BULL. AUSTRAL. MATH. SOC. 54A05, 54D99, 54B99, 54C99
VOL. 8 (1973), 443-459.

Solid convergence spaces

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The category of solid convergence spaces is introduced, and shown to lie strictly between the category of all convergence spaces and that of pseudo-topological spaces. A wide class of convergence spaces, including the c-embedded spaces of Binz, is then characterized in terms of this concept. Finally, several illustrative examples are given.

The categories L of convergence spaces, P of principal convergence spaces (also known as pseudo-topological spaces) and T of topological spaces are all more or less familiar. It is well known that any topological space is completely determined by the collection of all its open covers. After having introduced an analogous concept for convergence spaces (namely, indexed cover), we find instead that only some convergence spaces can be so determined. Such spaces are called solid. Some properties of solid spaces are listed, and it is shown that the inclusions $P \subseteq S \subseteq L$ are both proper, where S denotes the category of solid spaces.

In addition, using these ideas we discuss in more detail a more restricted class of spaces, obtaining as a special case the internal characterization of c-embedded convergence spaces given independently by Müller [6]. This states that a space is c-embedded iff it is Hausdorff, solid and ω-regular (this last term generalizing complete regularity for topological spaces). Finally, examples are given showing among other things that these three conditions are independent of one another.

1. Introductory concepts

The reader is assumed to be familiar with the concepts of convergence

Received 31 January 1973.

https://doi.org/10.1017/S0004972700042738 Published online by Cambridge University Press

structure (Limitierung) and convergence space (Limesraum), as well as with initial and final convergence structures. See [4] or [3; paragraphs 0.1, 0.1, 0.3(1) and 0.3(2) respectively], for example. Convergence spaces will usually be denoted by symbols such as X, or (X, γ) if the convergence structure γ is to be explicitly mentioned. For brevity, the word 'convergence' is often deleted from all the above terms.

DEFINITIONS. (i) Let X be a set, ϕ a filter on X and $A \subseteq X$. We say that ϕ *has a trace on* A if $F \cap A$ is non-void, for all $F \in \phi$. In this case, the symbol $\phi \cap A$ denotes the filter $\{F \cap A : F \in \phi\}$ on A.

(ii) Suppose a space (X, γ) be given. Then the sentence '*$x\phi$ is a pair from* (X, γ) ' is defined to mean that $x \in X$ and ϕ is a filter on X such that ϕ converges to x in (X, γ).

(iii) A collection Σ of subsets of a space (X, γ) is said to be an *indexed cover for* (X, γ) if it is the image of a map associating with each pair $x\phi$ from (X, γ) a subset $S_{x\phi}$ of X, such that $S_{x\phi} \in \phi$.

(iv) The space (X, γ) is called the *special inductive limit of the family* $\{(X_i, \gamma_i) : i \in I\}$ of convergence spaces if the following conditions all hold:

1. (X_i, γ_i) is a subspace of (X, γ), for all $i \in I$,
2. the collection $\{X_i : i \in I\}$ is directed upwards by inclusion, with $X = \cup X_i$, and
3. γ is the final structure on X induced by the family $\{j_i : X_i \to X\}_{i \in I}$ of inclusion maps.

REMARK 1.1. Let X and Y be convergence spaces and $f : X \to Y$ a continuous map. Then each indexed cover Σ for Y induces an indexed cover $f^{-1}(\Sigma)$ for X. Namely, if $x\phi$ is a pair from X then $f(\phi)$ converges to $f(x)$ in Y, since f is continuous. Now by defining $T_{x\phi} = f^{-1}\left(S_{f(x)f(\phi)}\right)$ and

$$f^{-1}(\Sigma) = \{T_{x\phi} : x\phi \text{ is a pair from } X\},$$

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we construct the indexed cover $f^{-1}(\Sigma)$.

Recalling that a space (X, γ) is called *compact* if every ultrafilter on X converges to some point of (X, γ) , and that a subset A of X is *compact* if it is compact as a subspace of (X, γ) , we say that (X, γ) is *locally compact* if it possesses an indexed cover consisting entirely of compact subsets. Equivalently, (X, γ) is locally compact iff it is the special inductive limit of its compact subsets.

