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RATIO LIMIT THEOREMS

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Introduction. Let $\{f_n, \mathscr{B}_n, n \geqq 1\}$ be an adapted sequence of integrable random variables on the probability space $(\Omega, \mathscr{B}, P)$. Let us set $\sigma^2(f_{n+m}|\mathscr{B}_n) = E(f_{n+m}^2|\mathscr{B}_n) - E^2(f_{n+m}|\mathscr{B}_n)$. The following result can be immediately derived from Brown [**2**]:

THEOREM 1. *If*

(1) $\int f_n > 0$ *and* $E(f_n|\mathscr{B}_{n-1}) \geqq 0$ *for all* $n \geqq 1$, *and*

(2) $\sup_{n \geqq 1} \dfrac{\sigma^2(f_{n+m}|\mathscr{B}_{n-1})}{E(f_{n+m}|\mathscr{B}_{n-1})} \in \mathscr{L}_1$, *for all* $m \geqq 0$

then we have

($\bar{1}$) *There exists, modulo a set of measure zero, a partition of* Ω *into two sets:*

$$\Omega_0 = \bigcap_{m=0}^{\infty} \left\{ \sum_1^{\infty} E(f_{n+m}|\mathscr{B}_n) \text{ exists and is finite} \right\}$$

and

$$\Omega_\infty = \bigcap_{m=0}^{\infty} \left\{ \sum_1^{\infty} E(f_{n+m}|\mathscr{B}_n) = \infty \right\}.$$

($\bar{2}$) *On* Ω_∞ *and for each* $m \geqq 0$:

$$(*) \quad \lim_{N\to\infty} \frac{\sum_1^N E(f_{n+m}|\mathscr{B}_n)}{\sum_1^N E(f_{n+m}|\mathscr{B}_{n-1})} = 1.$$

This result is readily extracted from Brown's Theorem (1) and Corollary (2). Our condition (2) requires that $E(f_{n+m}|\mathscr{B}_n) = 0 \Rightarrow \sigma^2(f_{n+m}|\mathscr{B}_n) = 0$; whenever this occurs we will interpret the ratio in (2) as equal to the value one. We also impose, for normalization purposes, the condition $\mathscr{B}_0 = \{\emptyset, \Omega\}$. Our objective in this paper is to derive a sharper form of the theorem above by imposing a mixing condition on the variables. This condition is one possible generalization of (*)-mixing (see Stout [**5**]) and will have the following form:

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There exists f_∞ such that, given $\epsilon > 0$, there exists n_0 where $n \geqq n_0$ implies

$$\left| \frac{E(f_{n+m}|\mathscr{B}_n)}{\int f_{n+m}} - f_\infty \right| \leqq \epsilon + \phi(m)$$

where $\phi(m)$ is a sequence of constants, $\phi(m) \downarrow 0$.

This condition will be written

$$(^{**}) \quad \frac{E(f_{n+m}|\mathscr{B}_n)}{\int f_{n+m}} \to f_\infty.$$

This kind of mixing is met in the following circumstances. Suppose there exists a sequence of reals $\{\phi(m)\}$ where $\phi(m) \downarrow 0$ and suppose there exists an adapted sequence $\{\bar{f}_n, \mathscr{B}_n\}$ with

$$\left| \frac{E(f_{n+m}|\mathscr{B}_n)}{\int f_{n+m}} - \bar{f}_n \right| \leqq \phi(m).$$

It is then easy to see that $\{\bar{f}_n, \mathscr{B}_n\}$ is a positive supermartingale converging to some f_∞ satisfying (**).

Theorems and corollaries.

THEOREM 2. *Let the adapted sequence $\{f_n, \mathscr{B}_n\}$ on $(\Omega, \mathscr{B}, P)$ satisfy the following conditions:*

(1) $\int f_n > 0$ *and* $E(f_n|\mathscr{B}_{n-1}) \geqq 0$, *for all* $n \geqq 1$.

(2) *For all integers* $m \geqq 1$: $\sup_{n \geqq 0} \dfrac{\sigma^2(f_{n+m}|\mathscr{B}_n)}{E(f_{n+m}|\mathscr{B}_n)} \in \mathscr{L}_1$.

(3) $\dfrac{E(f_{n+m}|\mathscr{B}_n)}{\int f_{n+m}} \to f_\infty > 0$.

(4) *For all* $m \geqq 1$: $\lim_{N \to \infty} \dfrac{E(f_{N+m}|\mathscr{B}_N)}{\sum_1^N E(f_{n+m}|\mathscr{B}_n)} \to 0$.

Then we can conclude

(I) $P\{\sum f_n = \infty\} = 1 \Leftrightarrow \sum \int f_n = \infty$, *and* $P\{\sum f_n$ *exists and is finite*$\} = 1 \Leftrightarrow \sum \int f_n < \infty$.

