BULL. AUSTRAL. MATH. SOC. 15A03
VOL. 30 (1984), 411-420.

ELEMENTARY PROPERTIES OF VECTOR SPACE GRAPHS

GRACE ORZECH

Let S_Γ be a vector space graph. A graphic subspace S'_Γ of S_Γ need not be a direct summand with a graphic complement. A necessary and sufficient condition for the existence of a graphic complement is given. Also, it is shown that every graphic subspace S'_Γ possesses an o-special basis which extends to an o-special basis of S_Γ .

Introduction

In [2] Ribenboim introduced the category of vector space graphs. We recall that an object in this category is a quadruple $S_\Gamma = (S, V(S), o, t)$ where S is a vector space, $V(S) \subseteq S$ is a subspace, and $o, t : S \to V(S)$ are linear transformations that restrict to the identity on $V(S)$. A morphism $f : S_\Gamma \to S'_\Gamma = (S', V(S'), o', t')$ is a linear transformation $f : S \to S'$ satisfying $o'f = fo$ and $t'f = ft$. We will call such a morphism a *graphic linear transformation*. A subspace $S' \subseteq S$ will be called a *graphic subspace* if $o(S') \cup t(S')$ is contained in S' . If S' is graphic we let $V(S') = V(S) \cap S'$ and $o' = o|S'$, $t' = t|S'$. Then $S'_\Gamma = (S', V(S'), o', t')$ is a vector space graph in its own right. Also, if S' is a graphic subspace of S then we let $V(S/S') = (V(S)+S')/S'$ and define $\bar{o}, \bar{t} : S/S' \to V(S/S')$ by $\bar{o}(s+S') = o(s) + S'$ and

Received 7 June 1984.

$\bar{t}(s+S') = t(s) + S'$. Then $(S/S', V(S/S'), \bar{o}, \bar{t})$ is a vector space graph which we denote by S_Γ/S'_Γ .

In this paper we consider ways in which vector space graphs are unlike ordinary vector spaces and we introduce conditions that enable us to mimic procedures used in ordinary linear algebra in spite of the observed differences.

1. o-special graphic bases and numerical invariants

Let $S_\Gamma = (S, V(S), o, t)$ be a vector space graph over a field K . Following Ribenboim's terminology we will say that a basis B of S is *graphic* if

$$o(B) \cup t(B) \subseteqq B \cup \{0\}$$

and *o-special* if $B \subseteqq \ker(o) \cup V(S)$. Every vector space graph has o-special graphic bases which are constructed as follows: let $B_L = \{l_1, \ldots, l_\lambda\}$ be a basis for $L_0 = \ker(o) \cap \ker(t)$ the subspace consisting of all loops with vertex 0 . Next choose a basis $B_0 = \{e_1, \ldots, e_\tau\}$ for any complement of L_0 in $\ker(o)$. It is easy to see that $t(B_0)$ is a basis for $t(\ker(o)) \subseteqq V(S)$. Finally choose a basis $B_V = \{v_1, \ldots, v_\mu\}$ for any complement of $t(\ker(o))$ in $V(S)$. It is easy to see that $B = B_L \cup B_0 \cup t(B_0) \cup B_V$ is a graphic o-special basis for S . The quadruple $(B \cup \{0\}, t(B_0) \cup B_V, o|B \cup \{0\}, t|B \cup \{0\})$ describes the finite directed graph in Figure 1.

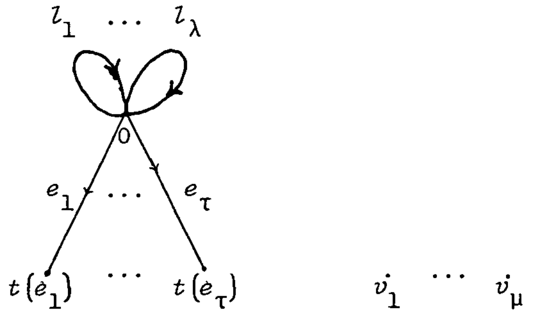

FIGURE 1

As Ribenboim remarked in [2], the numbers $\lambda = \dim_K(L_0)$, $\tau = \dim_K(t(\ker(o)))$, and $\tau + \mu = \dim_K(V(S))$ are numerical invariants of the vector space graph S_Γ.

DEFINITION 1.1. A vector space graph S_Γ for which $\dim_K(L_0) = \lambda$, $\dim_K(t(\ker(o))) = \tau$ and $\dim_K(V(S)) = \tau + \mu$ is said to be of *type* (λ, τ, μ). We denote this by writing $\text{typ}(S_\Gamma) = (\lambda, \tau, \mu)$.