Almost exactly as in topology, one proves that compact spaces are characterized by the Heine-Borel property. In fact, the following conditions on a space X are equivalent:

(i) X is compact,

(ii) every filter on X has a non-void cluster set, and

(iii) for each indexed cover Σ for X there is a finite subset Σ_0 of Σ , with $X = \cup\Sigma_0$.

(Fischer [4] proved the equivalence of (i) and (ii).)

PROPOSITION 1.2. *The special inductive limit of a family of locally compact spaces is locally compact. More generally, suppose that a collection* $\{X_\lambda : \lambda \in \Lambda\}$ *of locally compact spaces is given, and for each* $\lambda \in \Lambda$ *, a map* $f_\lambda : X_\lambda \to X$ *. Then* (X, γ) *is locally compact, where* γ *denotes the final structure on* X *induced by the collection* $\{f_\lambda : \lambda \in \Lambda\}$.

The proof of this proposition is omitted, being straightforward though slightly messy. (Note that as there are k-spaces which are not locally compact, this proposition is not true in the category of topological spaces.)

2. Solid spaces and solidifications

Using indexed covers, we associate with each space its solidification and its strong solidification, the space being called solid if it coincides with its solidification. It is shown later that not all spaces are solid, but that many are, including all topological spaces.

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To each space $X = (X, \gamma)$ are assigned its *solidification* $sX = (X, \sigma\gamma)$ and its *strong solidification* $s'X = (X, \sigma'\gamma)$. These are defined as follows:

(1) $\phi \to x$ in sX if for each indexed cover Σ for X there is a finite set, say $\phi_1, \ldots, \phi_n$, of filters on X , all converging to x in X , such that

$$\bigcup_1^n S_{x\phi_i} \in \phi ;$$

(2) $\phi \to x$ in $s'X$ if for each indexed cover Σ for X there is a finite set, say $x_1\phi_1, \ldots, x_n\phi_n$, of pairs from X such that

$$x \in \bigcap_1^n S_{x_i\phi_i} \text{ and } \bigcup_1^n S_{x_i\phi_i} \in \phi .$$

The basic properties of solidifications are now summarised, those marked * being also true of strong solidifications.

THEOREM 2.1. *(i) All solidifications are solid, that is,* $s^2 = s$.

(ii) If* $f : X \to Y$ *is continuous, then so is* $f : sX \to sY$.

*(iii)** sA *is a subspace of* sX *, if* A *is a subspace of* X .

(iv) If* X *is the special inductive limit of the family* $\{X_i : i \in I\}$ *, then* sX *is the special inductive limit of* $\{sX_i : i \in I\}$.

(v) If X *is* T_1 *then* sX *and* $s'X$ *coincide.*

Proof. *(i)* Left to the reader as an exercise.

(ii) Let $x\phi$ be a pair from sX . It must be shown that $f(\phi)$ converges to $f(x)$ in sY . Now by Remark 1.1, each cover Σ for Y induces an indexed cover $f^{-1}(\Sigma)$ for X . By definition of sX , there are filters $\phi_1, \ldots, \phi_n$, converging to x in X , with $T_{x\phi_1} \cup \ldots \cup T_{x\phi_n} \in \phi$. But

$$f\left(\bigcup_1^n T_{x\phi_i}\right) \subseteq \bigcup_1^n S_{f(x)f(\phi_i)} ,$$

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the former set belonging to $f(\phi)$. Thus the latter also belongs to $f(\phi)$, showing that $f(\phi) \to f(x)$ in sY , as required.

(iii) By part *(ii)*, the inclusion $j : sA \to sX$ is continuous. On the other hand, suppose that $x \in A$ and ϕ is a filter on A such that $j(\phi) \to x$ in sX . To prove the claim, it is enough to show that $\phi \to x$ in sA . Let Σ be a cover for A . We extend Σ to a cover $\underline{\Sigma}$ for X as follows: for any pair $y\psi$ from X ,

(α) if $y \in A$ and ψ has a trace on A , let $\underline{S}_{y\psi} = S_{y\psi\cap A} \cup (X\backslash A)$,

(β) if $y \in A$ and ψ does not have a trace on A , let $\underline{S}_{y\psi} = X\backslash A$, and

(γ) if $y \notin A$, let $\underline{S}_{y\psi} = X$.

