Canad. Math. Bull. Vol. 19 (3), 1976

ON COMMUTATOR EQUALITIES AND STABILIZERS IN FREE GROUPS

BY

R. G. BURNS, C. C. EDMUNDS, AND I. H. FAROUQI

ABSTRACT. A simple proof is given of a result of Hmelevskiĭ on the solutions of the equation $[x, y]=[u, v]$ over a free group for any specified u, v. To illustrate, the equation is solved explicitly for $(u, v)=(a, b)$, (a^2, b), $([a, b], c)$ (where a, b, c freely generate the free group) and thence stabilizers of the corresponding commutators in the automorphism group of this free group are determined.

In this note we apply an argument of Mal'cev [4] to give a short, elementary proof of a result of Hmelevskiĭ [3] on the solutions over a free group F of the two variable equation

$$[x, y]=[u, v], \tag{1}$$

where u, v are any specified elements of F and x, y are the unknowns. (Here $[g, h]$ denotes $g^{-1}h^{-1}gh$.) We illustrate the result by carrying out the solution when (u, v) takes the values (a, b), (a^2, b), $([a, b], c)$, (where $\{a, b, c \cdots\}$ is a set of free generators of F) and use these results to determine the stabilizers of the corresponding commutators in the automorphism groups of the appropriate free factors of F.

Let $|g|$ denote the length of $g \in F$ as a reduced word in $a, b, \ldots$. We say that two pairs in $F \times F$ are *connected* if one can be obtained from the other by means of a finite sequence of elementary transformations of the form $(s, t) \to (s, s_1 t)$ or $(s, t) \to (t_1 s, t)$ where s_1 and t_1 centralize s and t respectively. The pairs are *strongly connected* if they are connected by a sequence of such transformations with $s_1 = s^{\pm 1}$ and $t_1 = t^{\pm 1}$.

The following is a slight improvement of Hmelevskiĭ's result.

THEOREM 1. *(Cf. Hmelevskiĭ [3], first assertion of Theorem 1 (II).) If $u, v \in F$ are such that $[u, v] \neq 1$, then every solution $(\hat{x}, \hat{y})$ of (1) is strongly connected to some solution (x, y) of (1) with both $|x|$, $|y| \leq |[u, v]| - 3$.*

Note that if (p, q) is connected to (r, s) then $[p, q]=[r, s]$. Also note that the (finitely many) solutions (x, y), in the theorem, can be obtained systematically, and that if $[u, v]=1$, then as is well known, u and v are powers of a common element of F. It seems likely that, more generally, if $p, q, r, s \in F$ satisfy $[p, q]=[r, s]$, then (p, q) and (r, s) are connected.

Received by the editors September 23, 1974 and, in revised form, September 2, 1975.

https://doi.org/10.4153/CMB-1976-041-7 Published online by Cambridge University Press

Let $g, h, \ldots$ belong to a group G. We write $\operatorname{stab}_G g$ for the stabilizer of g in $\operatorname{aut} G$, the automorphism group of G, $\langle g, h, \ldots\rangle$ for the subgroup of G generated by $g, h, \ldots,$ and g^h for $h^{-1}gh$. Write $F_2=\langle a, b\rangle$, $F_3=\langle a, b, c\rangle$. Define $\alpha, \beta \in \operatorname{aut} F$ by $\alpha: b \to ab$, $x \to x$ for generators $x \neq b$, $\beta: a \to ba$, $x \to x$ for generators $x \neq a$, and write α_2, β_2; α_3, β_3 for the restrictions of α, β to F_2, F_3.

THEOREM 2. I. *The equation* (1) *implies that* (x, y) *is connected to* (u, v) *if* (u, v) *is any of* (a, b), (a^2, b), $([a, b], c)$.

II. (i) (*Cf. Mal'cev* [4].) $\operatorname{Stab}_{F_2}[a, b]=\langle\alpha_2, \beta_2\rangle$ (*and has a presentation* $\langle c_1, d_1 \mid c_1^2=d_1^3\rangle$ *where we may take* $c_1=\alpha_2^{-2}\beta_2$, $d_1=\alpha_2^{-1}\beta_2$).

(ii) $\operatorname{Stab}_{F_2}[a^2, b]=\langle\alpha_2, \lambda_2\rangle$ *where* λ_2 *is defined by* $a \to a^{-ba^2}$, $b \to b^{-ba^2}$.

