such that $\lim_{n\to\infty} \|P_\mu y_n - \alpha y_n\| = 0$. Thus $\alpha = \lim_{n\to\infty}\langle P_\mu y_n, y_n\rangle$ and so

$$\int_G \|\pi(g)y_n - \alpha y_n\|^2 \mu(dg) = \int_G \big(2 - \bar\alpha\langle \pi(g)y_n, y_n\rangle - \alpha\langle y_n, \pi(g)y_n\rangle\big)\,\mu(dg)$$
$$= 2 - \bar\alpha\langle P_\mu y_n, y_n\rangle - \alpha\langle y_n, P_\mu y_n\rangle \xrightarrow[n\to\infty]{} 0.$$

By the basic integration theory there exists a subsequence $x_n = y_{m_n}$ and a Borel set $A \subseteq G$ such that $\mu(A) = 1$ and $\lim_{n\to\infty} \|\pi(g)x_n - \alpha x_n\| = 0$ for every $g \in A$.

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Due to the regularity of μ, A can be chosen σ-compact. We also have $\lim_{n\to\infty} \|\pi(g)x_n - \alpha^{-1}x_n\| = 0$ for $g \in A^{-1}$, because $\|\pi(g)x_n - \alpha^{-1}x_n\| = \|\pi(g^{-1})x_n - \alpha x_n\|$. Then straightforward induction yields

$$\lim_{n\to\infty} \|\pi(g)x_n - \alpha^k x_n\| = 0 \tag{2.1}$$

for every $k \in \mathbb{Z}$ and $g \in A^k$. This, in turn, implies that for $g \in \bigcup_{k=1}^{\infty}(A^{-k}A^k \cup A^kA^{-k})$,

$$\lim_{n\to\infty} \|\pi(g)x_n - x_n\| = 0. \tag{2.2}$$

Since $\{g \in G \; ; \; \lim_{n\to\infty} \|\pi(g)x_n - x_n\| = 0\}$ is a subgroup of G, it follows that Equation (2.2) remains true for all $g \in N = \operatorname{gp}\left(\bigcup_{k=1}^{\infty}(A^{-k}A^k \cup A^kA^{-k})\right)$. Note that N is a σ-compact subgroup and $A \subseteq gN$ for every $g \in A$.

Next, let $D = \operatorname{gp}(A)$. Then D is a σ-compact subgroup which satisfies (i). By Lemma 2.1, $N \trianglelefteq D$ and D/N is cyclic. It is easy to see that for every $a \in A$, aN is a generator of D/N. Equations (2.1) and (2.2) imply that for $k \in \mathbb{Z}$ and $g \in a^kN$,

$$\lim_{n\to\infty} \|\pi(g)x_n - \alpha^k x_n\| = 0, \tag{2.3}$$

which is equivalent to

$$\lim_{n\to\infty} \langle \pi(g)x_n, x_n\rangle = \alpha^k. \tag{2.4}$$

As $D = \bigcup_{n\in\mathbb{Z}} a^kN$, it follows from (2.4) that the limit $\chi(g) = \lim_{n\to\infty}\langle \pi(g)x_n, x_n\rangle$ exists for every $g \in D$ and is a character on D with $\chi(g) = \alpha^k$ for $g \in a^kN$. Then (ii) follows immediately from Equation (2.3).

It remains to prove (iii). Clearly, $N \subseteq \operatorname{Ker}\chi$. Hence, $D/\operatorname{Ker}\chi$ is cyclic because D/N is. Furthermore, $\operatorname{Ker}\chi$ must be a union of cosets of N and as N is σ-compact, so is $\operatorname{Ker}\chi$. Finally, $\mu(a\operatorname{Ker}\chi) = 1$ because $a\operatorname{Ker}\chi \supseteq aN \supseteq A$, and since $\overline{\operatorname{Ker}\chi} \trianglelefteq G$, $\overline{\operatorname{Ker}\chi} \supseteq N_\mu$. ■

Remark 2.3 (a) Adaptedness of μ implies that D is dense in G.

(b) If $\rho(P_\mu) = 1$ then $\sigma(P_\mu)$ contains a unimodular element and Theorem 2.2 applies.

(c) Since $\overline{\operatorname{Ker}\chi} \supseteq N_\mu$, when μ is aperiodic Theorem 2.2 implies Theorem 2.3 of Lin and Wittmann [17]: there exists a dense subgroup $H(= \operatorname{Ker}\chi)$ such that $\lim_{n\to\infty} \|\pi(g)x_n - x_n\| = 0$ for every $g \in H$.