Since $j(\phi) \to x$ in sX , there are filters $\phi_1, \ldots, \phi_n$ converging to x in X , with $\underline{S}_{x\phi_1} \cup \ldots \cup \underline{S}_{x\phi_n} \in j(\phi)$. Without loss of generality, we suppose that ϕ_i has a trace on A if $i \leq m$, and that ϕ_i does not have a trace on A if $i > m$. Thus as

$$\bigcup_1^m S_{x\phi_i\cap A} = A \cap \left(\bigcup_1^n \underline{S}_{x\phi_i}\right) \in \phi$$

and the filters $\phi_i \cap A$ all converge to x in A , we have shown that $\phi \to x$ in sA .

(iv) Let X' be the inductive limit of the collection $\{sX_i : i \in I\}$. By *(ii)*, the inclusion maps $j_i : sX_i \to sX$ are all continuous, and consequently, so is $id : X' \to sX$. Conversely, let $\phi \to x$ in sX . Since X is an inductive limit, every convergent filter on X has a base on some X_i . Thus the family $\{X_i : i \in I\}$ can be made into an indexed cover for X , by means of the axiom of choice. Hence $x \in X_i \in \phi$, for some index i . But sX_i is a subspace of sX , showing that $\phi \cap X_i \to x$ in sX_i . It follows that

$$\phi = j_i(\phi\cap X_i) \to x \text{ in } X' .$$

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This proves that $X' = sX$; that is, sX is the special inductive limit of the family $\{sX_i : i \in I\}$.

(v) We recall that a space X is T_1 if $\dot{y} \to x$ in X implies $x = y$, for all $x, y \in X$. (The symbol $\dot{y}$ denotes the ultrafilter $\{A : A \subseteq X$ and $y \in A\}$.) Trivially, for all spaces X the identity maps $X \to sX \to s'X$ are both continuous.

On the other hand, let X be T_1 and $\phi \to x$ in $s'X$. With each indexed cover Σ for X is associated a cover Σ^* : for each pair $y\psi$ from X ,

(α) if $y = x$, then $S^*_{y\psi} = S_{y\psi}$, and

(β) otherwise, $S^*_{y\psi} = S_{y\psi} \backslash \{x\}$.

(That this is a cover for X follows from the fact that in a T_1 space one point sets are closed.) Using this construct, one can now easily show that $\phi \to x$ in sX .

The next theorem is mostly an immediate corollary of the foregoing.

THEOREM 2.2. *(i)* Subspaces, products, and special inductive limits of solid spaces are solid.*

(ii) In the category of T_1 convergence spaces, solidity and strong solidity are equivalent.

(iii) Every principal convergence space, and in particular every topological space, is solid.

Proof. To prove that products of solid spaces are solid, it is only necessary to use the universal property of products, and Theorem 2.1 *(ii)*, which is left as an exercise for the reader.

Finally, a space X is called principal if for each point $x \in X$ there is a filter ϕ_x on X such that $\phi \to x$ in X exactly when $\phi \geq \phi_x$. Let X be principal, and $\phi \to x$ in sX . For each $F \in \phi_x$, an indexed cover Σ^F is defined by

$$S^F_{y\psi} = X \text{ , if } \psi \to y \text{ in } X \text{ and } y \neq x \text{ ,}$$

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and

$$S^F_{x\psi} = F \text{ , if } \psi \to x \text{ in } X \text{ .}$$

Using this cover, one can see that $F \in \phi$. Consequently $\phi \geq \phi_x$, showing that X is solid.

3. *A*-embedded and *c*-embedded spaces

The main result of this section is an internal characterization of *c*-embedded convergence spaces resembling that obtained independently by Müller [6]. As a corollary, we give a simple condition for the special inductive limit of a family of *c*-embedded spaces to be *c*-embedded.

In what follows, any statement referring to the 'scalar field $\mathbb{F}$ ' is true for both the real field $\mathbb{R}$ and the complex field $\mathbb{C}$. Further, $\mathbb{F}$ will always carry its usual metric topology, generated by the closed unit ball $\mathbb{D}$.

