Canad. J. Math. Vol. **57** (1), 2005 pp. 3–16

Enriques Diagrams and Adjacency of Planar Curve Singularities

Maria Alberich-Carramiñana and Joaquim Roé

Abstract. We study adjacency of equisingularity types of planar complex curve singularities in terms of their Enriques diagrams. The goal is, given two equisingularity types, to determine whether one of them is adjacent to the other. For linear adjacency a complete answer is obtained, whereas for arbitrary (analytic) adjacency a necessary condition and a sufficient condition are proved. We also obtain new examples of exceptional deformations, *i.e,* singular curves of type $\mathcal{D}'$ that can be deformed to a curve of type $\mathcal{D}$ without $\mathcal{D}'$ being adjacent to $\mathcal{D}$.

Introduction

A class of reduced (germs of) planar curve singularities $\mathcal{D}'$ is said to be *adjacent* to the class $\mathcal{D}$ when every member of the class $\mathcal{D}'$ can be deformed into a member of the class $\mathcal{D}$ by an arbitrarily small deformation. If this can be done with a linear deformation, then we say that $\mathcal{D}'$ is *linearly adjacent* to $\mathcal{D}$. We shall focus on the equisingularity (or topological equivalence, see for instance [3, 26, 28]) classes, and we will call them simply *types*. The Enriques diagrams introduced by Enriques in [7, IV.I] represent the types: two reduced curves are equisingular at O if and only if their associated Enriques diagrams are isomorphic (see [3, 3.9]).

In [1] Arnold classified critical points of functions with modality at most two and described some adjacencies between them, introducing the so-called *series* of types A, D, E, J, W, X and Z. This was later generalized to other sequences of singularities (see [23, 27]). Apart from the sequences of adjacencies satisfied by singularities in these, only some particular cases of adjacency seem to be known, obtained using explicit deformations (see for instance [6], and references therein). On the other hand, the semicontinuity of some numerical invariants such as the genus discrepancy δ, the Milnor number μ or the singularity spectrum (see [24]) provide necessary conditions for adjacency. These are topological invariants which do not determine the type of the singularity [25]. Here, instead of numerical invariants, the Enriques diagram (which does determine the type) is used, providing a necessary condition and a sufficient condition for adjacency. In some cases we can show two types not to be adjacent as a consequence of our results, although their adjacency is not discarded by their numerical invariants (such as the spectrum, as in example 3).

In the case of *linear* adjacency we obtain a complete answer, namely, we determine all linear adjacencies in terms of Enriques diagrams. Non-linear adjacencies are

Received by the editors October 14, 2002; revised March 10, 2003.

Both authors completed this work as researchers of the *Programa Ramón y Cajal* of the Ministerio de Ciencia y Tecnología, and were also supported in part by CAICYT PB98-1185, Generalitat de Catalunya 2000SGR-00028 and EAGER, European Union contract HPRN-CT-2000-00099.

AMS subject classification: 14B07, 32S30.

a much subtler subject, as shown by the fact that the types do not form a stratification of $\mathbb{C}[[x, y]]$ (see [6, 13, 18] or example 4 below); this suggests that a complete understanding of analytic adjacencies can only be achieved by considering analytic moduli of singularities, rather than equisingularity classes alone. Non-linear deformations also allow us to consider the possibility of adjacencies from germs to multigerms; our method works equally well in the more general setting (one just needs to consider Enriques diagrams with several roots, as in [19]). However, we shall skip this in the exposition in order to keep notations simpler.

We close the paper giving a sufficient condition for adjacency of ordinary singularities to arbitrary singularities (Corollary 3.5). This condition is "asymptotically proper", by which we mean that it can be equivalently stated as an inequality $\boldsymbol{\mu}_0 \geq c\boldsymbol{\mu}-$terms of lower order, where $\boldsymbol{\mu}_0$ is the Milnor number of the ordinary singularity, $\boldsymbol{\mu}$ is the Milnor number of the singularity to which it deforms, and c is a universal constant.

We present a purely combinatorial definition of Enriques diagrams that was used by Kleiman and Piene [15] to list all equisingularity types with codimension up to 8, which is needed for the enumeration of 8-nodal curves (see also [11]).

A *tree* is a finite directed graph without loops; it has a single initial vertex or *root* and every other vertex has a unique immediate predecessor. If p is the immediate predecessor of the vertex q, we say that q is a successor of p. If p has no successors, then it is an extremal vertex. An *Enriques diagram* is a tree with a binary relation between vertices, called *proximity*, which satisfies:

1. The root is proximate to no vertex.
2. Every vertex that is not the root is proximate to its immediate predecessor.
3. No vertex is proximate to more than two vertices.
4. If a vertex q is proximate to two vertices, then one of them is the immediate predecessor of q and it is proximate to the other.
5. Given two vertices p, q with q proximate to p, there is at most one vertex proximate to both of them.

The vertices which are proximate to two points are called *satellite*, the other vertices are called *free*. We usually denote the set of vertices of an Enriques diagram $\mathbf{D}$ with the same letter $\mathbf{D}$.

To show graphically the proximity relation, Enriques diagrams are drawn according to the following rules:

1. If q is a free successor of p, then the edge going from p to q is smooth and curved and, if p is not the root, it has at p the same tangent as the edge joining p to its predecessor.
2. The sequence of edges connecting a maximal succession of vertices proximate to the same vertex p are shaped into a line segment, orthogonal to the edge joining p to the first vertex of the sequence.

An *isomorphism* of Enriques diagrams is a bijection i between the sets of vertices of the two diagrams so that q is proximate to p if and only if $i(q)$ is proximate to $i(p)$.

A *subdiagram* of an Enriques diagram $\mathbf{D}$ is a subtree $\mathbf{D}_0 \subset \mathbf{D}$ together with the induced proximity relation, such that the predecessors of every vertex $q \in \mathbf{D}_0$ belong to $\mathbf{D}_0$. An *admissible ordering* for an Enriques diagram $\mathbf{D}$ is a total ordering $\preceq$ for its

https://doi.org/10.4153/CJM-2005-001-1 Published online by Cambridge University Press

set of vertices refining the natural ordering of **D**.

Given an Enriques diagram **D** of n vertices with an admissible ordering $\preceq$, let $p_1, p_2, \dots, p_n$ denote its vertices, numbered according to $\preceq$. The *proximity matrix* of **D** is a square matrix $P = (p_{i,j})$ of order n, with

$$p_{i,j} = \begin{cases} 1 & \text{if } i = j, \\ -1 & \text{if } p_i \text{ is proximate to } p_j, \\ 0 & \text{otherwise.} \end{cases}$$

A *system of multiplicities* for (the vertices of) an Enriques diagram **D** is any map $\nu\colon \mathbf{D} \to \mathbb{Z}$. We will usually write $\nu_p = \nu(p)$. A pair $(\mathbf{D}, \nu)$, where **D** is an Enriques diagram and ν a system of multiplicities for it, is called a *weighted Enriques diagram*, and its *degree* is $\deg(\mathbf{D}, \nu) = \sum_{p\in\mathbf{D}} \nu_p(\nu_p + 1)/2$. A *consistent Enriques diagram* is a weighted Enriques diagram such that, for all $p \in \mathbf{D}$,

$$\nu_p \geq \sum_{q \text{ prox. to } p} \nu_q.$$

Note that if $(\mathbf{D}, \preceq)$ is an Enriques diagram of n vertices with an admissible ordering, then a system of multiplicities for **D** may be identified with a vector

$$\nu = (\nu_{p_1}, \nu_{p_2}, \dots, \nu_{p_n}) \in \mathbb{Z}^n,$$

$i = 1, \dots, n$; we shall use the notation $(\mathbf{D}, \preceq, \nu)$ for a weighted ordered Enriques diagram, where $\nu \in \mathbb{Z}^n$.