(d) The proof of Theorem 2.2 can be easily adapted to show that if α is a unimodular eigenvalue of P_μ with eigenvector x, then there exists a continuous character χ on G such that $N_\mu \subseteq \operatorname{Ker}\chi$, $\pi(g)x = \chi(g)x$ for every $g \in G$, and $\chi(g) = \alpha$ for every $g \in \operatorname{supp}\mu$.

Theorem 2.2 shows, in particular, that when D is considered a discrete group, the restriction of π to D weakly contains the 1-dimensional representation of D given by χ. Hence, we immediately obtain:

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Corollary 2.4 *If G is strongly identity excluding and π is irreducible of dimension greater than* 1*, then $\rho(P_\mu) < 1$.*

For a discrete group identity excluding and strongly identity excluding are equivalent properties simply because if an irreducible unitary representation π weakly contains a 1-dimensional representation given be a character χ, then $\tilde{\pi}(g) = \bar{\chi}(g)\pi(g)$ is an irreducible unitary representation weakly containing the 1-dimensional identity representation.

Corollary 2.5 *Let μ be an adapted probability measure on a countable group G. Then G is identity excluding if and only if $\rho(P_\mu) < 1$ for every irreducible unitary representation of dimension greater than* 1*.*

Proof It remains to prove the 'if' part. But if G is not identity excluding then there exists an irreducible unitary representation π different from the 1-dimensional identity representation Id which weakly contains Id. Thus there is a sequence of unit vectors x_n such that $\lim_{n\to\infty} \|\pi(g)x_n - x_n\| = 0$ for all $g \in G$. Of course, $\dim \pi > 1$. As $\|P_\mu x_n - x_n\| \leq \int_G \|\pi(g)x_n - x_n\|\mu(dg)$, dominated convergence yields $\lim_{n\to\infty} \|P_\mu x_n - x_n\| = 0$. Hence, $1 \in \sigma(P_\mu)$ and so $\rho(P_\mu) = 1$. ∎

The proof that a locally nilpotent group is strongly identity excluding is an easy modification of the proof [17, p. 133] that it is identity excluding:

Theorem 2.6 *Every locally compact nilpotent group is strongly identity excluding.*

Proof Let π be a continuous irreducible unitary representation of the locally compact nilpotent group G in $\mathfrak{H}$ and suppose that there exists a dense subgroup $D \subseteq G$ such that with D given discrete topology the restriction of π to D weakly contains the 1-dimensional representation defined by a character χ of D. Thus there is a net x_i of unit vectors in $\mathfrak{H}$ with $\lim_i \|\pi(g)x_i - \chi(g)x_i\| = 0$ for every $g \in D$. Now, $\tilde{\pi}(g) = \bar{\chi}(g)\pi(g)$, $g \in D$, is an irreducible representation of D and $\lim_i \|\tilde{\pi}(g)x_i - x_i\| = 0$ for every $g \in D$. Let $\tilde{D} = \tilde{\pi}(D)$. Then $\tilde{D}$ is a nilpotent subgroup of the unitary group of $\mathfrak{H}$ and $\tilde{D}$ acts irreducibly on $\mathfrak{H}$. Hence, by Schur's lemma every $z \in Z(\tilde{D})$ is a scalar multiple of the identity operator, $z = cI$. Since $\lim_i \|zx_i - x_i\| = 0$, it follow that $z = I$. So $Z(\tilde{D}) = \{I\}$ and as $\tilde{D}$ is nilpotent, $\tilde{D} = \{I\}$. Consequently, $\dim \mathfrak{H} = 1$. ∎

The conclusion of Theorem 2.2 can be strengthened when μ is a spread out measure, *i.e.*, for some n the n-th convolution power μ^n is nonsingular with respect to the Haar measure. The result is a generalization of Theorem 2.9 of Lin and Wittmann [17].

Lemma 2.7 *Let μ be a spread out probability measure on G. Then:*

(a) *If $A \subseteq G$ is a σ-compact set with $\mu(A) = 1$ then there exists $n \in \mathbb{N}$ such that A^n has nonempty interior.*

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(b) *For every* $\varepsilon > 0$ *there exists a neighbourhood* U *of* e *and* $n_0 \in \mathbb{N}$ *such that* $\|\delta_u * \mu^n - \mu^n\| < \varepsilon$ *for all* $u \in U$ *and* $n \geq n_0$.

Proof (a) There exists $k \in \mathbb{N}$ such that μ^k is nonsingular. Since $\mu^k(A^k) = 1$, A^k is not locally null. Hence, A^{2k} has nonempty interior by, *e.g.*, Lemma 5.8 in [12].