Let X be a convergence space, whose structure is γ . Then CX denotes the set of all continuous functions taking (X, γ) to $\mathbb{F}$, and $\omega\gamma$ the initial topology on X induced by the family CX , with ωX standing for the topological space $(X, \omega\gamma)$. Similarly if $A \subseteq CX$, one obtains the topology $\omega_A\gamma$ and the space $\omega_A X$. The space X is called ω_A*-regular* if $\phi^- \to x$ in X , for each pair $x\phi$ from X . (Here ϕ^- denotes that filter on X having $\{F^- : F \in \phi\}$ as base, with $^-$ being the closure operator in $\omega_A X$.)

For each subset A of CX there is a coarsest structure γ_c on A making the evaluation map $\Omega : A \times X \to \mathbb{F}$ continuous, where $\Omega(f, x)$ is defined to be $f(x)$, for all $f \in A$ and $x \in X$, [3]. This structure is called the structure of continuous convergence. For brevity, the spaces (A, γ_c) and (CX, γ_c) are denoted by A_c and C_cX . Clearly A_c is a subspace of C_cX . Note that $\theta \to f$ in A_c exactly when $\Omega(\theta\times\phi) \to f(x)$ in $\mathbb{F}$, for each pair $x\phi$ from X .

We now give the first of a sequence of technical lemmas.

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LEMMA 3.1. *The spaces* C_cX , C_csX *and* $C_cs'X$ *are the same, for all convergence spaces* X .

Proof. Since the space F is both Hausdorff and topological, it is strongly solid. Thus by Theorem 2.1 *(ii)*, the sets CX, CsX and $Cs'X$ are equal. Also, the identity maps $X \to sX \to s'X$ being continuous, so are the maps $id^* = id : C_cs'X \to C_csX$ and $id^* = id : C_csX \to C_cX$. (For any spaces Y and Z , a continuous map $f : Y \to Z$ induces a continuous map $f^* : C_cZ \to C_cY$, defined by $f^*(g) = g \circ f$, for all $g \in CZ$, [3].)

Finally, it must be shown that $id : C_cX \to C_cs'X$ is continuous. To prove this, it is enough to show that $\Omega(\theta\times\phi) \to \Omega(f, x) = f(x)$ in F , for each pair $f\theta$ from C_cX and $x\phi$ from $s'X$. However, since $\theta \to f$ in C_cX , for each pair $y\psi$ from X and each positive real number ε , there are $T_{\varepsilon y\psi} \in \theta$ and $P_{\varepsilon y\psi} \in \psi$ such that

$$\Omega\left(T_{\varepsilon y\psi}\times P_{\varepsilon y\psi}\right) \subseteq f(y) + \varepsilon D .$$

In this way, for each $\varepsilon > 0$ we construct a cover Σ_ε for X . As a result, there are pairs $x_1\phi_1, \ldots, x_n\phi_n$ from X with $x \in \bigcap_1^n P_{\varepsilon x_i\phi_i}$ and $P_\varepsilon = \bigcup_1^n P_{\varepsilon x_i\phi_i} \in \phi$. Now let $T_\varepsilon = \bigcap_1^n T_{\varepsilon x_i\phi_i}$. Clearly $T_\varepsilon \in \theta$, and

$$\Omega\left(T_\varepsilon\times P_\varepsilon\right) \subseteq f(x) + 2\varepsilon D .$$

This being true for each $\varepsilon > 0$, we see that $\Omega(\theta\times\phi) \to f(x)$ in F , completing our proof.

Let us fix now on a space $X = (X, \gamma)$ and a subset A of CX . As in [*1*], there are continuous maps $i : X \to C_cC_cX$ and $i_A : X \to C_cA_c$, the former defined by $i(x)(f) = f(x)$, for all $f \in CX$ and $x \in X$, and the latter by $i_A = j_A^* \circ i$, where $j_A : A_c \to C_cX$ is the inclusion map. For simplicity, both $i(x)$ and $i_A(x)$ are usually shortened to $\hat{x}$; that is, $\hat{x}(f) = f(x)$, for all f .

By the previous lemma, it does not matter whether we regard A_c as a subspace of $C_c X$; or $C_c sX$, or $C_c s'X$. This proves:

LEMMA 3.2. *The maps* $i : sX \to C_c C_c X$, *and* $i : s'X \to C_c \dot{C}_c X$ *are continuous, as are* $i_A : sX \to C_c A_c$ *and* $i_A : s'X \to C_c A_c$.

