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GENERIC MATRIX SIGN-STABILITY

BY
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ABSTRACT. A new concept of generic sign-stability is proposed, and a necessary and sufficient condition for this property is given. This result shows that the condition proposed by Quirk and Ruppert [12] is correct almost everywhere, and helps to clarify the counterexample presented by Jeffries [4].

1. **Introduction**. In qualitative matrix analysis, a set of equations is analysed based solely on the qualitative information, i.e., the signs, $+$, $-$, or 0, of the related elements (coefficients, partial derivatives, etc.) of that system. In this paper, we are concerned with the problem of qualitative stability (or, equivalently, sign-stability) of the following linear system.

$$\dot{x} = Ax \tag{1}$$

Throughout this paper, a matrix A is said, following Jeffries et al. [5], to be *stable* (*semistable*, resp.)[1], if $\mathrm{Re}[\sigma(A)] < 0$ ($\mathrm{Re}[\sigma(A)] \leq 0$, resp.), where $\sigma(A)$ denotes the set of all eigenvalues of A, $\mathrm{Re}[\cdot]$ stands for the real part of a complex number, and the last inequality should be interpreted as a requirement for each element of $\sigma(A)$.

The system (1) is said to be a *qualitative system* if each entry of A is specified only up to its signs, $+$, $-$, or, 0. Such a matrix is referred to as a *sign matrix*. A qualitative system (or, the sign matrix A itself) is *sign-stable* (*sign-semistable*, resp.) if it is stable (semistable, resp.) for all numerical matrices $\bar{A} \in Q_A$. Here, Q_A denotes the set of all matrices of the sign pattern A, i.e.,

$$Q_A = \{\bar{A} \in R^{n \times n} | \mathrm{sgn}(\bar{A}) = A\}$$

where $\mathrm{sgn}(\cdot)$ is an (element-wise) sign operator defined by

$$\mathrm{sgn}(a) = \begin{cases} + & \text{if} \quad a > 0 \\ - & \text{if} \quad a < 0 \\ 0 & \text{if} \quad a = 0 \end{cases}$$

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[1]This definition of stability and semistability is not standard in control literature. The dynamical system (1) is *asymptotically stable iff* A is stable in this sense. However, (1) may be *marginally stable* or *unstable* if A has pure imaginary eigen-values. See, e.g., Luenberger [8] for more details.

The purpose of this paper is to reexamine the conditions under which a qualitative matrix A is sign-stable. A necessary and sufficient condition for sign-stability was first presented by Quirk and Ruppert [12], but later this proved incorrect by the counterexample of Jeffries [4]. We prove in this paper that the condition proposed by Quirk and Ruppert is necessary and sufficient for the system to be *generically sign-stable*, which is a slightly weaker requirement than sign-stability.

Before concluding this section, let us introduce some terminology and notation. Let $\boldsymbol{n}$ stand for $\{1, 2, \ldots, n\}$. The *graph* G_A of an $n \times n$ sign matrix A consists of a set of *nodes* $V = \{1, 2, \ldots, n\}$ and a set of *directed arcs* $E = \{(i, j) \in V \times V | a_{ij} \neq 0\}$. An *elementary cycle* of G_A is a set of arcs of the form $(i_1, i_2), (i_2, i_3), \ldots, (i_k, i_1)$ with $\{i_1, i_2, \ldots, i_k\}$ being a set of k distinct nodes. The *length* of this cycle is k. An elementary cycle of length k is also referred to as a *k-cycle*.

2. **Statement of the problem**. A criterion for sign-semistability can be stated as the following theorem.

THEOREM 1 (Quirk–Ruppert–Maybee)[2]: *A sign matrix $A = (a_{ij})$ is sign-semistable iff the following conditions are all satisfied.*

(i) $a_{ii} \leq 0$ *for all* $i \in \boldsymbol{n}$

(ii) $a_{ij}a_{ji} \leq 0$ *for all* $i, j \in \boldsymbol{n}$, $i \neq j$

(iii) *there exists no k-cycle of length* $k \geq 3$ *in* G_A.

It is also known that, if all diagonal elements are strictly negative, i.e., (i)′ $a_{ii} < 0$ for all $i \in \boldsymbol{n}$ holds, then (ii) and (iii) are necessary and sufficient for A to be sign-stable (See Theorem 3 of [12]).

Quirk and Ruppert [12] further claimed that (i)–(iii) plus the following (iv) and (v) are both necessary and sufficient for A to be sign-stable.

(iv) $a_{ii} < 0$ *for some* $i \in \boldsymbol{n}$

(v) $\det(A) \neq 0$

Indeed, the necessity of these two additional conditions is quite obvious. However, these conditions are not sufficient for A to be sign-stable, as was shown by the counterexample of Jeffries [4]. Although a complete characterization of sign-stability has been obtained by Jeffries et al. [5] using the notion of R_A-coloring, this paper intends to clarify the above counterexample by showing that the conditions (i)–(v) are necessary and sufficient for (1) to be *generically sign-stable* (but not necessarily to be sign-stable).

For more rigorous arguments, let us introduce some concepts and terminology of elementary algebraic geometry [6]. Consider an $n \times n$ sign matrix A with $\mu(A)$ non-zero indeterminate entries (and $n^2 - \mu(A)$ fixed zeros). We can identify each $\bar{A} \in Q_A$ with a point in the positive orthant

$$R^{\mu(A)+} = \{\xi = (\xi_1, \xi_2, \ldots, \xi_{\mu(A)}) | \xi_i > 0,\ i = 1, 2, \ldots, \mu(A)\}.$$

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[2]This is Theorem 1 of [5], which is credited to Quirk–Ruppert–Maybee. See also [9] and [12].