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(II) *If* $\sum \int f_n = \infty$, *then*

$$\frac{\sum_1^N f_n}{\sum_1^N \int f_n} \rightarrow f_\infty \, P \text{ a.e.}$$

Proof. From Condition (3), we can find n_0, m_0 so large that $m \geqq m_0$ implies

$$0 < f_\infty - \epsilon \leqq \frac{\sum_{n_0}^N E(f_{n+m}|\mathscr{B}_n)}{\sum_{n_0}^N \int f_{n+m}} \leqq f_\infty + \epsilon < \infty.$$

From this it follows that $\sum_1^\infty \int f_n = \infty \Leftrightarrow \sum_1^\infty E(f_{n+m}|\mathscr{B}_n) = \infty$, and then using Theorem 1 we see that (I) obtains. Let us now assume $\sum \int f_n = \infty$. Returning to the inequality above, we can choose N so large that

$$0 < (1 - \epsilon)f_\infty - 3\epsilon \leqq \frac{\sum_1^N E(f_{n+m}|\mathscr{B}_n)}{\sum_1^N \int f_{n+m}} \leqq (1 + \epsilon)f_\infty + 3\epsilon < \infty$$

since all individual terms $E(f_{n+m}|\mathscr{B}_n)$, $\int f_{n+m}$ are finite. Now

$$\frac{\sum_1^N f_n}{\sum_1^N \int f_n} = \left[\prod^{m-1} \frac{\sum_1^N E(f_{n+k}|\mathscr{B}_n)}{\sum_1^N E(f_{n+k+1}|\mathscr{B}_n)}\right] \cdot \frac{\sum_1^N E(f_{n+m}|\mathscr{B}_n)}{\sum_1^N \int f_{n+m}}.$$

Given our last inequality, the ratio on the left will be close to f_∞ provided the product in brackets above is close to one, and for this it suffices that for each fixed k we have

$$\frac{\sum_1^N E(f_{n+k}|\mathscr{B}_n)}{\sum_1^N E(f_{n+k+1}|\mathscr{B}_n)} \rightarrow 1,$$

and given (*) from Theorem 1, this will happen provided

$$\frac{E(f_{N+k}|\mathscr{B}_N)}{\sum_1^N E(f_{n+k}|\mathscr{B}_n)} \rightarrow 0$$

which is our Condition (4). This completes the proof.

Remarks. Note that

$$\frac{E(f_{n+m}^2|\mathscr{B}_n)}{E(f_{n+m}|\mathscr{B}_n)} = \frac{\sigma^2(f_{n+m}|\mathscr{B}_n)}{E(f_{n+m}|\mathscr{B}_n)} + E(f_{n+m}|\mathscr{B}_n).$$

Therefore, if we were to replace Condition (2) of our theorem by

(2′) $\sup_n \dfrac{E(f_{n+m}{}^2|\mathscr{B}_n)}{E(f_{n+m}|\mathscr{B}_n)} \in \mathscr{L}_1$ *for all* $m \geqq 1$,

then Condition (4) would be automatically satisfied, given II.

COROLLARY 1. (Borel-Cantelli type result). *Let* $\{B_n, \mathscr{B}_n\}$ *be an adapted sequence of sets on* $(\Omega, \mathscr{B}, P)$ *where*

(1) $P(B_n) \geqq 0$ *for all* $n \geqq 1$, *and*

(2) $\dfrac{P(B_{n+m}|\mathscr{B}_n)}{P(B_{n+m})} \rightarrow f_\infty > 0.$

Then

(I) $P\{\sum I_{B_n} = \infty\} = 1 \Leftrightarrow \sum P(B_n) = \infty$, *and*

$P\{\sum I_{B_n} < \infty\} = 1 \Leftrightarrow \sum P(Bn) < \infty.$

(II) *If* $\sum P(B_n) = \infty$, *then*

$$\frac{\sum_1^N I_{B_n}}{\sum_1^N P(B_n)} \rightarrow f_\infty.$$

COROLLARY 2. (A probabilistic L'Hospital's Rule). *Let* $\{f_n, \mathscr{B}_n\}$, $\{g_n, \mathscr{B}_n\}$ be *adapted sequences on* $(\Omega, \mathscr{B}, P)$ *which satisfy conditions* (1) *through* (4) *of Theorem* 2. *Then if* $\sum \int f_n = \sum \int g_n = \infty$, *we have*

$$\overline{\lim_N} \frac{\sum_1^N f_n}{\sum_1^N g_n} = \overline{\lim_N} \frac{\sum_1^N \int f_n}{\sum_1^N \int g_n} \cdot \frac{f_\infty}{g_\infty}$$

and

$$\underline{\lim_N} \frac{\sum_1^N f_n}{\sum_1^N g_n} = \underline{\lim_N} \frac{\sum_1^N \int f_n}{\sum_1^N \int g_n} \cdot \frac{f_\infty}{g_\infty}.$$

An application. Suppose the adapted sequence $\{f_n, \mathscr{B}_n\}$ where $\int f_n > 0$ is (*) mixing in the following sense: There exists $\{\phi(m)\}$, a sequence of constants with $1 > \phi(m) \downarrow 0$ and

$$|P\{f_{n+m} \in A|\mathscr{B}_n\} - P\{f_{n+m} \in A\}| \leqq \phi(m)P\{f_{n+m} \in A\}$$

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for all Borel sets A. Then, provided second moments exist, one has, for $l = 1, 2$:

$$\left| \frac{E(f_{n+m}{}^l | \mathscr{B}_n)}{\int f_{n+m}{}^l} - 1 \right| \leqq \phi(m).$$

This observation leads to

COROLLARY 3. *Let $\{f_n, \mathscr{B}_n\}$ be (*) mixing as defined above. Assume the following conditions:*

(1) $\int f_n > 0, \int f_n{}^2 < \infty$ *for all n, and*

(2) $$\sup_n \frac{\int f_n{}^2}{\int f_n} < \infty.$$

Then we can conclude:

(I) $P\{\sum f_n = \infty\} = 1 \Leftrightarrow \sum \int f_n = \infty$, *and*

$P\{\sum f_n$ *exists and is finite*$\} = 1 \Leftrightarrow \sum \int f_n < \infty$.

(II) *If $\sum \int f_n = \infty$, then*

$$\frac{\sum_1^N f_n}{\sum_1^N \int f_n} \to 1 \; P \text{ a.e.}$$

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