REMARK 1.2. If $\text{typ}(S_\Gamma) = (\lambda, \tau, \mu)$, then $\dim_K(S) = \lambda + 2\tau + \mu$.

In the study of ordinary vector spaces the notions of irreducible, indecomposable, and completely reducible as defined in [1, Chapter II] are trivial. For vector space graphs the situation is a little more complicated.

Vector space graphs of types $(1, 0, 0)$ and $(0, 0, 1)$ are 1-dimensional as vector spaces and so have no nontrivial subspaces, graphic or otherwise. On the other hand, if $\text{typ}(S_\Gamma) = (0, 1, 0)$, then S is 2-dimensional as a vector space and has a 1-dimensional graphic subspace $V(S)$. Since S contains no loops at 0, any vector subspace, E, satisfying $V(S) \oplus E = S$ contains an edge with at least one nonzero vertex v. Since $V(S) \cap E = \{0\}$, v is not in E and so E is not a graphic subspace. This proves the next statement.

REMARK 1.3. A vector space graph of type $(0, 1, 0)$ is not completely reducible.

2. Criteria for the existence of graphic complements

We wish to provide a more elaborate example of the fact that vector space graphs need not be completely reducible.

EXAMPLE 2.1. Let S_Γ be a vector space graph of type $(0, 2, 2)$ with o-special basis $B = \{e_1, e_2, t(e_1), t(e_2), v_1, v_2\}$. Let S'_Γ and S''_Γ be graphic subspaces with o-special bases $B' = \{e_1, t(e_1), v_1, v_2\}$ and $B'' = \{e_1, t(e_1), t(e_2), v_1\}$, respectively. Both these subspaces are of type $(0, 1, 2)$, but S'_Γ has a graphic complement in S_Γ while S''_Γ

does not. To see this observe that $S_\Gamma = S'_\Gamma \oplus e_2K \oplus t(e_2)K$ and that $e_2K \oplus t(e_2)K$ is a graphic subspace of S_Γ of type $(0, 1, 0)$. On the other hand, suppose $S_\Gamma = S''_\Gamma \oplus T_\Gamma$. Since e_2 is not in S''_Γ, we can write $e_2 = a_1e_1 + a_2t(e_1) + a_3t(e_2) + a_4v_1 + x$ where x is a nonzero element of T. From this we see that

$$x = -a_1e_1 + e_2 - a_2t(e_1) - a_3t(e_2) - a_4v_1$$

and so $t(x) = -(a_1+a_2)t(e_1) + (1-a_3)t(e_2) - a_4v_1$. This expression places $t(x)$ in S''. But then $t(x)$ is in $S'' \cap T = \{0\}$. Thus $a_1 = -a_2$, $1 = a_3$, $a_4 = 0$, and so $x = -a_1(e_1-t(e_1)) + (e_2-t(e_2))$ which is manifestly in $\ker(t)$. Moreover, $o(x) = a_1t(e_1) - t(e_2)$ which is in $S'' \cap T = \{0\}$. But this would mean that x is in $L_0 = \{0\}$, contradicting our assumption.

The next two propositions provide some criteria for the existence of graphic complements.

PROPOSITION 2.2. *A graphic subspace S'_Γ has a graphic complement S''_Γ in S_Γ if and only if $t'(\ker(o')) = t(\ker(o)) \cap S'$.*

Proof. Assume $t'(\ker(o')) = t(\ker(o)) \cap S'$. Let $B' = B'_L \cup B'_0 \cup t'(B'_0) \cup B'_V$ be an o-special basis for S' and let $B_L \supseteq B'_L$ and $B_0 \supseteq B'_0$ be such that B_L is a basis for L_0 and $B_L \cup B_0$ is a basis for $\ker(o)$. Let $\langle X\rangle$ denote the subspace spanned by X. Then $t(B_0) \cup B'_V$ is linearly independent because

$$\begin{aligned}\langle t(B_0)\rangle \cap \langle B'_V\rangle &= t(\ker(o)) \cap (S' \cap \langle B'_V\rangle) \\ &= t'(\ker(o')) \cap \langle B'_V\rangle \\ &= \langle t'(B'_0)\rangle \cap \langle B'_V\rangle = \{0\} .\end{aligned}$$

Let $B_V \supseteq B'_V$ be such that $t(B_0) \cup B_V$ is a basis for $V(S)$. It is easy to see that $(B_L \backslash B'_L) \cup (B_0 \backslash B'_0) \cup t(B_0 \backslash B'_0) \cup (B_V \backslash B'_V)$ is an o-special basis for a graphic complement of S'_Γ.