(b) See the proof of Theorem 2.11 in [16]. ■

Theorem 2.8 *Let* μ *be spread out. If* $\sigma(P_\mu)$ *contains a unimodular element* α *then there exists a continuous character* χ *on* G *such that:*

(i) π *weakly contains the* 1*-dimensional representation defined by* χ*;*
(ii) $\operatorname{Ker}\chi$ *is open in* G, $\operatorname{Ker}\chi \supseteq N_\mu$, $G/\operatorname{Ker}\chi$ *is cyclic and admits a generator* $a\operatorname{Ker}\chi$ *such that* $\mu(a\operatorname{Ker}\chi) = 1$ *and* $\chi(a) = \alpha$.

Proof Let x_n, D, a, and χ be as described in Theorem 2.2. To prove (ii) it suffices to show that $D = G$ and that $\operatorname{Ker}\chi$ is open. Openness of $\operatorname{Ker}\chi$ implies continuity of χ. Since D is σ-compact, Theorem 2.2(i) combined with Lemma 2.7(a) gives that D is open, and so also closed. Therefore $D = G$ by adaptedness. Similarly, Theorem 2.2(iii) combined with Lemma 2.7(a) yields that for some $n \in \mathbb{N}$, $(a\operatorname{Ker}\chi)^n = a^n\operatorname{Ker}\chi$ has nonempty interior and, hence, $\operatorname{Ker}\chi$ is an open subgroup.

To prove (i) we need to show that $\chi(g) = \lim_{n\to\infty}\langle\pi(g)x_n, x_n\rangle$ uniformly on compact subsets of G, or equivalently, that $\lim_{n\to\infty}\|\pi(g)x_n - \chi(g)x_n\| = 0$ uniformly on compact subsets.

From Theorem 2.2(ii) we know that $\lim_{n\to\infty}\|\pi(g)x_n - \chi(g)x_n\| = 0$ holds pointwise. Since $\|P_\mu^k x_n - \chi(a^k)x_n\| \leq \int_G \|\pi(g)x_n - \chi(g)x_n\|\mu^k(dg)$, the dominated convergence theorem yields

$$\lim_{n\to\infty}\|P_\mu^k x_n - \chi(a^k)x_n\| = 0 \tag{2.5}$$

for every $k \in \mathbb{N}$. Next,

$$\begin{aligned}
\|\pi(g)P_\mu^k x_n - \chi(g)P_\mu^k x_n\| &\leq \|\pi(g)P_\mu^k x_n - \pi(g)\chi(a^k)x_n\| \\
&\quad + \|\pi(g)\chi(a^k)x_n - \chi(g)\chi(a^k)x_n\| \\
&\quad + \|\chi(g)\chi(a^k)x_n - \chi(g)P_\mu^k x_n\| \\
&= 2\|P_\mu^k x_n - \chi(a^k)x_n\| + \|\pi(g)x_n - \chi(g)x_n\|. \qquad (2.6)
\end{aligned}$$

Combining (2.5) and (2.6) one gets

$$\lim_{n\to\infty}\|\pi(g)P_\mu^k x_n - \chi(g)P_\mu^k x_n\| = 0 \tag{2.7}$$

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for every $g \in G$ and $k \in \mathbb{N}$. Next,

$$\begin{aligned}\|\pi(g)x_n - \chi(g)x_n\| &= \|\pi(g)\chi(a^k)x_n - \chi(g)\chi(a^k)x_n\| \\ &\leq \|\pi(g)\chi(a^k)x_n - \pi(g)P_\mu^k x_n\| + \|\pi(g)P_\mu^k x_n - \chi(g)P_\mu^k x_n\| \\ &\quad + \|\chi(g)P_\mu^k x_n - \chi(g)\chi(a^k)x_n\| \\ &= 2\|P_\mu^k x_n - \chi(a^k)x_n\| + \|\pi(g)P_\mu^k x_n - \chi(g)P_\mu^k x_n\|. \end{aligned} \tag{2.8}$$

Let a compact $C \subseteq G$ and $\varepsilon > 0$ be given. By Lemma 2.7(b) we can find $k \in \mathbb{N}$ and a neighbourhood U of e such that $\|\delta_u * \mu^k - \mu^k\| < \frac{1}{2}\varepsilon$ for every $u \in U$, and that $U \subseteq \operatorname{Ker}\chi$. By compactness of C there exist $g_1, \ldots, g_s \in C$ with $C \subseteq \bigcup_{i=1}^s g_i U$. When $g \in C$ then $g \in g_i U$ for some i. Hence, using (2.8) we get

$$\begin{aligned}\|\pi(g)x_n - \chi(g)x_n\| &\leq 2\|P_\mu^k x_n - \chi(a^k)x_n\| + \|\pi(g)P_\mu^k x_n - \pi(g_i)P_\mu^k x_n\| \\ &\quad + \|\pi(g_i)P_\mu^k x_n - \chi(g_i)P_\mu^k x_n\| + \|\chi(g_i)P_\mu^k x_n - \chi(g)P_\mu^k x_n\| \\ &\leq 2\|P_\mu^k x_n - \chi(a^k)x_n\| + \|\delta_g * \mu^k - \delta_{g_i} * \mu^k\| \\ &\quad + \|\pi(g_i)P_\mu^k x_n - \chi(g_i)P_\mu^k x_n\| \\ &\leq 2\|P_\mu^k x_n - \chi(a^k)x_n\| + \tfrac{1}{2}\varepsilon + \max_{1\leq j\leq s} \|\pi(g_j)P_\mu^k x_n - \chi(g_j)P_\mu^k x_n\|. \end{aligned} \tag{2.9}$$