To every system of multiplicities ν for a diagram **D** we associate a *system of values*, which is another map $v\colon \mathbf{D} \to \mathbb{Z}$, defined recursively as

$$v_p = \begin{cases} \nu_p & \text{if } p \text{ is the root,} \\ \nu_p + \sum_{p \text{ prox. to } q} v_q & \text{otherwise.} \end{cases} \tag{1}$$

Giving a system of multiplicities for an Enriques diagram is equivalent to giving a system of values, as obviously one recovers ν_p from v_p using (1).

It is well-known that there exists a bijection $\{\text{types}\} \leftrightarrow \{$consistent Enriques diagrams with no extremal free vertices of multiplicity $\nu_p \leq 1\}$. Next we recall the basics on this, referring the reader to [3] for proofs.

Assume that O is a smooth point on a complex surface S, whose local ring's completion is isomorphic to $\mathbb{C}[[x, y]]$, and let $f \in \mathbb{C}[[x, y]]$ be the equation of a (germ of) curve with an isolated singularity at O.

Let K be a finite set of points equal or infinitely near to the smooth point O, such that for each $p \in K$, K contains all points to which p is infinitely near. Such a set is called a *cluster* of points infinitely near to O. A point $p \in K$ is said to be *proximate* to another $q \in K$ if it is infinitely near to q and lies on the *strict* transform of the exceptional divisor of blowing up q. One encodes all the (combinatorial) information on proximities between the points of a cluster in its Enriques diagram.

The *value* of a germ of curve at a point p of a cluster K is the multiplicity at p of the pullback of the germ of curve in the blown up surface containing p. Given a cluster K and a system of values $v\colon K \to \mathbb{Z}$ (associated to the system of multiplicities ν), the set $H_{K,\nu} \subset \mathbb{C}[[x,y]]$ of all equations of the germs of curve which have at every point $p \in K$ *value* at least v_p is a complete ideal.

If $C \subset S$ is a reduced curve going through O, then the set of singular points of C equal or infinitely near to O is a cluster K. The Enriques diagram of K, weighted with the multiplicities of C at the points of K, is a consistent Enriques diagram, with no extremal free vertices of multiplicity $\nu_p \leq 1$. Conversely, if $\mathbf{D}$ is a consistent Enriques diagram with no extremal free vertices of multiplicity $\nu_p \leq 1$, then there are germs of curve at O whose cluster of singular points has Enriques diagram isomorphic to $\mathbf{D}$, and two reduced curves are equisingular at O if and only if their associated Enriques diagrams are isomorphic.

1 Linear Adjacency

Let $I \subset \mathbb{C}[[x,y]]$ be an ideal. According to [3, 7.2.13], general members of I (by the Zariski topology of the coefficients of the series) define equisingular germs.

Lemma 1.1 *$(\mathbf{D}',\nu')$ is linearly adjacent to $(\mathbf{D},\nu)$ if and only if for every $f \in \mathbb{C}[[x,y]]$ defining a reduced germ of curve of type $(\mathbf{D}',\nu')$, there exists an ideal $I \subset \mathbb{C}[[x,y]]$ with $f \in I$ and whose general member defines a reduced germ of type $(\mathbf{D},\nu)$.*

Proof The "if" part of the claim is evident. To see the "only if" part, assume that f defines a reduced germ of type $(\mathbf{D}',\nu')$ that can be deformed to a reduced germ of type $(\mathbf{D},\nu)$ by a linear deformation $f + tg$, $g \in \mathbb{C}[[x,y]]$. This means that general members of the pencil $f + tg$ define reduced germs of type $(\mathbf{D},\nu)$. Hence general members of the ideal $I = (f,g)$ define germs of type $(\mathbf{D},\nu)$ as well ([3, 7.2]). ■

Proposition 1.2 *Let $(\mathbf{D},\mu)$ and $(\mathbf{D}',\mu')$ be weighted Enriques diagrams, with $(\mathbf{D}',\mu')$ consistent. The following are equivalent:*

1. *There are two clusters, K and K', whose Enriques diagrams are $\mathbf{D}$ and $\mathbf{D}'$, respectively, such that $H_{K',\mu'} \subseteq H_{K,\mu}$.*
2. *For every cluster K with Enriques diagram $\mathbf{D}$, there is a cluster K' with Enriques diagram $\mathbf{D}'$ such that $H_{K',\mu'} \subseteq H_{K,\mu}$.*
3. *For every cluster K' with Enriques diagram $\mathbf{D}'$, there is a cluster K with Enriques diagram $\mathbf{D}$ such that $H_{K',\mu'} \subseteq H_{K,\mu}$.*
4. *There exist isomorphic subdiagrams $\mathbf{D}_0 \subset \mathbf{D}$, $\mathbf{D}'_0 \subset \mathbf{D}'$ and an isomorphism*

$$i\colon \mathbf{D}_0 \longrightarrow \mathbf{D}'_0$$

such that the system of multiplicities ν for $\mathbf{D}$ defined as

$$\nu(p) = \begin{cases} \mu'(i(p)) & \text{if } p \in \mathbf{D}_0, \\ 0 & \text{otherwise} \end{cases}$$

https://doi.org/10.4153/CJM-2005-001-1 Published online by Cambridge University Press

has the property that the values v *and* v' *associated with the multiplicities* μ *and* ν *respectively satisfy* $v(p) \leq v'(p)\ \forall p \in \mathbf{D}$.

Proof Clearly both 3 and 2 imply 1. We shall prove that 1 implies 4 and that 4 implies both 2 and 3.

Let us first prove that 1 implies 4. So assume there are two clusters, K and K', whose Enriques diagrams are $\mathbf{D}$ and $\mathbf{D}'$ respectively, such that $H_{K',\mu'} \subseteq H_{K,\mu}$. The points common to K and K' clearly form a cluster, which we call K_0. The vertices in $\mathbf{D}$ and $\mathbf{D}'$ corresponding to points in K_0 form subdiagrams $\mathbf{D}_0$ and $\mathbf{D}'_0$, and the coincidence of points in K_0 determines an isomorphism $i\colon \mathbf{D}_0 \longrightarrow \mathbf{D}'_0$. It only remains to be seen that the values v and v' associated with the multiplicities μ and ν respectively (with ν as in the claim) satisfy $v(p) \leq v'(p)\ \forall p \in \mathbf{D}$. Now choose a germ $f \in H_{K',\mu'}$ having multiplicity exactly μ'_p at each point $p \in K'$ (such an f exists because $(\mathbf{D}', \mu')$ is consistent, see [3, 4.2.7]). This implies that f has value exactly $v'(p)$ at each point $p \in K$. Then $f \in H_{K,\mu}$ because $H_{K',\mu'} \subseteq H_{K,\mu}$, and the claim follows by the definition of $H_{K,\mu}$.