Let γ_A be the initial structure on X induced by the map i_A , and $c_A X$ denote the resulting space (X, γ_A) . (If $A = CX$, we write cX instead of $c_{CX} X$.) We call $c_A X$ *the A-embedded space associated with* X , and say that X is *A-embedded* if $\gamma = \gamma_A$. It is easy to see that $\phi \to x$ in $c_A X$ exactly when $\Omega(\theta \times \phi) \to f(x)$ in F , for each pair $f\theta$ from A_c . Clearly $id : X \to c_A X$ is continuous, showing that $Cc_A X$ is a subalgebra of CX . Despite this,

LEMMA 3.3. A_c *is a subspace of* $C_c c_A X$.

Proof. First we must show that $A \subseteq Cc_A X$. Take $f \in A$, and define $\hat{f} : C_c A_c \to \mathsf{F}$ by $\hat{f}(F) = F(f)$, for all $F \in CA_c$. By [1, Lemma 6] the map $\hat{f}$ is continuous. Thus $f : c_A X \to \mathsf{F}$ is continuous, since f is clearly the composite $\hat{f} \circ i_A$ of two continuous maps. That is, $f \in Cc_A X$, as claimed.

Next, since the identity $id : X \to c_A X$ is continuous, so is the inclusion $j = id^* : C_c c_A X \to C_c X$. Consequently, A inherits a finer structure from $C_c c_A X$ than from $C_c X$. To complete the proof, we need only show the converse, namely, that the inclusion $j' : A_c \to C_c c_A X$ is continuous. However, the diagram

$$\begin{array}{ccc} c_A X \times A_c & \xrightarrow{\tilde{j}'} & \mathsf{F} \\ \Big\downarrow{\scriptstyle i_A \times id} & \nearrow{\scriptstyle \Omega} & \\ C_c A_c \times A_c & & \end{array} \qquad \begin{array}{ccc} (x, f) & \longmapsto & f(x) \\ \Big\downarrow & \nearrow & \\ (\hat{x}, f) & & \end{array} ,$$

commutes. As a result, $\tilde{j}'$ is continuous, being the composite of two

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continuous maps. Thus by the universal property of the structure of continuous convergence [3, Satz 2], j' itself is continuous.

REMARK. This lemma clearly justifies our having called $c_A X$ the A-embedded space associated with X, for it shows that $c_A X$ is an A-embedded space.

The following two propositions are an attempt to characterize A-embedded spaces, but without complete success, for it is convenient to place certain algebraic restrictions on A.

PROPOSITION 3.4. *The space* $c_A X$ *is* ω_A*-regular and strongly solid.*

Proof. By Lemma 3.3, A_c is a subspace of $C_c c_A X$, and so $i_A : s' c_A X \to C_c A_c$ is continuous, by Lemma 3.2. It follows that $id : s' c_A X \to c_A X$ is also continuous, by the universal property of $c_A X$. Thus $c_A X$ is strongly solid.

Secondly, ω_A-regularity. Suppose that $\phi \to x$ in $c_A X$. It must be shown that $\phi^- \to x$ in $c_A X$ as well. For each pair $f\theta$ from A_c and each positive real number ε, there are $T_\varepsilon \in \theta$ and $F_\varepsilon \in \phi$ with

$$\Omega\left(T_\varepsilon \times F_\varepsilon\right) \subseteq f(x) + \varepsilon D .$$

Each $g \in A$ being continuous in the topology $\omega_A \gamma$, by its definition, if $g \in T_\varepsilon$ then

$$g\left(F_\varepsilon^-\right) \subseteq g\left(F_\varepsilon\right)^- \subseteq f(x) + \varepsilon D .$$

This shows that $\Omega\left(T_\varepsilon \times F_\varepsilon^-\right) \subseteq f(x) + \varepsilon D$. Thus $\phi^- \to x$ in $c_A X$, completing our proof.

We do not know if the converse is true in general; we can only prove it under certain conditions. We assume

(i) that A is a vector subspace of CX containing the constant function 1, and

(ii) that $|f| \in A$ for all $f \in A$.

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Under these conditions, if C is closed in $\omega_A X$ and $x \in X \setminus C$, then there is $f \in A$ such that $f = |f|$ and $0 \leq f(y) \leq 1$ for all $y \in X$, and in addition, $f(x) = 1$ and $f(y) = 0$ if $y \in C$.