https://doi.org/10.1017/S0004972700002124 Published online by Cambridge University Press

On the other hand assume that $S_\Gamma = S'_\Gamma \oplus S''_\Gamma$. Since $t'\{\ker(o')\} \subseteq t\{\ker(o)\} \cap S'$ is clear, it suffices to demonstrate the reverse inclusion. If y is in $t\{\ker(o)\} \cap S'$ then $y = t(e'+e'')$ with $o(e'+e'') = 0$, e' in S' , and e'' in S'' . Thus $y - t(e') = t(e'')$ is in $S' \cap S'' = \{0\}$ and $o(e') = -o(e'')$ which is in $S' \cap S'' = \{0\}$. This shows $y = t(e') = t'(e')$ where $o(e') = o'(e') = 0$. Hence y is in $t'\{\ker(o')\}$.

In much the same spirit we have a theorem about kernels of graphic linear transformations.

PROPOSITION 2.3. *Let* $F : S_\Gamma \to S'_\Gamma$ *be a graphic linear transformation.* $\ker(F)$ *is a graphic subspace of* S *and* $\ker(F)$ *has a graphic complement if and only if* $F^{-1}(L'_0) = L_0 + \ker(F)$.

Proof. Clearly $\ker(F)$ is graphic. We write

$$(\ker(F))_\Gamma = (\ker(F), V\{\ker(F)\}, o'', t'') .$$

Assume $\ker(F)$ has a graphic complement. Then

$$t\{\ker(o)\} \cap \ker(F) = t''\{\ker(o'')\} .$$

$F^{-1}(L'_0) \supseteq L_0 + \ker(F)$ is clear. If s is in $F^{-1}(L'_0)$ then

$$F\{t(s)-o(s)\} = t'\{F(s)\} - o'\{F(s)\} = 0 .$$

Also $t(s) - o(s) = t\{s-o(s)\}$. Thus $t(s) - o(s)$ is in

$$t\{\ker(o)\} \cap \ker(F) = t''\{\ker(o'')\}$$

and so there is an element u in $\ker(o'')$ such that $t(u) = t(s) - o(s)$. Clearly $s - u - o(s)$ is in L_0 , $u + o(s)$ is in $\ker(F)$ and $s = \{s-u-o(s)\} + \{u+o(s)\}$ is in $L_0 + \ker(F)$.

On the other hand, assume that $F^{-1}(L'_0) = L_0 + \ker(F)$. If x is in $t\{\ker(o)\} \cap \ker(F)$ then $F(x) = 0$ and $s = t(y)$ with $o(y) = 0$. It is easy to check that y must be in $F^{-1}(L'_0)$. Write $y = l + g$ where l is in L_0 and $F(g) = 0$. Then $x = t(y) = t(g) = t''(g)$ and $o''(g) = o(g) = o(y-l) = 0$. That is, x is in $t''\{\ker(o'')\}$. This suffices to show that $t(\ker(o)) \cap \ker(F) = t''(\ker(o''))$.

3. Extendibility of graphic bases

The problem of extending bases is related to the discussion in §2.

DEFINITION 3.1. If $S'_\Gamma \subseteqq S_\Gamma$ is a graphic subspace, then an o-special basis B' of S'_Γ is called *extendible* if there is an o-special basis B of S such that $B' \subseteqq B$.

First we give an example to show that not every o-special basis of a subspace is extendible.

EXAMPLE 3.2. Let S_Γ be a vector space graph of type $(0, 2, 1)$ and let $B = \{e_1, e_2, t(e_1), t(e_2), v\}$ be the o-special basis in Figure 2.

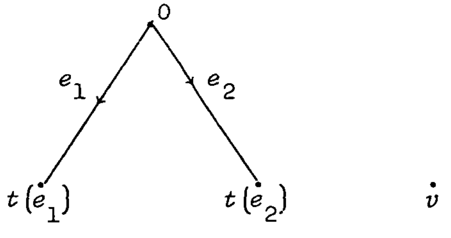

FIGURE 2

S_Γ has a subspace S'_Γ of type $(0, 1, 2)$ with o-special basis $B' = \{e_1, t(e_1), t(e_2), v\}$. Since $B' \subseteqq B$, B' is extendible.