Let us now prove that 4 implies 3. Assume that 4 holds, and let K' be a cluster whose Enriques diagram is $\mathbf{D}'$. We must prove the existence of a cluster K with Enriques diagram $\mathbf{D}$ such that $H_{K',\mu'} \subseteq H_{K,\mu}$. Let K_0 be the cluster formed by the points corresponding to vertices in $\mathbf{D}'_0$. Add to K_0 the points necessary to get a cluster K with Enriques diagram $\mathbf{D}$. Because of the hypothesis on the values v and v' and the characterization of $H_{K,\mu}$ (see for instance [3, 4.5.4]), $H_{K',\mu'} \subseteq H_{K,\mu}$.

In the same way it is proved that 4 implies 2. ∎

If the conditions of Proposition 1.2 are satisfied, we shall write $(\mathbf{D}', \mu') \geq (\mathbf{D}, \mu)$. Now we can prove our main result on linear adjacency. The interest of Proposition 1.2 and Theorem 1.3 lies in the fact that condition 4 of Proposition 1.2 can be checked directly on the Enriques diagrams, using their combinatorial properties, thus giving a practical means to decide whether a type is or is not linearly adjacent to another.

Theorem 1.3 *Let* $(\mathbf{D}, \mu)$, $(\tilde{\mathbf{D}}, \tilde{\mu})$ *be types.* $(\tilde{\mathbf{D}}, \tilde{\mu})$ *is linearly adjacent to* $(\mathbf{D}, \mu)$ *if and only if there exists a weighted consistent Enriques diagram* $(\mathbf{D}', \mu')$*, differing from* $(\tilde{\mathbf{D}}, \tilde{\mu})$ *at most in some free vertices of multiplicity one, satisfying* $(\mathbf{D}', \mu') \geq (\mathbf{D}, \mu)$.

Proof To prove the "if" part, given a reduced germ $f \in \mathbb{C}[[x, y]]$ defining a curve singularity of type $(\tilde{\mathbf{D}}, \tilde{\mu})$ we have to show the existence of an ideal $I \subset \mathbb{C}[[x, y]]$ containing f and whose general member defines a reduced germ of type $(\mathbf{D}, \mu)$, provided that $(\mathbf{D}', \mu') \geq (\mathbf{D}, \mu)$. Let $\tilde{K}$ be the cluster of singular points of f (whose Enriques diagram is $\tilde{\mathbf{D}}$). For each vertex p of $\mathbf{D}'$ not in $\tilde{\mathbf{D}}$, whose predecessor is denoted by q, choose a point on $f = 0$ on the first neighbourhood of the point corresponding to the vertex q. Then $\tilde{K}$ together with all these additional points (which are nonsingular, therefore free of multiplicity 1) form a cluster K' with Enriques diagram $\mathbf{D}'$, with $f \in H_{K',\mu'}$. As $(\mathbf{D}', \mu') \geq (\mathbf{D}, \mu)$, Proposition 1.2 says that there is a cluster K with Enriques diagram $\mathbf{D}$ such that $f \in H_{K',\mu'} \subseteq H_{K,\mu}$. On the other hand [3, 4.2.7] says that the general member of $H_{K,\mu}$ defines a germ of type $(\mathbf{D}, \mu)$, so we are done.

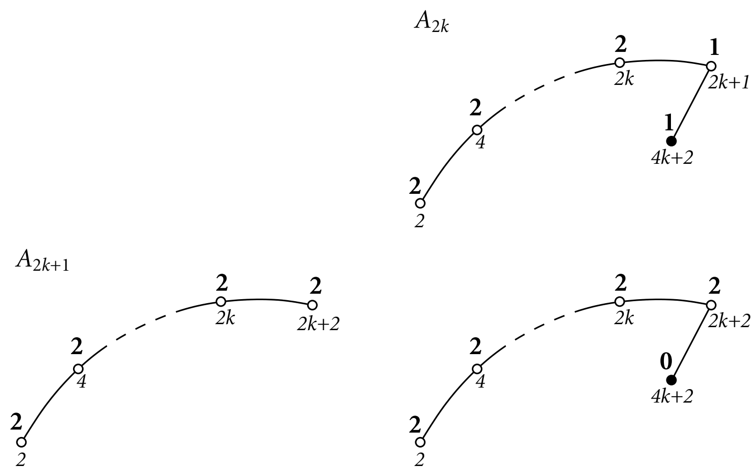

Figure 1: In white, the vertices of the isomorphic subdiagrams, in bold shape, the systems of multiplicities and, in italics, the systems of values. With notations as in Proposition 1.2, the top right diagram is $(\mathbf{D}, \mu)$, the bottom left is $(\mathbf{D}', \mu')$ and the bottom right is $(\mathbf{D}, \nu)$.

Let us now prove the "only if" part. So assume that for every $f \in \mathbb{C}[[x, y]]$ defining a reduced germ of type $(\tilde{\mathbf{D}}, \tilde{\mu})$, there exists an ideal $I \subset \mathbb{C}[[x, y]]$ with $f \in I$ whose general member defines a reduced germ of type $(\mathbf{D}, \mu)$. We first reduce to the case that I has no fixed part. Indeed, for n big enough and $h \in (x, y)^n$, the types of f and $f + h$ coincide (see for instance [3, 7.4.2]), and also the types of g and $g + h$ for g general in I, so we can take $I + (x, y)^n$ instead of I, and this has no fixed part. Then by [3, 7.2.13] the Enriques diagram of the weighted cluster $BP(I)$ of base points of I is $(\mathbf{D}, \mu)$ plus some free vertices of multiplicity one. Let K be the subcluster of $BP(I)$ whose Enriques diagram is $\mathbf{D}$. As $f \in I$, f goes through the weighted cluster $BP(I)$, and therefore $f \in H_{K,\mu}$. By [3, 4.5.4] this means that the *value* of f at each point $p \in K$ is at least ν_p. Add to the cluster $(\tilde{K}, \tilde{\mu})$ of singular points of f all points on $f = 0$ which belong to K, weighted with multiplicity 1 (these are all infinitely near points at which $f = 0$ is smooth). Then the resulting cluster (K', μ') satisfies $H_{K',\mu'} \subseteq H_{K,\mu}$ and by Proposition 1.2 we obtain $(\mathbf{D}', \mu') \geq (\mathbf{D}, \mu)$, where $\mathbf{D}'$ is the Enriques diagram of K'. ■

Example 1 Let A_k, D_k, E_k, $J_{k,p}$ and so on denote the types of germs of curve of Arnold's lists (*cf.* [1]). Then for every $k, d > 0$, A_{k+d} is linearly adjacent to A_k, D_{k+d} is linearly adjacent to D_k, E_{k+d} is linearly adjacent to E_k, $J_{k+d,p+d}$ is linearly adjacent to $J_{k+d,p}$ and to $J_{k,p+d}$ and so on. To see this from Theorem 1.3, without the need of explicit formulae, just take the weighted Enriques diagrams corresponding to each type, and apply Theorem 1.3. For instance, Figure 1 shows the Enriques diagrams corresponding to types A_{2k} and A_{2k+1} with the corresponding isomorphic subdiagrams, the multiplicities and the values involved. All other cases are handled similarly.

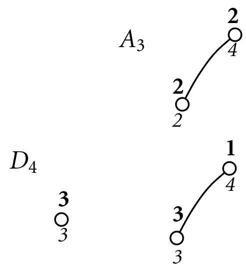

Figure 2: Diagrams corresponding to example 2.