Then (2.5) and (2.7) imply that $\sup_{g\in C} \|\pi(g)x_n - \chi(g)x_n\| < \varepsilon$ for large enough n. ∎

Note that by Remark 2.3d) the condition that $\rho(P_\mu) < 1$ for every continuous irreducible unitary representation of dimension greater than 1 is a weakening of the condition that for every continuous unitary representation the unimodular elements of $\sigma(P_\mu)$ are eigenvalues. The next corollary answers the question what is needed in order that this stronger condition holds.

Corollary 2.9 *Given an adapted probability measure μ on G consider the following two conditions:*

(1) *For every continuous unitary representation of G the unimodular elements of the spectrum of P_μ are eigenvalues.*
(2) *G has Property T.*

Then (1) *implies* (2). (1) *and* (2) *are equivalent when μ is spread out.*

Proof (1) $\Rightarrow$ (2): Suppose π is a continuous unitary representation of G which weakly contains the 1-dimensional identity representation Id. Arguing similarly as in the proof of Corollary 2.5, we obtain that $1 \in \sigma(P_\mu)$. So $P_\mu x = x$ for a nonzero $x \in \mathfrak{H}$. But this implies that $\pi(g)x = x$ for all $g \in G$ [6, Proposition 2.1]. Therefore π contains Id.

(2) $\Rightarrow$ (1) for spread out μ: This follows immediately from Theorem 2.8 and the fact that if a continuous unitary representation π of a group with Property T weakly contains a continuous 1-dimensional representation χ, then π contains χ. ∎

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3 The Asymptotic Behaviour of P_μ^n

In [6] Derriennic and Lin proved that if μ is an aperiodic spread out probability measure on a locally compact group then for every continuous unitary representation π the powers P_μ^n converge in the strong operator topology to the orthogonal projection onto the subspace of the fixed points of π. When G is abelian or compact the result remains true without the assumption that μ be spread out. According to [16, 17, 20] this assumption can be also disposed of for certain classes of noncompact nonabelian groups: [SIN] groups [16], metrizable nilpotent groups [17], some solvable algebraic groups [20]. We will see below that for nilpotent groups the assumption of metrizability is redundant. It remains an open question whether the result holds for any (regular) aperiodic probability measure on any locally compact group.

In [1] Akcoglu and Boivin showed that when T is any contraction in a Hilbert space then there exists another contraction R and an isometry U such that $s\text{-}\lim_{n\to\infty}(T^n - U^n R) = 0$. The result of Derriennic and Lin tells us what U and R are when $T = P_\mu$ where μ is spread out and aperiodic. This raises not only the question what happens when μ fails to be spread out but also the question what happens when μ is no longer assumed aperiodic.

The only known proof of the extension of the result of Derriennic and Lin to arbitrary aperiodic probability measures on nilpotent groups relies on the direct integral decomposition combined with the fact that nilpotent groups are identity excluding. Using our observation that nilpotent groups are strongly identity excluding more can be achieved: one can completely describe the asymptotic behaviour of the powers P_μ^n for any adapted probability measure μ.

To do this we will need the following version of a result of Lin and Wittmann [17, Theorem 2.2]. Recall that N_μ denotes the smallest closed normal subgroup of G such that μ is carried on a coset of N_μ. For a given continuous unitary representation in a Hilbert space $\mathfrak{H}$ we will write E_μ for the orthogonal projection onto the subspace $\mathfrak{N}_\mu = \{x \in \mathfrak{H} ; \pi(g)x = x \text{ for every } g \in N_\mu\}$.

Theorem 3.1 *Let $a \in G$ be an element with $\mu(aN_\mu) = 1$. The following two conditions are equivalent:*

(1) $s\text{-}\lim_{n\to\infty}\left(P_\mu^n - \pi(a)^n E_\mu\right) = 0$ *for every continuous unitary representation π;*
(2) $s\text{-}\lim_{n\to\infty} P_\mu^n = 0$ *for every continuous irreducible unitary representation π of dimension greater than* 1.