Consider $B'' = \{e_1, t(e_1), v+t(e_2), v\} \subseteqq S'_\Gamma$. B'' is an o-special basis of S'_Γ , but it is not extendible. Suppose

$$C = \{f_1, f_2, t(f_1), t(f_2), w\} ,$$

with f_1 and f_2 in $\ker(o)$, and w not in $t(\ker(o))$, is an o-special basis of S_Γ such that $B'' \subseteqq C$. Then e_1 is in $\{f_1, f_2\}$. Say $e_1 = f_1$. Then $t(f_1) = t(e_1)$. Moreover $\{v+t(e_2), v\} = \{t(f_2), w\}$. Since v is not in $t(\ker(o))$, v must be equal to w . Thus $v + t(e_2) = t(f_2)$. But then $v = t(f_2-e_2)$ would be in $t(\ker(o))$ which

https://doi.org/10.1017/S0004972700002124 Published online by Cambridge University Press

is a contradiction.

Our next task is to enunciate a criterion for extendibility. For the ensuing discussion we need to establish some notation.

Let S'_Γ be a graphic subspace of S_Γ. Given an o-special basis

$$B' = B'_L \cup B'_0 \cup t'(B'_0) \cup B'_V , \tag{3.3}$$

let

$$B'_{V1} = B'_V \cap t(\ker(o)) ,$$

$$B'_{V2} = B'_V \backslash B'_{V1} . \tag{3.4}$$

Finally let

$$\begin{aligned} V'_i &= \text{the subspace of } V(S') \text{ spanned by } B'_{Vi} \text{ where } i = 1, 2 , \\ V' &= \text{the subspace of } V(S') \text{ spanned by } B'_V . \end{aligned} \tag{3.5}$$

PROPOSITION 3.6. *If* S'_Γ *is a graphic subspace of* S_Γ *and* B' (*as in* (3.3)) *is an* o*-special basis of* S', *then the following are equivalent:*

(a) B' *is extendible;*

(b) $t(\ker(o)) \cap V'_2 = \{0\}$;

(c) $V' \cap t(\ker(o)) = V'_1$.

Proof. *(b)* and *(c)* are easily seen to be equivalent.

(b) ⇒ *(a)*. To obtain an o-special basis that extends B' we begin by extending B'_L to a basis B_L of L_0. It is easy to see that $B_L \cup B'_0$ is linearly independent for any choice of B_L.

For each v in B'_{V1}, choose e in $\ker(o)$ such that $t(e) = v$. Let C be the set of vectors chosen in this way. We claim that $B_L \cup B'_0 \cup C$ is a linearly independent subset of $\ker(o)$. For suppose

$$\sum_{l_i \text{ in } B_L} a_i l_i + \sum_{e_j \text{ in } B'_0} b_j e_j + \sum_{e_k \text{ in } C} c_k e_k = 0$$

with not all coefficients equal to 0 . At least one $c_k \neq 0$ in this case. By applying t , we get

$$\sum_{e_j \text{ in } B_0'} b_j t\, e_j + \sum_{v_k \text{ in } B_{V1}'} c_k v_k = 0 .$$

Since $t(B_0') \cup B_{V1}'$ is part of a basis, all c_k must be 0 .

Next extend $B_L \cup B_0' \cup C$ to a basis $B_L \cup B_0$ of $\ker(o)$. Since $B_0' \cup C \subseteqq B_0$, $t(B_0) \supseteqq t(B_0') \cup t(C) = t'(B_0') \cup B_{V1}'$.

Finally observe that $B_L \cup B_0 \cup t(B_0)$ is a basis for $\ker(o) \oplus t(\ker(o))$. Since $t(\ker(o)) \cap V_2' = \{0\}$, we can say that $B_L \cup B_0 \cup t(B_0) \cup B_{V2}'$ is linearly independent. We can extend $t(B_0) \cup B_{V2}'$ to a basis $t(B_0) \cup B_V$ of $V(S)$ thereby obtaining an o-special basis $B_L \cup B_0 \cup t(B_0) \cup B_V$ of S that contains B' .

(a) $\Rightarrow$ *(b)*. Suppose $0 \neq x$ is in $t(\ker(o)) \cap V_2'$ and suppose $B' \subseteqq B_L \cup B_0 \cup t(B_0) \cup B_V$, an o-special basis of S . Since $x = t(e)$ for some e in $\ker(o)$, x can be written as a linear combination of the $t(e_i)$'s where e_i is in $\ker(o)$. Because x is in V_2' it can also be written as a linear combination of the elements v_i' in B_{V2}' . Thus there is a dependence relation among vectors in $t(B_0) \cup B_{V2}'$, a subset of $t(B_0) \cup B_V'$, which in turn is part of the basis containing B' . This is impossible and so the proof is complete.