Example 2 The simplest example in which one needs to consider $(\mathbf{D}', \mu') \neq (\tilde{\mathbf{D}}, \tilde{\mu})$ is to prove that a triple point (D_4 in Arnold's notation) is linearly adjacent to the tacnode of type A_3 in Arnold's notation. Indeed, in this case $(\mathbf{D}', \mu')$ is obtained from the triple point by adding a free point with multiplicity 1 to it (see Figure 2).

Remark 1.4 In the proof of the "only if" part of Theorem 1.3 one just needs to assume that there exists a $f \in \mathbb{C}[[x, y]]$ defining a reduced germ of type $(\tilde{\mathbf{D}}, \tilde{\mu})$ and an ideal $I \subset \mathbb{C}[[x, y]]$, with $f \in I$, whose general member defines a reduced germ of type $(\mathbf{D}, \mu)$. It follows therefore that if a germ of curve of type $(\tilde{\mathbf{D}}, \tilde{\mu})$ can be deformed linearly to a germ of type $(\mathbf{D}, \mu)$, then $(\tilde{\mathbf{D}}, \tilde{\mu})$ is linearly adjacent to $(\mathbf{D}, \mu)$. In other words, there are no exceptional linear deformations.

Remark 1.5 In Theorem 1.3 the linear adjacency of the type $(\tilde{\mathbf{D}}, \tilde{\mu})$ to $(\mathbf{D}, \mu)$ is characterized through a third Enriques diagram $(\mathbf{D}', \mu')$ which differs from $(\tilde{\mathbf{D}}, \tilde{\mu})$ at most in some free simple vertices. A coarse *a priori* upper bound to the number of such extra vertices is the number of free vertices of $\mathbf{D}$ minus one. A finer upper bound can be given if we examine how to proceed in practise. The subdiagrams of $\mathbf{D}$ which are isomorphic to some subdiagram of $\tilde{\mathbf{D}}$ are partially ordered by inclusion. Consider a maximal subdiagram $\mathbf{D}_1$ of $\mathbf{D}$ such that there exists $i : \mathbf{D}_1 \cong \tilde{\mathbf{D}}_1$, $\tilde{\mathbf{D}}_1$ subdiagram of $\tilde{\mathbf{D}}$, satisfying the inequalities of values $\nu(p) \leq \tilde{\nu}(i(p))$ for all $p \in \mathbf{D}_1$, where ν and $\tilde{\nu}$ are the values associated with the multiplicities μ and $\tilde{\mu}$ respectively. Then $\mathbf{D}_1$ defines a unique maximal (by inclusion) subdiagram $\mathbf{D}_0$ of $\mathbf{D}$, $\mathbf{D}_1 \subseteq \mathbf{D}_0$ and such that $\mathbf{D}_0 - \mathbf{D}_1$ are free vertices. Check whether $(\mathbf{D}_0, \mu') \geq (\mathbf{D}, \mu)$, where $\mu'(p) = \tilde{\mu}(i(p))$ if $p \in \mathbf{D}_1$, and $\mu'(p) = 1$ if $p \in \mathbf{D}_0 - \mathbf{D}_1$. From Theorem 1.3 we obtain that there is linear adjacency if and only if $(\mathbf{D}_0, \mu') \geq (\mathbf{D}, \mu)$ for some $\mathbf{D}_1$, and in this case the diagrams $\tilde{\mathbf{D}}$ and $\mathbf{D}_0$, joined along $\mathbf{D}_1 \cong \tilde{\mathbf{D}}_1$, give the diagram $\mathbf{D}'$ we are looking for. Therefore an upper bound of the number of vertices in $\mathbf{D}' - \tilde{\mathbf{D}}$ is the maximum over such $\mathbf{D}_1$ of the number of free vertices of $\mathbf{D} - \mathbf{D}_1$ which are not preceded by any satellite vertex of $\mathbf{D} - \mathbf{D}_1$.

2 Non-Linear Adjacency

We have shown in the preceding section a criterion to decide whether a type is or is not linearly adjacent to another. Non-linear adjacencies are a much subtler subject, as shown by Example 4 below, and we cannot give a criterion to decide in all cases. However, we are able to give a necessary condition and a sufficient condition.

We say that a weighted Enriques diagram $(\mathbf{D}, \mu)$ is *tame* whenever it is consistent or the sequence of unloadings that determines, leading to a consistent Enriques diagram $(\mathbf{D}, \mu')$ is tame (see [3, 4.7]); the fact that $(\mathbf{D}, \mu)$ is tame or not depends on the multiplicities and the proximities between vertices of $\mathbf{D}$, *i.e.,* on the combinatorial properties of the weighted cluster. Moreover, $(\mathbf{D}, \mu)$ is tame if and only if for every cluster K with Enriques diagram $\mathbf{D}$ the condition $\dim \mathbb{C}[[x, y]]/H_{K,\mu} = \deg(\mathbf{D}, \mu)$ holds (see [3, 4.7.3]).

For every Enriques diagram $\mathbf{D}$, endowed with an admissible ordering $\preceq$ of the vertices, there is a variety $Cl(\mathbf{D}, \preceq)$ parameterizing all ordered clusters with ordered Enriques diagram $(\mathbf{D}, \preceq)$ (see [19]). In the sequel we shall make use of these spaces and the results on their relative positions in the variety of all clusters obtained in [19]. In particular, we write $(\mathbf{D}, \preceq) \rightsquigarrow (\mathbf{D}', \preceq')$ to mean $Cl(\mathbf{D}', \preceq') \subset \overline{Cl(\mathbf{D}, \preceq)}$.

We begin with a sufficient condition for adjacency.

Proposition 2.1 *Let $(\mathbf{D}, \mu)$, $(\tilde{\mathbf{D}}, \tilde{\mu})$ be types, and assume that there exist a weighted consistent Enriques diagram $(\mathbf{D}', \mu')$, differing from $(\tilde{\mathbf{D}}, \tilde{\mu})$ at most in some free vertices of multiplicity one, an Enriques diagram $\mathbf{D}_0$ with the same number of vertices as $\mathbf{D}$, and admissible orderings $\preceq$ and $\preceq_0$ of $\mathbf{D}$ and $\mathbf{D}_0$ respectively satisfying*

1. $(\mathbf{D}, \preceq) \rightsquigarrow (\mathbf{D}_0, \preceq_0)$,
2. $(\mathbf{D}_0, \preceq_0, \boldsymbol{\mu})$ *is tame, and*
3. $(\mathbf{D}', \mu') \geq (\mathbf{D}_0, \preceq_0, \boldsymbol{\mu})$,

where μ is the vector of multiplicities of $(\mathbf{D}, \mu)$ for the ordering $\preceq$ of $\mathbf{D}$. Then the type $(\tilde{\mathbf{D}}, \tilde{\mu})$ is adjacent to the type $(\mathbf{D}, \mu)$.