(iii) $\operatorname{Stab}_{F_3}[[a, b], c]=\langle\alpha_3, \beta_3, \mu_3, \nu_3\rangle$, *where* μ_3, ν_3 *are defined by:* $\mu_3: a \to a$, $b \to b$, $c \to [a, b]c$, $\nu_3: a \to b^{c[a,b]}$, $b \to a^{c[a,b]}$, $c \to c^{-c[a,b]}$. (*Note that* ν_3^{-2} *is conjugation by* $[[a, b], c]$.)

REMARKS.

1. That $\langle\alpha_2, \beta_2\rangle$ has the presentation $\langle c_1, d_1 \mid c_1^2=d_1^3\rangle$ is implicit in [2, p. 90]. We sketch a simple direct proof. It is straightforward to check that $(\alpha_2^{-1}\beta_2)^3= (\alpha_2^{-2}\beta_2)^2$. Now, as is well known (see [2]), $\langle\alpha_2, \beta_2\rangle$ has as quotient $SL_2(Z)$ generated by the automorphisms of the derived quotient of F_2 induced by α_2, β_2, and $SL_2(Z)$ has the presentation $\langle c_2, d_2 \mid c_2^2=d_2^3, c_2^4=1\rangle$. Hence the desired presentation will be established once it is shown that $\alpha_2^{-2}\beta_2$ has infinite order. This is a consequence of the easily verified fact that $(\alpha_2^{-2}\beta_2)^4$ is simply conjugation by $[b, a]$.

2. The equation $axbaya^{-1}x^{-1}b^{-1}a^{-1}y^{-1}=1$, studied by Appel [1] takes the form $[x, y]=[a^2, b]$ after a change of variables.

Write $g \wedge h$ for the meet, i.e. the largest common initial segment, of $g, h \in F$. Theorem 1 is a consequence of the following lemma.

LEMMA. (i) *Every pair* $(\hat{x}, \hat{y}) \in F \times F$, *with* $[\hat{x}, \hat{y}] \neq 1$, *is strongly connected to a pair* $(x, y) \in F \times F$ *where*

$$|x \wedge y^{-1}|, |x \wedge y|, |x^{-1} \wedge y| \leq \tfrac{1}{2}|x| \quad \text{and} \quad \tfrac{1}{2}|y|. \tag{2}$$

(ii) *Further, if we write*

$$x=px_1^{-1}=qx_2r^{-1}, \qquad y=qy_2^{-1}p^{-1}=ry_1 \tag{3}$$

in reduced form (*i.e. the factors are reduced words and no cancellation occurs between factors*) *where* $p=x \wedge y^{-1}$, $q=x \wedge y$, $r=x^{-1} \wedge y$, *then* $x_1, y_1, x_2, y_2 \neq 1$, *and thus* $x_1y_2x_2y_1$ *is the reduced form of* $[\hat{x}, \hat{y}](=[x, y])$.

REMARK. The pair (x, y) is not unique; for instance, $[a, bab]=[aba, b]$.

Proof of the lemma. The proof is much as in [4]. We prove part (i) by induction on $|\hat{x}|+|\hat{y}|$. If $|\hat{x}|+|\hat{y}|=2$ the lemma follows with $x=\hat{x}$ and $y=\hat{y}$.

https://doi.org/10.4153/CMB-1976-041-7 Published online by Cambridge University Press

Suppose $|\hat{x}|+|\hat{y}|>2$ and that the statement of the lemma holds for all pairs $(\hat{\hat{x}}, \hat{\hat{y}})$ with $|\hat{\hat{x}}|+|\hat{\hat{y}}|<|\hat{x}|+|\hat{y}|$ and $[\hat{x}, \hat{y}]\neq 1$. Suppose that at least one of $|\hat{x}\wedge\hat{y}^{-1}|$, $|\hat{x}\wedge\hat{y}|$, $|\hat{x}^{-1}\wedge\hat{y}|$ exceeds $\frac{1}{2}|x|$ or $\frac{1}{2}|y|$. Say, for instance, $|\hat{x}\wedge\hat{y}^{-1}|>\frac{1}{2}|x|$. Write $z=\hat{x}\wedge\hat{y}^{-1}$ and $\hat{x}=z\hat{x}_1$, $\hat{y}=\hat{y}_1z^{-1}$ in reduced form. (Thus $|z|>|\hat{x}_1|$.) Then the sequence