PROPOSITION 3.5. *Let A satisfy the above conditions. Then X is A-embedded if it is ω_A-regular and strongly solid.*

Proof. By continuity of $id : X \to c_A X$, to prove the proposition it is enough to show that every filter which converges in $c_A X$ also converges in X. Accordingly, let $\phi \to x$ in $c_A X$. Then every open neighbourhood of x in $\omega_A X$ belongs to ϕ, since $id : c_A X \to \omega_A X$ is continuous.

Let Σ be an indexed cover for X. By ω_A-regularity, $\bar{\psi} \to y$ in X whenever $\psi \to y$ in X. Thus we can choose a subset $C_{y\bar{\psi}}$ of $S_{y\bar{\psi}}$ which is closed in $\omega_A X$ and belongs to $\bar{\psi}$. Now we set $S'_{y\psi} = C_{y\bar{\psi}}$, for all pairs $y\psi$ from X, and so obtain an indexed cover Σ' for X refining Σ, and consisting entirely of sets closed in $\omega_A X$.

A filter θ converging to 0 in A_c is now constructed: let

$$B_{y\psi} = \{f \in A : f(S'_{y\psi}) \subseteq \mathrm{D}\},$$

for each pair $y\psi$ from X, and θ be that filter on A generated by

$$\{\varepsilon B_{y\psi} : y\psi \text{ is a pair from } X \text{ and } \varepsilon > 0\}.$$

This means that $\Omega(\theta \times \phi) \to 0$ in F, as $\phi \to x$ in $c_A X$. In particular, there are $T \in \theta$ and $F \in \phi$ such that

$$(*) \qquad \Omega(T \times F) \subseteq \mathrm{D}.$$

As $T \in \theta$, there is a positive real number ε and a finite collection $y_1\psi_1, \ldots, y_n\psi_n$ of pairs from X with $T \supseteq \bigcap_1^n \varepsilon B_{y_i\psi_i}$. Without loss of generality, we assume that if $i \leq m$ then x belongs to $S'_{y_i\psi_i}$, whereas $x \notin S'_{y_i\psi_i}$ if $i > m$. Consequently $X \setminus \bigcup_{m+1}^{n} S'_{y_i\psi_i}$ is an

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open neighbourhood of x in $\omega_A X$. Thus

$$G = F \setminus \bigcup_{m+1}^{n} S'_{y_i\psi_i} \in \phi .$$

Suppose if possible that there is a point $y \in G \setminus \bigcup_{1}^{m} S'_{y_i\psi_i}$. Then by our conditions on A , we can find a function $f \in A$, vanishing on $\bigcup_{1}^{n} S'_{y_i\psi_i}$, with $f(y) = 1$. Since all scalar multiples of f belong to T and y to F , this contradicts (*). Thus

$$G \subseteq \bigcup_{1}^{m} S'_{y_i\psi_i} .$$

From this it is clear that $\phi \to x$ in $s'X$, or in other words, $c_A X$ is the strong solidification of X . Hence X is A-embedded, being strongly solid.

Binz [1] introduced the idea of c-embedded convergence spaces, the definition reading that a space X is c-embedded if the map $i : X \to C_c C_c X$ is an embedding (he considered $\mathsf{F} = \mathsf{R}$ only, but the same class of spaces is obtained by using $\mathsf{F} = \mathsf{C}$). The preceding results allow us to characterize c-embedded spaces as follows:-

THEOREM 3.6. *A convergence space is c-embedded if and only if it is Hausdorff, ω-regular and solid.*

Proof. By Proposition 3.4, using $A = CX$, one sees that X is ω-regular and solid if it is c-embedded. Moreover, c-embedded spaces are known to be Hausdorff.

Conversely, suppose X to be ω-regular, solid and Hausdorff. The first condition implies that for all $x \in X$ and $y \in \{x\}^-$, the ultrafilter y converges to x in X . Thus $\{x\}^- = \{x\}$, since X is Hausdorff. However, $\{x\}^- = \{y \in X : f(y) = f(x)$, for all $f \in CX\}$, showing that if $x \neq y$ there is $f \in CX$ with $f(x) \neq f(y)$. Consequently the map $i : X \to C_c C_c X$ is injective.