Proof Let C be a germ of curve of type $(\tilde{\mathbf{D}}, \tilde{\mu})$; we have to see that there is a family of germs containing C whose general member is of type $(\mathbf{D}, \mu)$. Let $f \in \mathbb{C}[[x, y]]$ be an equation of C, and let $\tilde{K}$ be the cluster of singular points of C. For each vertex p of $\mathbf{D}'$ not in $\tilde{\mathbf{D}}$, whose predecessor is denoted by q, choose a point on C on the first neighbourhood of the point corresponding to the vertex q. Then $\tilde{K}$, together with all these additional points (which are nonsingular, therefore free of multiplicity 1) form a cluster K' with Enriques diagram $\mathbf{D}'$, with $f \in H_{K',\mu'}$. As $(\mathbf{D}', \mu') \geq (\mathbf{D}_0, \preceq_0, \mu)$, Proposition 1.2 says that there is a cluster K_0 with Enriques diagram $\mathbf{D}_0$ such that $f \in H_{K',\mu'} \subseteq H_{K_0,\preceq_0,\mu}$. The hypothesis $\mathbf{D} \rightsquigarrow \mathbf{D}_0$ says that we can deform K_0 to a family K_t of clusters, $t \in \Delta \subset \mathbb{C}$, where Δ is a suitably small disc, such that for $t \neq 0$ the cluster K_t has Enriques diagram $\mathbf{D}$. Now the $H_{K_t,\mu}$ form a family of linear subspaces of $\mathbb{C}[[x, y]]$ with constant codimension (because $(\mathbf{D}_0, \preceq_0, \boldsymbol{\mu})$ is tame and $(\mathbf{D}, \mu)$ is consistent) and therefore determine a family of germs which contain f and whose general member has type $(\mathbf{D}, \mu)$, as wanted. ∎

If needed, it is not hard to obtain from the family described in the proof of Proposition 2.1 a one-dimensional family C_t with the desired properties and $C_0 = C$, even explicitly. For the particular case when $\mathbf{D}$ is unibranched, the reader may find details on the family $H_{K_t,\mu}$, with explicit equations, in [20, 3].

Note that, as in the linear case, the interest of Proposition 2.1 lies in the fact that the conditions can be checked directly on the Enriques diagrams using their combinatorial properties. This is always true for the conditions that $(\mathbf{D}_0, \preceq_0, \boldsymbol{\mu})$ is tame and $(\tilde{\mathbf{D}}, \tilde{\mu}) \geq (\mathbf{D}_0, \preceq_0, \boldsymbol{\mu})$. The condition $(\mathbf{D}, \preceq) \rightsquigarrow (\mathbf{D}_0, \preceq_0)$ is more difficult to handle, but in some cases (such as when $\mathbf{D}$ has no satellite points or when it is unibranched) it can also be determined from the combinatorial properties of $\mathbf{D}$ and $\mathbf{D}_0$ (see [19]) using proximity matrices.

It is not to be expected that Proposition 2.1 gives all existing adjacencies; on the other hand, it simplifies a great deal the search for such adjacencies without the need for explicit equations, and this for singularities of arbitarily high multiplicity. Confining ourselves to the easier cases, it may be interesting to note that by checking E. Brieskorn's lists [2], one sees that Proposition 2.1 implies all adjacencies between 1-modular singularities of curves except one (namely, that $S_{2,5,6} = W_{13}$ is adjacent to $T_{2,3,8} = J_{2,2}$).

Next we prove a necessary condition for adjacency.

Proposition 2.2 *Let $(\mathbf{D}, \mu)$, $(\tilde{\mathbf{D}}, \tilde{\mu})$ be types such that there exists a family of curves C_t, $t \in \Delta \subset \mathbb{C}$, whose general members are of type $(\mathbf{D}, \mu)$ and with C_0 of type $(\tilde{\mathbf{D}}, \tilde{\mu})$, and let $\preceq$ be any admissible ordering of $\mathbf{D}$. Then there exist a weighted consistent Enriques diagram $(\mathbf{D}', \mu')$ differing from $(\tilde{\mathbf{D}}, \tilde{\mu})$ at most in some free vertices of multiplicity one, an Enriques diagram $\mathbf{D}_0$ with the same number of vertices as $\mathbf{D}$ and an admissible ordering $\preceq_0$ of $\mathbf{D}_0$, such that*

1. $(\mathbf{D}', \mu') \geq (\mathbf{D}_0, \preceq_0, \boldsymbol{\mu})$*, and*
2. *the matrix $P_0^{-1}P$, where P and P_0 are the proximity matrices of $(\mathbf{D}, \preceq)$ and $(\mathbf{D}_0, \preceq_0)$ respectively, has no negative entries.*

Proof Let $S_t \longrightarrow \operatorname{Spec} \mathbb{C}[[x, y]]$ be a desingularization of the family C_t, $t \neq 0$ ([29], see also [26]). Because of the universal property of the space X_{n-1} of all ordered clusters of n points (see [14, 19]) this induces a family of clusters K_t (parameterized by a possibly smaller punctured disc $\Delta' \setminus \{0\}$) which can be uniquely extended taking $K_0 = \lim_{t\to 0} K_t$ (X_{n-1} is projective and therefore complete). All clusters of this family except maybe K_0 have type $\mathbf{D}$, and for all $t \in \Delta'$, it is easy to see that C_t goes through the weighted cluster (K_t, μ). Taking $\mathbf{D}_0$ to be the Enriques diagram of K_0, both claims follow (see [19] for the second claim). ■

Obviously this implies:

Corollary 2.3 *Let $(\mathbf{D}, \mu)$, $(\tilde{\mathbf{D}}, \tilde{\mu})$ be types such that $(\tilde{\mathbf{D}}, \tilde{\mu})$ is adjacent to $(\mathbf{D}, \mu)$, and let $\preceq$ be any admissible ordering of $\mathbf{D}$. Then there exist a weighted consistent Enriques diagram $(\mathbf{D}', \mu')$, differing from $(\tilde{\mathbf{D}}, \tilde{\mu})$ at most in some free vertices of multiplicity one, an Enriques diagram $\mathbf{D}_0$ with the same number of vertices as $\mathbf{D}$ and an admissible ordering $\preceq_0$ of $\mathbf{D}_0$, such that*

1. $(\mathbf{D}', \mu') \geq (\mathbf{D}_0, \preceq_0, \mu)$*, and*
2. *the matrix $P_0^{-1}P$, where P and P_0 are the proximity matrices of $(\mathbf{D}, \preceq)$ and $(\mathbf{D}_0, \preceq_0)$ respectively, has no negative entries.*

https://doi.org/10.4153/CJM-2005-001-1 Published online by Cambridge University Press

Again, the interest of Corollary 2.3 lies in the fact that the conditions can be checked directly on the Enriques diagrams, using their combinatorial properties. Thus we prove, for example, that some types (including all irreducible curve singularities with a single characteristic exponent m/n with $n < m < 2n$) allow only linear adjacencies (which in turn implies that they allow no exceptional deformations, after Remark 1.4):

Corollary 2.4 *Let* $(\mathbf{D}, \mu)$, $(\tilde{\mathbf{D}}, \tilde{\mu})$ *be types such that* $(\tilde{\mathbf{D}}, \tilde{\mu})$ *is adjacent to* $(\mathbf{D}, \mu)$*, and suppose that* $\mathbf{D}$ *has at most two free vertices. Then* $(\tilde{\mathbf{D}}, \tilde{\mu})$ *is linearly adjacent to* $(\mathbf{D}, \mu)$.