Proof (1) $\Rightarrow$ (2) Since $N_\mu \trianglelefteq G$, $\mathfrak{N}_\mu = \{x \in \mathfrak{H} ; \pi(g)x = x \text{ for every } g \in N_\mu\}$ is a π-invariant subspace of $\mathfrak{H}$. Hence, if π is irreducible then $\mathfrak{N}_\mu = \{0\}$ or $\mathfrak{N}_\mu = \mathfrak{H}$. If $\mathfrak{N}_\mu = \mathfrak{H}$ then $N_\mu \subseteq \operatorname{Ker}\pi$ and so π would give rise to an irreducible representation of G/N_μ. But this is impossible because $\dim\pi > 1$ and G/N_μ is abelian by Proposition 1.6 in [6]. Hence, $\mathfrak{N}_\mu = \{0\}$ and so $s\text{-}\lim_{n\to\infty} P_\mu^n = 0$.

(2) $\Rightarrow$ (1) Since $\mathfrak{N}_\mu$ is an invariant subspace on which $P_\mu = \pi(a)$, it suffices to prove that $s\text{-}\lim_{n\to\infty} P_\mu^n = 0$ for every continuous unitary representation π with $\mathfrak{N}_\mu = \{0\}$.

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Suppose first that G is second countable (which is the same as metrizable because G is σ-compact). Then to apply the direct integral argument of Lin and Wittmann [17, proof of Theorem 2.2] one just needs to observe that the only 1-dimensional representations φ that can possibly occur in the direct integral decomposition of π are those with $N_\mu \not\subseteq \operatorname{Ker} \varphi$ (because $\mathfrak{N}_\mu = \{0\}$). For such 1-dimensional representations one has $\|P_\mu\| = \|\int_G \varphi(g)\mu(dg)\| < 1$ and so $\lim_{n\to\infty} \|P_\mu^n\| = 0$. This proves the claim for second countable groups.

When G is not necessarily second countable, then G, being σ-compact, contains arbitrarily small compact normal subgroups K with G/K second countable [10, Theorem 8.7]. Let $x \in \mathfrak{H}$ and $\varepsilon > 0$ be given. Then we can find a compact normal subgroup K such that G/K is second countable and $\|x - \pi(k)x\| \le \frac{1}{2}\varepsilon$ for all $k \in K$. Let ω_K be the normalized Haar measure of K and consider P_{ω_K}, the ω_K-average of π. We have $\|(I - P_{\omega_K})x\| \le \frac{1}{2}\varepsilon$. Moreover, P_{ω_K} is the orthogonal projection onto a π-invariant subspace $\mathfrak{H}'$. $\mathfrak{H}'$ is the subspace of the fixed points of K. Let π' and P'_μ denote the restrictions of π and P_μ to $\mathfrak{H}'$. Then π' can be considered a continuous unitary representation of the second countable group G/K. Moreover, every continuous irreducible unitary representation of G/K can be lifted to a continuous irreducible unitary representation of G and so assumption (2) is true for G/K and the adapted measure μ' which is the canonical image of μ in G/K. By the result for second countable groups it follows that $s\text{-}\lim_{n\to\infty} P_{\mu'}^n = 0$ where $P_{\mu'}$ is the μ'-average of π'. But $P_{\mu'} = P'_\mu$. Hence, $\|P_\mu^n x\| \le \|P_\mu^n(I - P_{\omega_K})x\| + \|P_\mu^n P_{\omega_K} x\| \le \frac{1}{2}\varepsilon + \|P_{\mu'}^n P_{\omega_K} x\| \le \varepsilon$ for large enough n. ■

Using [17, Theorem 2.1] it can be seen that Theorem 3.1 implies the result of Lin and Wittmann [17, Theorem 2.2] (without their assumption of metrizability): when μ is aperiodic then P_μ^n converges strongly for every continuous unitary representation if and only if P_μ^n converges strongly for every continuous irreducible unitary representation.

Combining Theorem 3.1 and Corollary 2.4 we obtain:

Theorem 3.2 *If G is strongly identity excluding then $s\text{-}\lim_{n\to\infty}\big(P_\mu^n - \pi(a)^n E_\mu\big) = 0$ for every continuous unitary representation π and every $a \in G$ with $\mu(aN_\mu) = 1$.*

Theorem 3.3 *If μ is a spread out probability measure and G has Property T, then $\lim_{n\to\infty} \|P_\mu^n - \pi(a)^n E_\mu\| = 0$ for every continuous unitary representation and every $a \in G$ with $\mu(aN_\mu) = 1$.*

Proof As $\mathfrak{N}_\mu$ is invariant under π, so is $\mathfrak{N}_\mu^\perp$. Let π' and P'_μ denote the restrictions of π and P_μ to $\mathfrak{N}_\mu$, and π'' and P''_μ their restrictions to $\mathfrak{N}_\mu^\perp$. Since ${P'_\mu}^n = \pi'(a)^n$, it suffices to show that $\lim_{n\to\infty} \|{P''_\mu}^n\| = 0$.