Next, by Theorem 2.2 *(ii)*, X is strongly solid, and so (taking

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$A = CX$) we see that X is CX-embedded. This, combined with the fact that the map i is injective, proves that i is an embedding.

NOTE. The topological space wX can be characterised internally as follows: its topology is the topology associated with the finest uniformity on X with the property that ϕ is Cauchy for each pair $x\phi$ from X. Consequently the characterization of c-embedded spaces given above is also internal.

It is known [1, Korollar 22, 23] that subspaces and products of c-embedded spaces are c-embedded. We investigate in a subsequent paper the problem of when the special inductive limit of c-embedded spaces is c-embedded, being content at this point just to state a simple condition for this to be so.

COROLLARY 3.7. *Let X be the special inductive limit of the family $\{X_i : i \in I\}$ of c-embedded spaces. Then X is c-embedded if*

(i) each X_i is closed in wX, and

(ii) each X_i is a subspace of cX.

The proof is omitted, being trivial. In case each X_i is compact c-embedded, we can do rather better. However, such spaces are compact Hausdorff topological spaces [2, Satz 9], so that one is actually discussing the special inductive limit of a family of compact Hausdorff topological spaces in this case.

COROLLARY 3.8. *Let X be the special inductive limit of the family $\{X_i : i \in I\}$ of compact Hausdorff topological spaces. Then X is locally compact, and the following statements are equivalent:*

(i) X is c-embedded,

(ii) each X_i is a subspace of cX,

(iii) each X_i is a subspace of wX, and

(iv) CX distinguishes the points of X (that is, for all $x, y \in X$, if $x \neq y$ there is $f \in CX$ with $f(x) \neq f(y)$).

Proof. The equivalence of *(i)* with *(iv)* was first noted by Kutzler.

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The implications *(i)* ⇒ *(iv)*, *(iv)* ⇒ *(iii)* and *(iii)* ⇒ *(ii)* are all trivial, the first being always true, and the second because *(iv)* implies that ωX is Hausdorff. Last, *(ii)* ⇒ *(i)*. Suppose X is not c-embedded. Then it is not ω-regular, by Theorem 3.6. This means that $\phi^{-} \not\to x$ in X, for some pair $x\phi$ from X with $x \in X_i \in \phi$, say. It follows that X_i^{-} properly contains X_i, and hence that ωX can not be Hausdorff (for if it were, the compact subset X_i would be closed). Thus there are distinct points y and z of X with $f(y) = f(z)$ for all $f \in CX$, and $\dot{y} \to z$ in cX by Proposition 3.4. Now both y and z belong to $X_{i'}$, for some $i' \in I$, and so it is not possible for $X_{i'}$ to be a subspace of cX. This completes the proof.

We show in the next section that there are spaces which do not satisfy these conditions.

4. Various examples

In this section examples are given showing that the three conditions, ω-regularity, Hausdorff-ness, and solidity, mentioned in Theorem 3.6 are independent of each other. One of these examples perforce is of a non-solid convergence space. Thus the category S of solid spaces is a proper subcategory of L. Similarly, examples of solid but not principal spaces are easy to find, since every c-embedded space is solid but not necessarily principal.

EXAMPLE 4.1. The two point indiscrete topological space is solid and ω-regular, but not Hausdorff.

EXAMPLE 4.2. There are Hausdorff topological spaces which are not regular. Such a space is solid, but not ω-regular.

EXAMPLE 4.3. Let X be the set $\mathbb{N} \cup \{\infty\}$, and Φ the set of all non-trivial ultrafilters on X together with $\dot{\infty}$. A convergence structure γ on X is defined by

$$\phi \to n \text{ in } (X, \gamma) \text{ iff } \phi = \dot{n} ,$$

and

$$\phi \to \infty \text{ in } (X, \gamma) \text{ iff } \phi = \phi_1 \wedge \ldots \wedge \phi_n ,$$

for some finite subset $\phi_1, \ldots, \phi_n$ of Φ .

