ON SEMIGROUPS OF TRANSFORMATIONS ACTING TRANSITIVELY ON A SET

E.J. Tully, Jr.*

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We call a semigroup S transitive if S is isomorphic to a semigroup T of transformations of some set M into itself, where T acts on M transitively, that is in such a manner that for all $x, y \in M$ we have $x\pi = y$ for some transformation $\pi \in T$. In [4] the author showed that S is transitive if and only if there exists a right congruence σ (i.e., an equivalence relation for which $a \sigma b$ always implies $ac \sigma bc$ for all $c \in S$) on S, satisfying:

(1) There exists a left identity modulo σ, that is an element e such that $ea \sigma a$ for all $a \in S$.

(2) Each σ-class meets each right ideal, or, equivalently, for all $a, b \in S$ we have $ac \sigma b$ for some $c \in S$.

(3) The relation σ contains (i.e., is less fine than) no left congruence except the identity relation (in which each class consists of a single element).

This was used to obtain a much simpler condition [4, Theorem 3.4, p.538] for the special case where S contains a minimal right ideal.

The purpose of the present note is to obtain a somewhat complicated necessary and sufficient condition, which does not involve right congruences, for the transitivity of an arbitrary semigroup. This condition asserts that there exists a subset A of S which meets each right ideal, such that every pair a, b of

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distinct elements of S has a left multiple consisting of a pair ca, cb which cannot be joined to each other by successive left multiplications and divisions by elements of A. This is then used to obtain a simpler sufficient condition for transitivity.

LEMMA. Suppose σ is a right congruence on a semigroup S, and (1) is satisfied. Then (3) is equivalent to

(3') For every pair a, b of distinct elements of S, there exists $x \in S$ such that xa and xb are in different σ-classes.

Proof. First suppose (3') holds, and (3) is false. Then there is a left congruence ρ contained in σ, with $a \rho b$ for some $a \neq b$. Let x be as in (3'). Then $xa \rho xb$, and hence $xa \sigma xb$. This contradicts (3'). Conversely, suppose (3) holds, and (3') is false. Then for some $a \neq b$ we have $xa \sigma xb$ for all x. Define ρ to be the smallest left congruence for which $a \rho b$. Explicitly, ρ is given by: $c \rho d$ if and only if either $c = d$ or there exist $x_0, \ldots, x_n \in S$ with $x_0 = c$, $x_n = d$, and, for each i, either $\{x_{i-1}, x_i\} = \{a, b\}$ or $\{x_{i-1}, x_i\} = \{y_i a, y_i b\}$ for some $y_i \in S$. Now ρ is contained in σ. For $y_i a \sigma y_i b$ by the falsity of (3'), and, using (1), we can obtain $a \sigma ea \sigma eb \sigma b$. But ρ is not the identity relation. This contradicts (3).

THEOREM. A semigroup S is transitive if and only if there exists a subset A of S which meets each right ideal, and satisfies

(4) If $a, b \in S$, and for each $c \in S$ there exist $x_0, \ldots, x_n \in S$ with $x_0 = ca$, $x_n = cb$, and, for each i, either $x_{i-1} = y_i x_i$ or $x_i = y_i x_{i-1}$ for some $y_i \in A$, then $a = b$.

Proof. First suppose that S is transitive. Let σ be a right congruence satisfying (1), (2) and (3'). Choose A to be a σ-class containing a left identity modulo σ. It is easy to see that every element of A is a left identity modulo σ. By (2), A meets each right ideal. We now prove (4). Suppose each left multiple ca, cb of a pair a, b could be joined by a chain as in (4). Then, for all i, $x_i \sigma x_{i-1}$, since y_i is a left identity modulo σ. Hence $ca \sigma cb$ for all c. By (3') $a = b$.

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Conversely, suppose such a subset A exists. Define a right congruence σ by: $x \sigma y$ if and only if there exist $x_0, \ldots, x_n \in S$ with $x_0 = x$, $x_n = y$, and, for each i, either $x_{i-1} = y_i x_i$ or $x_i = y_i x_{i-1}$ for some $y_i \in A$. Any element of A is clearly a left identity modulo σ. Now let $a, b \in S$ be given. Since A meets each right ideal, we have $ay \in A$ for some $y \in S$. Hence $ayb \, \sigma \, b$. This proves (2). Finally, (3') clearly reduces in the present context to (4). Thus σ satisfies (1), (3) and (3'), so that S is transitive.

Following Dubreil [1], we call a subsemigroup T of S <u>left unitary</u> if $a, ab \in T$ implies $b \in T$. It is easy to see from the preceding proof, that we could, in the statement of the theorem, impose upon A the additional requirement that it be a left unitary subsemigroup of S.

COROLLARY. <u>Suppose</u> S <u>contains a left unitary subsemigroup</u> T <u>which meets each right ideal, and satisfies</u>

(5) If $a, b \in S$ and $a \neq b$, then there exist $x, y \in S$ such that either $xay \in T$, $xby \notin T$ or $xay \notin T$, $xby \in T$.

<u>Then</u> S <u>is transitive.</u>

<u>Proof.</u> We need only show that T satisfies (4). Suppose $a \neq b$, and each pair ca, cb can be joined by a chain as in (4). Let x, y be as in (5). Then, in particular, xa, xb can be joined by a chain as in (4). Hence, xay, xby can also be joined by such a chain. Since T is a left unitary subsemigroup, this implies that xay, xby are either both in T or both outside T, contradicting (5).

Teissier [2] and the present author [3, 5] have studied the following condition on a subset A of a semigroup:

(5') If $a, b \in S$ and $a \neq b$, then there exist $x, y \in S^1$ such that either $xay \in A$, $xby \notin A$ or $xay \notin A$, $xby \in A$, where S^1 denotes S with an identity element adjoined.

Teissier [2] showed that (5') is equivalent to the assertion that the identity relation is the only (two-sided) congruence for which A is a union of congruence classes. Condition (5') is slightly weaker than (5). However, if desired, (5) in the corollary could, by a slight modification of the proofs, be replaced by (5').

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As an example of the application of the corollary, let S be the free semigroup with two generators, so that S consists essentially of all finite sequences of 0's and 1's, with juxtaposition as the semigroup operation. It is already known [4, ex. 2, p. 540] that S is transitive. Let T be the subsemigroup generated by the sequence 0 together with all sequences consisting of n 1's, followed by 0, followed by n - 1 entries chosen arbitrarily. (For example, 00111011 $\in$ T, 11100 $\notin$ T.) Then T is left unitary, meets each right ideal, and satisfies (5). This supplies a new proof of the transitivity of S.

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University of California, Davis

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