Proof If p is a satellite vertex of $\mathbf{D}$, then there are at least two vertices in $\mathbf{D}$ preceding it (namely, the two vertices to which p is proximate). Therefore, if $\mathbf{D}$ has only one free vertex, then it consists of the root alone, and if it has two free vertices, they must be the root and another vertex which is the unique one which has the root as immediate predecessor. Under these conditions, it is not hard to see that, given any admissible ordering $\preceq$ on $\mathbf{D}$, if $(\mathbf{D}_0, \preceq_0)$ is an ordered Enriques diagram such that the matrix $P_0^{-1}P$ has no negative entries where P and P_0 are the proximity matrices of $(\mathbf{D}, \preceq)$ and $(\mathbf{D}_0, \preceq_0)$ respectively, then $(\mathbf{D}, \preceq) = (\mathbf{D}_0, \preceq_0)$. Now the claim follows from Corollary 2.3 and Theorem 1.3. ■

Example 3 Let $(\tilde{\mathbf{D}}, \tilde{\mu})$ be a single point of multiplicity 7, *i.e.,* the type of an ordinary singularity of multiplicity 7, and let $(\mathbf{D}, \mu)$ be the Enriques diagram of the type of a singularity with two tangent branches, each one with a single characteristic exponent $\frac{3}{2}$ and $\frac{5}{3}$, respectively. Applying Theorem 1.3, $(\tilde{\mathbf{D}}, \tilde{\mu})$ is not linearly adjacent to $(\mathbf{D}, \mu)$ and hence neither is it adjacent by Corollary 2.4. However their adjacency is not discarded by their spectra.

It is well known (see [6, 13, 18]) that the equisingularity classes do not form a stratification of $\mathbb{C}[[x, y]]$ (*i.e.,* there exist types $(\mathbf{D}, \mu)$, $(\tilde{\mathbf{D}}, \tilde{\mu})$ and curves of type $(\tilde{\mathbf{D}}, \tilde{\mu})$ that can be deformed to curves of type $(\mathbf{D}, \mu)$ without $(\tilde{\mathbf{D}}, \tilde{\mu})$ being adjacent to $(\mathbf{D}, \mu)$, which are called *exceptional deformations*). This fact can be proved using the result, proved in [19], that the varieties $Cl(\mathbf{D})$ do not form a stratification of the space of all clusters (*i.e.,* there exist $\mathbf{D}$, $\mathbf{D}'$ with $Cl(\mathbf{D}') \cap \overline{Cl(\mathbf{D})} \neq \varnothing$ and $\mathbf{D} \not\to \mathbf{D}'$). In the example explicitly stated in [19, 3], $\mathbf{D}$ has two roots; this would correspond to adjacencies from germs to 2-germs. Here we present another example which deals with clusters with only one root, and provides infinitely many new exceptional deformations of germs of plane curve singularities:

Example 4 Let $(\mathbf{D}, \mu)$, $(\mathbf{D}', \mu')$, $(\tilde{\mathbf{D}}, \tilde{\mu})$ be the Enriques diagrams of Figure 3 (all systems of multiplicities μ and μ' such that $(\mathbf{D}, \mu)$ is consistent and $(\mathbf{D}', \mu')$ is tame work). It is not hard to prove, using the method of [19, 3], that there exist clusters K and K' with Enriques diagram $\mathbf{D}'$ such that K' can be deformed to clusters with Enriques diagram $\mathbf{D}$ and K can not. If C is a curve of type $(\tilde{\mathbf{D}}, \tilde{\mu})$, and (K', μ') is the cluster (of type $(\mathbf{D}', \mu')$) formed by the singular points and the two first nonsingular points on each branch of C, then it can be deformed to curves of type $(\mathbf{D}, \mu)$, using the method of the proof of Proposition 2.1. On the other hand, $(\tilde{\mathbf{D}}, \tilde{\mu})$ is not

https://doi.org/10.4153/CJM-2005-001-1 Published online by Cambridge University Press

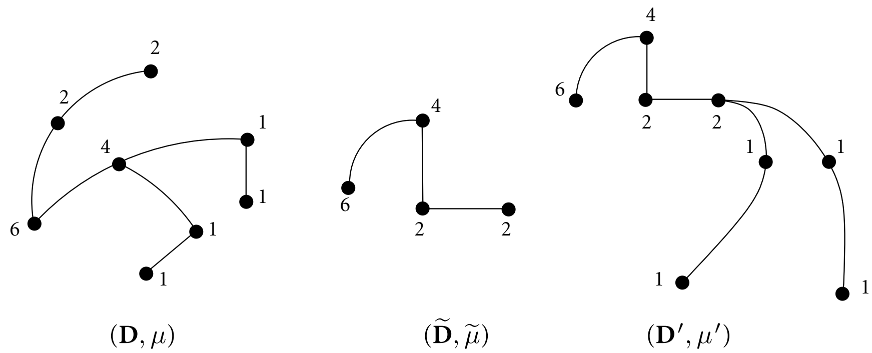

Figure 3: Enriques diagrams corresponding to the types of example 4.

adjacent to $(\mathbf{D}, \mu)$; this can be proved using that K cannot be deformed to clusters with Enriques diagram $\mathbf{D}$ or, more easily, by observing that both types have the same codimension.

3 Non-Linear Adjacency via Hilbert Schemes

Non-linear adjacency can be approached using Hilbert schemes instead of varieties of clusters. In fact, it is possible to give a characterization of all adjacencies in terms of the relative positions of some subschemes of the Hilbert scheme of points on a surface. However, these relative positions are in general not known, so the answer obtained using Hilbert schemes is theoretical and not easy to put into practice, in contrast with the criteria given above, which are combinatorial and can be effectively applied.

As customary, $\operatorname{Hilb}^n R$ will denote the Hilbert scheme parameterizing ideals of colength n in $R = \mathbb{C}[[x, y]]$. We consider also the "nested Hilbert scheme" $Z_{n_1,n_2}R \subset (\operatorname{Hilb}^{n_1} R) \times (\operatorname{Hilb}^{n_2} R)$ studied by J. Cheah, which parameterizes pairs of ideals (I_1, I_2) with $I_1 \supset I_2$ (see [4, 5]). For every type $(\mathbf{D}, \mu)$, let $\operatorname{Hilb}^{\mu}_{\mathbf{D}} R$ be the subset of $\operatorname{Hilb}^n R$ parameterizing the ideals $H_{K,\mu}$ where K are clusters with Enriques diagram $\mathbf{D}$, and $n = \deg(\mathbf{D}, \mu)$. It is known that $\operatorname{Hilb}^{\mu}_{\mathbf{D}} R$ is a locally closed irreducible subscheme of $\operatorname{Hilb}^n R$ (see [15, 17, 16], for example); $\overline{\operatorname{Hilb}^{\mu}_{\mathbf{D}} R}$ will denote its closure in $\operatorname{Hilb}^n R$.

Theorem 3.1 *Let $(\mathbf{D}, \mu)$, $(\tilde{\mathbf{D}}, \tilde{\mu})$ be types. $(\tilde{\mathbf{D}}, \tilde{\mu})$ is adjacent to $(\mathbf{D}, \mu)$ if and only if there exists a weighted consistent Enriques diagram $(\mathbf{D}', \mu')$, differing from $(\tilde{\mathbf{D}}, \tilde{\mu})$ at most in some free vertices of multiplicity one, satisfying $\operatorname{Hilb}^{\mu'}_{\mathbf{D}'} R \subset \pi' \pi^{-1}(\overline{\operatorname{Hilb}^{\mu}_{\mathbf{D}} R})$, where π and π' are the projections of $Z_{n,n'}R$ onto $\operatorname{Hilb}^n R$ and $\operatorname{Hilb}^{n'} R$ respectively, and $n = \deg(\mathbf{D}, \mu)$, $n' = \deg(\mathbf{D}', \mu')$.*

To prove Theorem 3.1 we use the following lemma:

Lemma 3.2 *Let $(\mathbf{D}, \mu)$, $(\mathbf{D}', \mu')$ be types such that $(\mathbf{D}', \mu')$ is adjacent to $(\mathbf{D}, \mu)$.*

https://doi.org/10.4153/CJM-2005-001-1 Published online by Cambridge University Press

Then for every $f \in \mathbb{C}[[x,y]]$ defining a reduced germ of curve of type $(\mathbf{D}',\mu')$, there exists an ideal $I \in \overline{\mathrm{Hilb}^{\mu}_{\mathbf{D}} R}$ with $f \in I$.