Clearly (X, γ) is compact and Hausdorff. Moreover, $\phi_\infty =: \wedge\Phi$ is the neighbourhood filter of ∞ in the usual one point compactification (X, τ) of $\mathbb{N}$. It is not hard to check that if $\phi_1, \ldots, \phi_{n+1}$ is a finite subset of Φ , then $\bigwedge_1^{n+1} \phi_i$ is strictly coarser than $\bigwedge_1^{n} \phi_i$. Thus (X, γ) is not principal and in particular, ϕ_∞ does not converge to ∞ in (X, γ) . Now let Σ be an indexed cover for (X, γ) . We construct a cover Σ' by setting

$$S'_{x\phi} = \begin{cases} S_{x\phi} \text{ , if } x = \infty \text{ , and } \phi \to \infty \text{ ,} \\ \{x\} \text{ , if } x \in \mathbb{N} \text{ .} \end{cases}$$

By compactness, there is a finite collection of pairs, say $x_1\phi_1, \ldots, x_n\phi_n$, from (X, γ) such that $X = \bigcup_1^n S'_{x_i\phi_i}$. We assume that $x_i = \infty$ if $i \le m$, and that $x_i \in \mathbb{N}$ if $i > m$. It follows that

$$\bigcup_1^m S_{x_i\phi_i} = \bigcup_1^m S'_{x_i\phi_i} \supseteq \{p \in X : p \ge k\} \text{ ,}$$

for some $k \in \mathbb{N}$. Hence $\bigcup_1^m S_{x_i\phi_i} \in \phi_\infty$. This shows that $\phi_\infty \to \infty$ in $(X, \sigma\gamma)$; that is (X, γ) is not solid, and (X, τ) is its solidification.

In fact, (X, γ) is also ω-regular. To see this, note first that $C(X, \gamma) = C(X, \tau)$, by Theorem 2.1 *(ii)*. This implies that the topologies τ and $\omega\gamma$ coincide, being compact Hausdorff. Secondly, a subset A of X is τ-closed if and only if it is

(i) finite, or

(ii) contains ∞ .

Suppose $\phi \to x$ in (X, γ) . First, if $x \in \mathbb{N}$ then $\phi = \dot{x} = \phi^-$ (the $^-$ denoting the τ-closure operator), and so $\phi^- \to x$ in (X, γ) in this

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case. Secondly, if $x = \infty$ then the remark above shows that $\phi^{-} = \phi \wedge \dot{\infty}$. Thus ϕ^{-} converges in this case as well.

To summarise, (X, γ) is compact, Hausdorff, and ω-regular, but not solid.

The next two examples are of spaces to which Corollary 3.8 applies. In addition, the first is a solid space which is not principal.

EXAMPLE 4.4. Let τ be the usual Euclidean topology on R^2 and its subsets, and let $X = \{\underline{0}\} \cup \left\{\underline{x} \in R^2 : x_1 > 0 \text{ and } |\underline{x}| \le 1\right\}$. For all $n \in N$ we put $X_n = \{\underline{x} \in X : |x_2| \le nx_1\}$, observing that (X_n, τ) is compact. Thus Corollary 3.8 applies to the special inductive limit (X, γ) of the family $\{(X_n, \tau) : n \in N\}$. However, the continuity of $id : (X, \gamma) \to (X, \tau)$ ensures that $C(X, \gamma) \supseteq C(X, \tau)$, the inclusion being actually proper. Hence condition *(iv)* of that corollary is satisfied, showing the c-embeddedness of (X, γ). In particular, (X, γ) is solid. It is left to the reader to prove that (X, γ) is not principal.

EXAMPLE 4.5. Let X be the Urysohn-Hewitt 'spiral staircase' from $-\infty$ to $+\infty$ [5], and τ its topology. The space (X, τ) is compactly generated, but $C(X, \tau)$ does not distinguish the points $-\infty$ and $+\infty$. Let (X, γ) be the special inductive limit of the compact subsets of (X, τ). Clearly $C(X, \gamma) = C(X, \tau)$, and so (X, γ) is not c-embedded, again by Corollary 3.8.

As a final remark, we point out that though more filters may converge in the solidification of a space than in the space, this is not true for ultrafilters. In fact, if X is a space and ϕ an ultrafilter on X then the following statements are equivalent:

(i) $\phi \to x$ in X,

(ii) $\phi \to x$ in sX, and

(iii) $\phi \to x$ in $s'X$.

This is however not true of the principal space associated with X, as a slight modification of Example 4.3 will show. The reader will easily verify these facts for himself.

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