Suppose that $\lim_{n\to\infty} \|{P''_\mu}^n\| \neq 0$. Then by Theorem 2.8 π'' weakly contains a 1-dimensional representation given by a continuous character χ with $\operatorname{Ker} \chi \supseteq N_\mu$. Property T implies that there exists a nonzero $x \in \mathfrak{N}_\mu^\perp$ such that $\pi''(g)x = \chi(g)x$

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for all $g \in G$. In particular, $\pi''(g)x = x$ for all $g \in N_\mu$. So $x \in \mathfrak{N}_\mu$ which is a contradiction. ■

Motivated by the result of Derriennic and Lin one may wonder whether, at least for spread out μ, the asymptotic behaviour described in Theorem 3.2 can be proven for arbitrary locally compact groups, in particular, for any adapted μ on any countable G. The following example shows that this is not the case. The example was suggested to us by M.-D. Choi.

Example 3.4 Let $\{e_j\}_{j\in\mathbb{Z}}$ be the standard basis of $\mathfrak{H} = l^2(\mathbb{Z})$ and let u and v be the unitary operators in $\mathfrak{H}$ defined by

$$ue_j = e_{j+1}, \quad j \in \mathbb{Z},$$
$$ve_j = e_{j+1}, \quad j \in \mathbb{Z} - \{0\},$$
$$ve_0 = -e_1.$$

Define G to be the group generated by u and v. When $g \in G$, then g has the form $g = u_1^{\varepsilon_1} u_2^{\varepsilon_2} \cdots u_n^{\varepsilon_n}$ where $u_i \in \{u, v\}$ and $\varepsilon_i \in \{\pm 1\}$. So $ge_j = \varepsilon e_{j+k}$ with $\varepsilon = \pm 1$ and $k = |\{i \ ; \ \varepsilon_i = 1\}| - |\{i \ ; \ \varepsilon_i = -1\}|$. In particular, it follows that every element of the derived group $[G, G]$ is diagonalized by the basis $\{e_j\}_{j\in\mathbb{Z}}$. Thus $[G, G]$ is abelian and so G is solvable. It is routine to show that every linear operator in $\mathfrak{H}$ which commutes with both u and v is a multiple of the identity operator. Hence, the natural representation $\pi(g) = g$ of G in $\mathfrak{H}$ is irreducible. Let $\mu = \frac{1}{2}\delta_u + \frac{1}{2}\delta_v$. Then $P_\mu = \frac{1}{2}u + \frac{1}{2}v$ and it follows that $P_\mu^n e_1 = e_{n+1}$ for every $n \in \mathbb{N}$. Thus $s\text{-}\lim_{n\to\infty} P_\mu^n \neq 0$ and the conclusion of Theorem 3.2 is false here.

Remark 3.5 The asymptotic behaviour of Theorem 3.2 can be established also under different hypotheses. For example, when the measure $\tilde{\mu}(A) = \mu(A^{-1})$ is spread out and defines an ergodic random walk, or when G is second countable and μ defines a recurrent random walk.

4 Countable Amenable Identity Excluding Groups

The goal of this section is to provide a characterization of countable amenable identity excluding groups and discuss some of their properties.

Let G be a group. An element $g \in G$ is called an FC-*element* [22, Chapter 4.3] if the conjugacy class of g is finite. The set $F(G)$ of FC-elements is a characteristic subgroup of G, called the FC-*centre*. A group which coincides with its FC-centre is called an FC-*group*; a group for which $F(G)$ is trivial (*i.e.*, every nontrivial conjugacy class is infinite) is called an ICC *group*.

The class of FC-groups includes abelian groups and direct products of finite groups. It is closed with respect to forming subgroups and quotients; moreover, extensions of finite groups by FC-groups are FC-groups. Standard examples of ICC groups are free groups on 2 or more generators and groups of finite permutations of infinite sets. Certain solvable groups are ICC groups, *e.g.*, the $ax + b$ group and

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the group from our Example 3.4. Moreover, every infinite simple group is an ICC group.[1]