COROLLARY 3.7. *If S_Γ' has a graphic complement then every o-special basis of S_Γ' is extendible.*

Proof. S_Γ' has a graphic complement implies that

$$t'(\ker(o')) = t(\ker(o)) \cap S' .$$

Thus

$$t(\ker(o)) \cap V_2' = t(\ker(o)) \cap (S' \cap V_2') = t'(\ker(o')) \cap V_2' = \{0\} .$$

https://doi.org/10.1017/S0004972700002124 Published online by Cambridge University Press

Finally we will show that a subspace need not have a graphic complement in order to contain an extendible o-special basis.

PROPOSITION 3.8. *Every graphic subspace* $S'_\Gamma \subseteq S_\Gamma$ *possesses an extendible* o*-special basis.*

Proof. Let $B' = B'_L \cup B'_0 \cup t(B'_0) \cup B'_{V1} \cup B'_{V2}$ be any o-special basis of S'. If B' is not extendible then $W = t(\ker(o)) \cap V'_2 \neq \{0\}$. Let C_{V1} be a basis for W and extend C_{V1} to a basis $C_{V1} \cup C_{V2}$ for V'_2. Now let $B''_L = B'_L$, $B''_0 = B'_0$, and $B''_V = B'_{V1} \cup C_{V1} \cup C_{V2}$. It is not hard to see that $B'' = B''_L \cup B''_0 \cup t(B''_0) \cup B''_V$ is an o-special basis for S'. Moreover $B''_{V1} = B'_{V1} \cup C_{V1}$ while $B''_{V2} = C_{V2}$. Thus $t(\ker(o)) \cap V''_2 = \{0\}$. By Proposition 3.6, B'' is extendible.

PROPOSITION 3.9. *Let* S'_Γ *be a graphic subspace of* S_Γ *and suppose* $\operatorname{typ}(S_\Gamma) = (\lambda, \tau, \mu)$ *and* $\operatorname{typ}(S'_\Gamma) = (\lambda', \tau', \mu')$. *Also let* $\sigma = \dim(V' \cap t(\ker(o)))$. *Then*

$$\operatorname{typ}(S_\Gamma/S'_\Gamma) = (\lambda-\lambda'+\sigma,\ \tau-\tau'-\sigma,\ \mu-\mu'+\sigma) .$$

Proof. Let $B' = B'_L \cup B'_0 \cup t'(B'_0) \cup B'_V$ be an o-special basis of S'. Then $\sigma = \dim(V'_1) = \operatorname{card}(B'_{V1})$. The elements of B'_{V1} can be written $t(f_1), \ldots, t(f_\sigma)$ where each f_i is in $\ker(o)$. There exists an o-special basis B for S of the form

$$B = B'_L \cup \{l_1, \ldots, l_{\lambda-\lambda'}\} \cup B'_0 \cup \{f_1, \ldots, f_\sigma\} \cup \{e_1, \ldots, e_{\tau-\tau'-\sigma}\}$$
$$\cup\ t(B'_0) \cup B'_{V1} \cup \{t(e_1), \ldots, t(e_{\tau-\tau'-\sigma})\} \cup B'_{V2} \cup \{v_1, \ldots, v_{\mu-(\mu'-\sigma)}\}$$

where $l_1, \ldots, l_{\lambda-\lambda'}$ are in $L\backslash L'$, $e_1, \ldots, e_{\tau-\tau'-\sigma}$ are in $\ker(o)$, and $v_1, \ldots, v_{\mu-\mu'+\sigma}$ are in $V(S)$ but not in $t(\ker(o))$. It is now easy to see that S_Γ/S'_Γ has o-special basis $\bar{B}_L \cup \bar{B}_0 \cup \bar{t}(\bar{B}_0) \cup \bar{B}_V$ with

$$\bar{B}_L = \{\bar{l}_1, \ldots, \bar{l}_{\lambda-\lambda'}, \bar{f}_1, \ldots, \bar{f}_\sigma\} ,$$

$$\bar{B}_0 = \{\bar{e}_1, \ldots, \bar{e}_{\tau-\tau'-\sigma}\} ,$$

https://doi.org/10.1017/S0004972700002124 Published online by Cambridge University Press

$$\bar{B}_V = \{\bar{v}_1, \ldots, \bar{v}_{\mu-\mu'+\sigma}\} .$$

References

[1] C.W. Curtis and I. Reiner, *Representation theory of finite groups and associative algebras* (Interscience, New York, 1962).

[2] P. Ribenboim, "Vector space graphs", *Nanta Math.* **12** (1979), 125-132.

Department of Mathematics and Statistics,
Queen's University,
Kingston,
Ontario,
Canada K7L 3N6.

https://doi.org/10.1017/S0004972700002124 Published online by Cambridge University Press