Proof Let $f \in \mathbb{C}[[x,y]]$ be a germ of equation of a curve of type $(\mathbf{D}',\mu')$. Because of the adjacency, there exists a family of germs $f_t, t \in \Delta \subset \mathbb{C}$, whose general members are of type $(\mathbf{D},\mu)$ and with $f_0 = f$. Let $S_t \longrightarrow \operatorname{Spec} \mathbb{C}[[x,y]]$ be a desingularization of the family f_t, $t \neq 0$ ([29, 26]). Because of the universal property of the space of all clusters (see [14, 19]) this induces a family of clusters K_t (parameterized by a possibly smaller punctured disc $\Delta' \setminus \{0\}$). Now the $I_t = H_{K_t,\mu}$ form a (complex) one-dimensional family inside $\mathrm{Hilb}^{\mu}_{\mathbf{D}} S$ which can be uniquely extended with $I_0 = \lim_{t\to 0} I_t$. It is easy to see that, for all $t \in \Delta'$, $f_t \in I_t$, so the claim follows for $I = I_0$. ∎

Proof of Theorem 3.1 The "if" part of the claim is proved in a similar way to the proof of Proposition 2.1; we leave the details for the reader to check. For the "only if" part of the claim, we show that assuming $(\tilde{\mathbf{D}},\tilde{\mu})$ is adjacent to $(\mathbf{D},\mu)$ and that there exists no consistent weighted Enriques diagram $(\mathbf{D}',\mu')$ differing from $(\tilde{\mathbf{D}},\tilde{\mu})$ only in free vertices of multiplicity one, in the conditions of the claim, leads to contradiction.

The second assumption means that for every consistent Enriques diagram $(\mathbf{D}',\mu')$ differing from $(\tilde{\mathbf{D}},\tilde{\mu})$ only in free vertices of multiplicity one, there are clusters K' with $H_{K',\mu'} \in \mathrm{Hilb}^{\mu'}_{\mathbf{D}'} R \setminus \pi'\pi^{-1}(\overline{\mathrm{Hilb}^{\mu}_{\mathbf{D}} R})$. Consider the sequence of weighted Enriques diagrams defined as follows. $(\mathbf{D}_1,\mu_1)$ is obtained from $(\tilde{\mathbf{D}},\tilde{\mu})$ by adding

$$\tilde{\mu}_p - \sum_{q \text{ prox. to } p} \tilde{\mu}_q$$

free successors of multiplicity 1 to each $p \in \tilde{\mathbf{D}}$, and for $k > 1$, $(\mathbf{D}_k,\mu_k)$ is obtained from $(\mathbf{D}_{k-1},\mu_{k-1})$ by adding a free successor of multiplicity 1 to each extremal vertex (which will be free of multiplicity 1). Obviously $(\mathbf{D}_{k-1},\mu_{k-1})$ is a subdiagram of $(\mathbf{D}_k,\mu_k)$ for all $k > 1$, and it is not hard to see that the map $F_k\colon \mathrm{Hilb}^{\mu_k}_{\mathbf{D}_k} R \longrightarrow \mathrm{Hilb}^{\mu_{k-1}}_{\mathbf{D}_{k-1}} R$ defined by sending H_{K,μ_k} to $H_{\check{K},\mu_{k-1}}$, where $\check{K}$ is the subcluster of K with diagram $\mathbf{D}_{k-1}$, satisfies

$$F_k\big(\mathrm{Hilb}^{\mu_k}_{\mathbf{D}_k} R \setminus \pi_k\pi^{-1}\big(\overline{\mathrm{Hilb}^{\mu}_{\mathbf{D}} R}\big)\big) = \mathrm{Hilb}^{\mu_{k-1}}_{\mathbf{D}_{k-1}} R \setminus \pi_{k-1}\pi^{-1}\big(\overline{\mathrm{Hilb}^{\mu}_{\mathbf{D}} R}\big).$$

Therefore we can construct a sequence of clusters $K_1, K_2, \dots$ such that $H_{K_k,\mu_k} \in \mathrm{Hilb}^{\mu_k}_{\mathbf{D}_k} R \setminus \pi_k\pi^{-1}(\overline{\mathrm{Hilb}^{\mu}_{\mathbf{D}} R})$ and each K_k is obtained from K_{k-1} by adding in the first neighbourhood of each extremal point a free point of multiplicity one. But then there exists a reduced germ f of type $(\tilde{\mathbf{D}},\tilde{\mu})$ belonging to all H_{K_k,μ_k} (see [3, 5.7]]). Now by Lemma 3.2, there exists an ideal $I \in \overline{\mathrm{Hilb}^{\mu}_{\mathbf{D}} R}$ with $f \in I$; as $\dim_{\mathbb{C}} \mathbb{C}[[x,y]]/I = n$, we must have $I \supset (x,y)^n$ also. On the other hand, applying [3, 5.7.1] and [3, 7.2.16], for k big enough, we infer that $H_{K_k,\mu_k} \subset (f) + (x,y)^n$, which implies $H_{K_k,\mu_k} \subset I$, a contradiction. ∎

Remark 3.3 Linear adjacencies may also be dealt with using Hilbert schemes; indeed, with notations as above, $(\tilde{\mathbf{D}},\tilde{\mu})$ is linearly adjacent to $(\mathbf{D},\mu)$ if and only if there

https://doi.org/10.4153/CJM-2005-001-1 Published online by Cambridge University Press

exists a weighted consistent Enriques diagram $(\mathbf{D}', \mu')$, differing from $(\tilde{\mathbf{D}}, \tilde{\mu})$ at most in some free vertices of multiplicity one, satisfying $\mathrm{Hilb}^{\mu'}_{\mathbf{D}'} R \subset \pi' \pi^{-1} \mathrm{Hilb}^{\mu}_{\mathbf{D}} R$. Again this criterion is hard to apply, in contrast to the purely combinatorial we gave before. We skip the proof, which adds no new ideas to what was already done.

There exist a few cases in which the criterion of Theorem 3.1 can be effectively applied. For types $(\mathbf{D}, \mu)$ where $\mathbf{D}$ has three vertices or less, the closure of $\mathrm{Hilb}^{\mu}_{\mathbf{D}} R$ is known, due to the works [9, 10] of Évain; so in this case the Hilbert scheme method does give a characterization of adjacencies. Another particular situation which we would like to mention is an example due to H. Russell (see [21]) in which the study of the Hilbert scheme provides a previously known example of exceptional deformation, (*cf.* [6]).