For any group there exists a transfinite series of characteristic subgroups

$$\{e\} = F_0(G) \le F_1(G) \le F_2(G) \le \cdots \le F_\alpha(G) \le \cdots$$

indexed by ordinals α, such that $F_{\alpha+1}(G) = \{g \in G \; ; \; gF_\alpha(G) \in F(G/F_\alpha)\}$ for every α (in particular, $F_1(G) = F(G)$), and that for every limit ordinal α, $F_\alpha(G) = \bigcup_{\beta<\alpha} F_\beta(G)$. This series is called the *upper* FC*-central series* of G [22, Chapter 4.3]. Since the cardinality of G cannot be exceeded, there exists an ordinal γ whose cardinal number is at most that of G and such that $F_\alpha(G) = F_\gamma(G)$ for every $\alpha \ge \gamma$. The smallest γ with this property is called the *length of the upper* FC*-central series* and the terminal subgroup $\hat{F}(G) = F_\gamma(G)$ is called the FC*-hypercentre* of G. If G coincides with its hypercentre, it is called FC*-hypercentral*. An FC-hypercentral group whose upper FC-central series has finite length is called FC*-nilpotent*.

Remark 4.1 $G/\hat{F}(G)$ is an ICC group. A nontrivial FC-hypercentral group has a nontrivial FC-centre. A group is FC-hypercentral if and only if it does not admit a nontrivial ICC group as a quotient.

Of course, every FC-group and every nilpotent group is FC-nilpotent. Direct products of nilpotent groups are FC-hypercentral but they need not be FC-nilpotent [22, Corollary 2, p. 131]. Subgroups, quotients, and finite extensions of FC-hypercentral (resp., FC-nilpotent) groups are FC-hypercentral (resp., FC-nilpotent). Extensions of finite groups by FC-hypercentral (resp., FC-nilpotent) groups are FC-hypercentral (resp., FC-nilpotent). The class of FC-hypercentral groups is closed under forming direct products.

Theorem 4.2 ([8, 18]) *The following conditions are equivalent for a finitely generated group G:*

(i) *G is* FC*-hypercentral.*
(ii) *G is* FC*-nilpotent.*
(iii) *G contains a nilpotent subgroup of finite index.*

As the example of an infinite direct product of finite nonabelian simple groups shows, in general, an FC-nilpotent group need not contain any nilpotent subgroups of finite index.

Recall that a (discrete) group is said to have *polynomial growth* if for every finite subset $A \subseteq G$ there exists a polynomial p such that $|A^n| \le p(n)$ for every $n \in \mathbb{N}$. According to a theorem of Gromov [9] a finitely generated group has polynomial growth if and only if it contains a nilpotent subgroup of finite index. We thus have:

Corollary 4.3 *Every* FC*-hypercentral group has polynomial growth. A finitely generated group has polynomial growth if and only if it is* FC*-nilpotent.*

[1]This is an immediate consequence of the following characterization of ICC groups communicated to us by László Babai: G is ICC if and only if it has no nontrivial finite normal subgroups and every normal subgroup of finite index has trivial centre.

We note that there exist groups of polynomial growth, even locally finite groups, which are not FC-hypercentral; such is the group of finite permutations of an infinite set.

Theorem 4.4 *Every* FC*-hypercentral group is identity excluding.*

Proof Let π be an irreducible unitary representation of an FC-hypercentral group G in a Hilbert space $\mathfrak{H}$. Suppose that π weakly contains the 1-dimensional identity representation Id, *i.e.*, there is a net x_i of unit vectors in $\mathfrak{H}$ with $\lim_i \|\pi(g)x_i - x_i\| = 0$ for every $g \in G$. Now, the group $\tilde{G} = \pi(G)$ is FC-hypercentral. Let $u \in F(\tilde{G})$, C be the (finite) conjugacy class of u, and let $T = \frac{1}{|C|}\sum_{c\in C} c$. As for every $g \in \tilde{G}$ the inner automorphism $g \cdot g^{-1}$ permutes the elements of C, it follows that $gTg^{-1} = T$ for all $g \in \tilde{G}$. Since $\tilde{G}$ acts irreducibly on $\mathfrak{H}$ this means that $T = \tau I$ for some $\tau \in \mathbb{C}$. But $|\tau - 1| = \|Tx_i - x_i\| \le \frac{1}{|C|}\sum_{c\in C}\|cx_i - x_i\| \to 0$. Hence, $\tau = 1$, *i.e.*, $\frac{1}{|C|}\sum_{c\in C} cx = x$ for every $x \in \mathfrak{H}$. This implies that $cx = x$ for all $c \in C$ and $x \in \mathfrak{H}$, in particular, $u = I$. Thus $F(\tilde{G})$ is trivial, and so $\tilde{G}$ is trivial by Remark 4.1. This means that $\pi = \mathrm{Id}$. ∎

Theorem 4.5 *A countable group is amenable and identity excluding if and only if it is* FC*-hypercentral.*

Proof Recall that every group of polynomial growth is amenable. Hence, in view of Theorem 4.4 and Corollary 4.3 it remains to show that every countable amenable identity excluding group G is FC-hypercentral. Now, if G is not FC-hypercentral, then by Remark 4.1 G admits a nontrivial ICC group as a quotient. Therefore it suffices to show that a nontrivial countable amenable ICC group is not identity excluding. The following (standard) proof was communicated to us by E. Kaniuth.