$$(\hat{x}, \hat{y})=(z\hat{x}_1, \hat{y}_1z^{-1})\to((\hat{y}_1z^{-1})(z\hat{x}_1), \hat{y}_1z^{-1})$$
$$\to(\hat{y}_1\hat{x}_1, (\hat{y}_1\hat{x}_1)^{-1}(\hat{y}_1z^{-1}))=(\hat{y}_1\hat{x}_1, \hat{x}_1^{-1}z^{-1})$$

shows that $(\hat{x}, \hat{y})$ is strongly connected to $(\hat{y}_1\hat{x}_1, \hat{x}_1^{-1}z^{-1})$. Since

$$|\hat{y}_1\hat{x}_1|+|\hat{x}_1^{-1}z^{-1}|=|\hat{y}_1|+2|\hat{x}_1|+|z|<|\hat{y}_1|+|\hat{x}_1|+2|z|=|\hat{x}|+|\hat{y}|,$$

the inductive hypothesis implies that $(\hat{y}_1\hat{x}_1, \hat{x}_1^{-1}z^{-1})$ is strongly connected to a pair (x, y) satisfying (2). Thus $(\hat{x}, \hat{y})$ is strongly connected to (x, y). The remaining five cases are similar: we omit the details.

For the second part of the lemma note that if $y_2=1$ then by (2) and (3) $|p|=|q|$. Since p and q are both initial segments of x it follows that $p=q$, whence $y=1$ contradicting $[\hat{x}, \hat{y}]\neq 1$. Similarly $x_2\neq 1$. That $x_1\neq 1\neq y_1$ is clear from (2).

Proof of Theorem 2. We omit the proof of Part I which is a routine, but lengthy check. (It is not difficult to devise ways of shortening the algorithm.) Statement (i) of Part II is immediate from Part I. We give a sketch only of the proofs of (ii) and (iii).

(ii) If $\varphi\in F_2$ fixes $[a^2, b]$ then $[a^2, b]=[(a\varphi)^2, b\varphi]$, whence by Part I, (a^2, b) and $((a\varphi)^2, b\varphi)$ are connected. Thus the problem becomes first that of detecting among the $(w, z)\in F_2\times F_2$ connected to (a^2, b) those in which w is a square. Denote by

$$(a^2, b)\to(w_1, z_1)\to(w_2, z_2)\to\cdots\to(w_n, z_n)=(w, z), \tag{4}$$

a shortest sequence of elementary transformations connecting (a^2, b) to (w, z). We shall prove, by induction on n, that if w is a square in $\langle a, b\rangle$ then w is conjugate to $a^{\pm 2}$. This is clear if $n=1$. Suppose $n>1$ and that the claim holds for shorter sequences. Let (w_r, z_r) be the first pair after (a^2, b) in the sequence (4) such that at least one of the entries is a proper power. We may suppose $r>1$ since if $r=1$ then $(w_1, z_1)=(a^2, a^sb)$, and we get the desired result by applying the inductive hypothesis to the subsequence of (4) starting with (w_1, z_1). Then $\langle w_r, z_r\rangle=\langle a^2, b\rangle$ whence w_r, z_r are certainly primitive in the free group $\langle a^2, b\rangle$. Now it is easily proved (by considering the amalgamated product $(\langle a\rangle*\langle a_1, b\rangle; a^2=a_1)$) that the only elements primitive in $\langle a^2, b\rangle$ that are proper powers in $\langle a, b\rangle$ are conjugates of $a^{\pm 2}$. The inductive hypothesis applied to the shorter sequence $(w_r, z_r)\to\cdots\to(w_n, z_n)$, then establishes the

https://doi.org/10.4153/CMB-1976-041-7 Published online by Cambridge University Press

desired conclusion. Thus $w = g^{-1}a^{\pm 2}g$, for some $g \in \langle a, b\rangle$ and therefore $a\varphi = g^{-1}a^{\pm 1}g$. If we write $z = g^{-1}z_1 g$, then $[a^2, b] = [w, z] = g^{-1}[a^{\pm 2}, z_1]g$. Rewriting z_1 as $a^k z_2 a^\ell$ where z_2 begins in $b^{\pm 1}$ and ends in $b^{\pm 1}$, we obtain $[a^{\pm 2}, z_1] = a^{\mp 2-\ell}z_2^{-1}a^{\pm 2}z_2 a^\ell$, which has $a^{\mp 2}z_2^{-1}a^{\pm 2}z_2$ as a cyclically reduced form. It follows that $z_2 = b^{\pm 1}$. It is then not difficult to find the possible values for g, and thence deduce that $\varphi \in \langle \alpha_2, \lambda_2\rangle$ as required.