Another case in which the criterion of Theorem 3.1 can be applied to obtain nontrivial results is the study of adjacencies of ordinary singularities. Indeed, these are related to linear systems of singular curves in the affine (or projective) plane and we can use what is known on linear systems to obtain a significant sufficient condition. Recall that the regularity of a scheme Z defined by an $\mathfrak{m}$-primary ideal $I \subset \mathbb{C}[[x, y]]$ is the minimal d such that the natural map $\mathbb{C}[x, y]_d \longrightarrow \mathbb{C}[[x, y]]/I_Z$ is onto, where $\mathbb{C}[x, y]_d$ denotes the vector space of all polynomials of degree $\leq d$ (corresponding to the linear system of all degree d curves). By semicontinuity, we also know that the regularity of the schemes $Z \in \mathrm{Hilb}^{\mu}_{\mathbf{D}}(\mathbb{P}^2)$ achieves its minimum in a dense open set. Then we have:

Proposition 3.4 *Let* $(\mathbf{D}, \mu)$ *be a type and let* d *be the minimal regularity of a scheme* $Z \in \mathrm{Hilb}^{\mu}_{\mathbf{D}}$. *Then every singularity of multiplicity at least* $d + 1$ *is adjacent to* $(\mathbf{D}, \mu)$.

Proof Clearly it is enough to see that the ordinary point of multiplicity $d + 1$ is adjacent to $(\mathbf{D}, \mu)$. By Theorem 3.1 then, it will be enough to show that in $\overline{\mathrm{Hilb}^{\mu}_{\mathbf{D}}}$ there are schemes contained in a fat point of multiplicity $d + 1$. Now, [8, Proposition 4] shows how to construct a scheme $C \in \overline{\mathrm{Hilb}^{\mu}_{\mathbf{D}}}$, defined by an homogeneous ideal I_C, with the same regularity as Z. I_C being homogeneous, its regularity is the minimal d such that $I_C \supset \mathfrak{m}^{d+1}$, and the claim follows. ■

The abundant literature on linear systems allows us to obtain sufficient conditions for adjacency which can be seen to be "asymptotically proper" (see [12, 22]) when expressed in terms of the Milnor number. For instance, we get:

Corollary 3.5 *An ordinary singularity of multiplicity* $m \geq 5$ *is adjacent to*

1. *every* A_k*-singularity with* $\binom{m+1}{2} \geq \left\lfloor \frac{3k}{2} \right\rfloor + 2$,
2. *every* D_k*-singularity with* $\binom{m+1}{2} \geq \left\lfloor \frac{3k+1}{2} \right\rfloor$,
3. *every singularity with Milnor number* μ *such that* $m \geq 3\sqrt{\mu} - 1$.

Proof Use the regularity results of [20, 22] together with Proposition 3.4. ■

References

[1] V. I. Arnold, *Local normal forms of functions*, Invent. Math. **35** (1976), 87–109.
[2] E. Brieskorn, *Die hierarchie der 1-modularen singularitäten. (German. English summary)*, Manuscripta Math. **27** (1979), 183–219.
[3] E. Casas-Alvero, *Singularities of plane curves*, London Math. Soc. Lecture Notes Series, no. 276, Cambridge University Press, 2000.
[4] J. Cheah, *The cohomology of smooth nested Hilbert schemes of points*, Ph.D. thesis, Chicago, 1994.
[5] ———, *The virtual hodge polynomials of nested Hilbert schemes and related varieties*, Math. Z. **227** (1998), no. 3, 479–504.
[6] A. du Plessis and C. T. C. Wall, *The geometry of topological stability*, London Mathematical Society Monographs. New Series, vol. 9, Oxford Science Publications. The Clarendon Press, Oxford University Press, New York, 1995.
[7] F. Enriques and O. Chisini, *Lezioni sulla teoria geometrica delle equazioni e delle funzioni algebriche*, N. Zanichelli, Bologna, 1915.
[8] L. Évain, *Calculs des dimensions de systèmes linéaires de courbes planes par collisions de gros points*, C. R. Acad. Sci. Paris **325, I** (1997), 1305–1308.
[9] ———, *Collisions de trois gros points sur une surface algébrique*, Ph.D. thesis, Université de Nice, 1997.
[10] ———, *Compactification of configuration spaces via Hilbert schemes*, preprint (2001), http://xxx.lanl.gov/abs/math/0107041.
[11] A. Granja and T. Sánchez-Giralda, *Enriques graphs of plane curves.*, Comm. Algebra **20** (1992), no. 2, 527–562.
[12] G.M. Greuel, C. Lossen, and E. Shustin, *Plane curves of minimal degree with prescribed singularities*, Invent. Math. **133** (1998), 539–580.
[13] S. M. Guseĭn-Zade and N. N. Nekhoroshev, *Contiguities of singularities A_k to points of the singularity stratum* $\mu =$ constant, Functional Anal. Appl. **17** (1983), no. 4, 312–313.
[14] B. Harbourne, *Complete linear systems on rational surfaces*, Trans. A.M.S. **289** (1985), 213–226.
[15] S. Kleiman and R. Piene, *Enumerating singular curves on surfaces*, Proc. Conference on Algebraic Geometry: Hirzebruch 70 (Warsaw 1998), vol. 241, A.M.S. Contemp. Math., 1999, pp. 209–238.
[16] C. Lossen, *The geometry of equisingular and equianalytic families of curves on a surface*, Ph.D. thesis, Universität Kaiserslautern, 1998.
[17] A. Nobile and O. Villamayor, *Equisingular stratifications associated to families of planar ideals*, J. Alg. **193** (1997), 239–259.
[18] F. Pham, *Remarque sur l'équisingularité universelle*, Preprint, Université de Nice (1970).
[19] J. Roé, *Varieties of clusters and Enriques diagrams*, Math. Proc. Cambridge Philos. Soc., to appear, http://xxx.lanl.gov/abs/math/0108023.
[20] ———, *On the conditions imposed by tacnodes and cusps*, Trans. A.M.S. **353** (2001), no. 12, 4925–4948.
[21] H. Russell, *Counting singular plane curves via Hilbert schemes*, Preprint (2000), http://xxx.lanl.gov/abs/math/0011214.
[22] E. Shustin, *Analytic order of singular and critical points*, preprint (2002), http://arXiv.org/abs/math.AG/0209043.
[23] D. Siersma, *Periodicities in Arnold's lists of singularities*, Real and Complex Singularities, Oslo 1976 (P. Holm, ed.), Sijthoff & Noordhoof, 1977, pp. 497–524.
[24] J. H. M. Steenbrink, *Semicontinuity of the singularity spectrum*, Invent. Math. **79** (1985), no. 3, 557–565.
[25] ———, *The spectrum of hypersurface singularities*, Actes du Colloque de Theorie de Hodge (Luminy, 1987), Astérisque, vol. 179–180, 1989, pp. 163–184.
[26] J. Wahl, *Equisingular deformations of plane algebroid curves*, Trans. A.M.S. **193** (1974), 143–170.
[27] C. T. C. Wall, *Notes on the classification of singularities*, Proc. London Math. Soc. (3) **48** (1984), no. 3, 461–513.
[28] O. Zariski, *Studies in equisingularity I*, Amer. J. Math. **87** (1965), 507–536.
[29] ———, *Studies in equisingularity II*, Amer. J. Math. **87** (1965), 972–1006.

Departament de Matemàtica Aplicada I
Universitat Politècnica de Catalunya
Av. Diagonal 647
08028-Barcelona, Spain
E-mail: maria.alberich@upc.es

Departament de Matemàtiques
Universitat Autonòma de Barcelona
Edifici C, 08193-Bellaterra
Barcelona, Spain
E-mail: jroe@mat.uab.es