Let G be a nontrivial countable ICC group. Consider the left regular representation λ of G and the associated representation Λ of the group C^*-algebra $C^*(G)$. It is well known that Λ is a factor representation. Now, the kernel of a factor representation is a prime ideal [7, 5.7.6] and since $C^*(G)$ is separable, the kernel of Λ is also a primitive ideal [7, 3.9.1]. Thus there exists a irreducible representation Π of $C^*(G)$ which is weakly equivalent to Λ. This means that the irreducible representation π of G associated with Π is weakly equivalent to λ. But when G is amenable then λ weakly contains Id. Hence π also weakly contains Id and so G is not identity excluding. ∎

Corollary 4.6 *The following conditions are equivalent for a finitely generated group G:*

(i) *G is amenable and identity excluding;*
(ii) *G has polynomial growth;*
(iii) *G contains a nilpotent subgroup of finite index;*
(iv) *G is* FC*-nilpotent.*

Our last goal is to show that every adapted random walk on every countable amenable identity excluding group is ergodic.

Let G be a countable group. Recall that a random walk of law μ on G is called *ergodic* if for every $g \in G$, $\lim_{n\to\infty} \|\frac{1}{n}\sum_{i=1}^{n}\mu^i - \delta_g * \frac{1}{n}\sum_{i=1}^{n}\mu^i\| = 0$. An ergodic

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random walk is necessarily adapted (*i.e.*, μ is adapted) and it is well known that ergodic random walks can exist only on amenable groups. However, while on some amenable groups every adapted random walk is ergodic, others admit both ergodic and non-ergodic adapted random walks [15]. The most general class of countable groups for which the former is known to be true is the class of finite extensions of nilpotent groups, which is a proper subclass of the class of countable amenable identity excluding groups.

A bounded function $h\colon G \to \mathbb{C}$ is called *μ-harmonic* if for every $g \in G$, $h(g) = \int_G h(gg')\mu(dg')$. Ergodicity of the random walk of law μ is equivalent to the condition that every bounded μ-harmonic function is constant [21]. Let $\mathcal{H}_\mu$ denote the space of the bounded μ-harmonic functions. $\mathcal{H}_\mu$ is a subspace of $L^\infty(G)$ which is invariant under the usual left action of G on $L^\infty(G)$. The representation of G given by this action on $\mathcal{H}_\mu$ will be called the *μ-representation*.

Lemma 4.7 *Let μ be a probability measure on a countable group G such that $G = S_\mu S_\mu^{-1}$ where S_μ denotes the semigroup generated by the support of μ. It follows that the* FC*-centre $F(G)$ is contained in the kernel of the μ-representation.*

Proof By [2, Théorème IV.1] or [11, Proposition 3.1 and p. 117], for $h \in \mathcal{H}_\mu$ and $g \in F(G)$ we have $h(tg) = h(t)$ for every $t \in G$. Since $F(G) \trianglelefteq G$, for every $h \in \mathcal{H}_\mu$ and $g \in F(G)$ we obtain: $(gh)(t) = h(g^{-1}t) = h(tt^{-1}g^{-1}t) = h(t)$ for all $t \in G$. Thus g is contained in the kernel of the μ-representation. ■

Theorem 4.8 *Every adapted random walk on a countable amenable identity excluding group is ergodic.*

Proof Let N be the kernel of the μ-representation, $\varphi\colon G \to G/N$ the canonical homomorphism, and μ' the measure $\mu'(A) = \mu\big(\varphi^{-1}(A)\big)$ on G/N. μ' is adapted. Since G/N has polynomial growth, by [12, Theorem 5.6] we have that $G/N = S_{\mu'}S_{\mu'}^{-1}$ where $S_{\mu'}$ is the semigroup generated by supp μ'. Hence, by Lemma 4.7, $F(G/N)$ is contained in the kernel of the μ'-representation of G/N.

Now, when $h \in \mathcal{H}_\mu$ then $h = h' \circ \varphi$ for some $h' \in \mathcal{H}_{\mu'}$. Hence, $\varphi^{-1}\big(F(G/N)\big)$ must be contained in the kernel of the μ-representation, *i.e.*, in N. But this means that $F(G/N)$ is trivial and so G/N is trivial by Remark 4.1. So $N = G$ which implies that every $h \in \mathcal{H}_\mu$ is constant. ■

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School of Mathematics and Statistics
Carleton University
Ottawa, Ontario
K1S 5B6
e-mail: wjaworsk@math.carleton.ca