(iii) Reasoning as above we see that it suffices to elicit from among the $(w, z) \in F \times F$ strongly connected to $([a, b], c)$ those where w is a commutator. Write $[a, b] = t$. Since t, c freely generate $\langle t, c\rangle$ it follows that a word w in these free generators which is a commutator in $\langle a, b, c\rangle$, must have zero exponent sum on c. Write $\boldsymbol{\tau}$, $\boldsymbol{\gamma}$ for the automorphisms of $\langle t, c\rangle$ defined by $\tau: t \to t,\ c \to tc$; $\gamma: t \to ct,\ c \to c$. We shall now characterize those $\boldsymbol{\varphi} \in \langle \tau, \gamma\rangle$ with the property that the exponent sum on c in $t\varphi$ is zero. As in Remark 1 we go to the group $SL_2(Z)$ of automorphisms induced by $\langle \tau, \gamma\rangle$ on the derived quotient of $\langle t, c\rangle$. The desired φ correspond to matrices of the form

$$\begin{pmatrix} 1 & 0 \\ * & 1 \end{pmatrix} \quad or \quad \begin{pmatrix} -1 & 0 \\ * & -1 \end{pmatrix} \tag{5}$$

where * can take any integral value. Using the presentation $\langle c_1, d_1 \mid c_1^2 = d_1^3, c_1^4 = 1\rangle$ of $SL_2(Z)$, where

$$c_1 = \begin{pmatrix} 1 & 1 \\ -2 & -1 \end{pmatrix}, \qquad d_1 = \begin{pmatrix} 1 & 1 \\ -1 & 0 \end{pmatrix}$$

(note that c_1 is induced by $\tau^{-2}\gamma$ and d_1 by $\tau^{-1}\gamma$), it is easily checked that the elements of the form (5) are precisely $(d_1c_1^{-1})^n$, $(d_1c_1^{-1})^n c_1^2$ (n integral). The preimages of these elements in $\langle \tau, \gamma\rangle$ will be the only candidates for φ. From Remark 1 we deduce that φ has the form $\tau^m(\tau^{-2}\gamma)^{2n}$ (m, n integral). Now τ is induced modulo $\langle \alpha_3, \beta_3\rangle$ by μ_3 (and only by μ_3; the verification of this is not difficult). Hence we need only check to see if $(\tau^{-2}\gamma)^2$ is indeed induced by automorphisms of $\langle a, b, c\rangle$. One computes that $(\tau^{-2}\gamma)^2$ maps t to t^{-ct} and c to c^{-ct}. Finally it is not difficult to check (using II(i)) that, modulo $\langle \alpha_3, \beta_3\rangle$ the only element of $\mathrm{aut}\langle a, b, c\rangle$ sending t to t^{-ct} and c to c^{-ct} is the automorphism ν_3 described in the theorem.

ACKNOWLEDGEMENTS. We thank D. Solitar for helpful comments. This paper was written at the Summer Research Institute of the C.M.C., Calgary, 1974.

REFERENCES

1. K. I. Appel, *On two-variable equations in free groups*, Proc. Amer. Math. Soc. **21** (1969), 179–185.

2. H. M. S. Coxeter and W. O. J. Moser, *Generators and relations for discrete groups*, Springer, 1965.

https://doi.org/10.4153/CMB-1976-041-7 Published online by Cambridge University Press

3. Ju. I. Hmelevskiĭ, *Systems of equations in a free group*. I, Izv. Akad. Nauk SSSR, Ser. **Mat. 35,** No. 6 (1971) (A.M.S. translations: Math. USSR Izvestija **5,** No. 6 (1971), 1245–1276.)

4. A. I. Mal'cev, *On the equation* $zxyx^{-1}y^{-1}z^{-1}=aba^{-1}b^{-1}$ *in a free group*, Algebra i Logika (Seminar) **1,** No. 5 (1962), 45–50.

York University
Toronto
Mount Saint Vincent University
Halifax
University of Karachi
Karachi

https://doi.org/10.4153/CMB-1976-041-7 Published online by Cambridge University Press