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Thus, we write $\bar{A} = \bar{A}(\xi)$, and $Q_A = \{\bar{A}(\xi) | \xi \in R^{\mu(A)+}\}$. A subset V of $R^{\mu(A)+}$ is said to be an *algebraic variety* if for some real-coefficient polynomials $p_1(\cdot), p_2(\cdot), \ldots,$ $p_m(\cdot)$ defined over $R^{\mu(A)+}$, V can be written as

$$V = \{\xi \in R^{\mu(A)+} | p_i(\xi) = 0, \qquad i = 1, 2, \ldots, m\}$$

An algebraic variety V is *proper* if $\phi \neq V \neq R^{\mu(A)+}$. It is well known that for a proper algebraic variety V, $R^{\mu(A)+} \backslash V$ is open and dense in $R^{\mu(A)+}$, and furthermore the Lebesgue measure of V is zero [13].

A $\{0, 1\}$-valued function π defined over $R^{\mu(A)+}$, $\pi : R^{\mu(A)} \to \{0, 1\}$, is said to be a *property* of A. Here, $\pi(\xi) = 1$ (0, resp.) means that the property holds (fails) at the parameter point $\xi \in R^{\mu(A)+}$. π is said to be a *generic property* [13] if there exists a proper algebraic variety V such that $\pi(\xi) = 1$ holds for all $\xi \in R^{\mu(A)+} \backslash V$. Since the Lebesgue measure of a proper algebraic variety is zero, a generic property can be regarded as a property that holds almost everywhere in the defining parameter space, or equivalently, for almost all $\bar{A}$ in Q_A.

3. **Generic sign-stability**. Consider an $n \times n$ sign matrix A. A is said to be *reducible* if there exists a permutation matrix P such that

$$P'AP = \begin{bmatrix} A_{11} & 0 \\ A_{21} & A_{22} \end{bmatrix}$$

with square matrices A_{11} and A_{22}. A is *irreducible* if it is not reducible. This is equivalent to the requirement that G_A is strongly connected [1].

If A is reducible, it can be put into block lower triangular form with diagonal blocks being irreducible by an appropriate permutation of columns and rows. Since the stability of A is completely determined by the stability of each diagonal block, we can restrict ourselves to the irreducible case. Thus, without loss of generality, we can assume that A is irreducible. The main result of this paper is the following theorem.

THEOREM 2. *An irreducible sign-matrix $A = (a_{ij})$ is generically sign-stable iff the conditions* (i)−(v) *are all satisfied.*

To prove the necessity of (i)−(iii), let us introduce the notion of the restriction of a sign matrix $A = (a_{ij})$ to a set of indices $S \subseteq \boldsymbol{n} \times \boldsymbol{n}$ by $A|_S = (a_{ij|S})$, where

$$a_{ij|S} = \begin{cases} a_{ij} & \text{if} \quad (i, j) \in S \\ 0 & \text{otherwise} \end{cases}$$

Then, we obtain the following lemma.

LEMMA. *Let A be an $n \times n$ sign matrix. A is not generically sign-semistable, if for some subset $S \subseteq \boldsymbol{n} \times \boldsymbol{n}$ and a (numerical) matrix $\bar{A}$,*

(a) *$\bar{A}$ has a simple eigenvalue with strictly positive real part; and*

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(b) $\mathrm{sgn}(\bar{A}) = A|_S$

are both satisfied.

This lemma is proved in the Appendix. We are now ready to prove the necessity part of Theorem 2.

PROOF OF THEOREM 2 (Necessity). If either one of (i)−(iii) is violated, we have, after appropriate renumbering of the columns and rows of A, (α) $a_{11} > 0$, (β) $a_{12}a_{21} > 0$, or (γ) $a_{12}a_{23} \ldots a_{k-1k}a_{k1} \neq 0$ $(k \geq 3)$. Let S be either $\{(1,1)\}$, $\{(1,2),(2,1)\}$, or $\{(1,2),(2,3),\ldots,(k,1)\}$, depending on the cases (α)−(γ), and define $\bar{A} = (\bar{a}_{ij})$ by

$$\bar{a}_{ij} = \begin{cases} 1 & \text{if} \quad (i,j) \in S \\ 0 & \text{otherwise} \end{cases}$$

Then, $\bar{A}$ has a simple eigenvalue $\bar{\lambda}$ with strictly positive real part, which $\bar{\lambda} = 1$ in case of (α) and (β), and $\exp(2\pi\, i/k)$ in case of (γ). This is a contradiction, due to the above lemma.

The necessity of (iv) and (v) is obvious. □

To prove the sufficiency part of Theorem 2, we need to use some results from graph and control theories. First, without loss of generality, we can assume that A is of the form

$$A = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}$$

with A_{11} being a $p \times p$ matrix with $\mathrm{diag}(A_{11}) < 0$, and A_{22} being an $(n-p) \times (n-p)$ matrix with $\mathrm{diag}(A_{22}) = 0$ $(0 < p \leq n)$. Correspondingly, the set of nodes V of G_A is divided into $V_1 = \{1,2,\ldots,p\}$ and $V_2 = \{p+1,\ldots,n\}$. Removing from G_A the set of all arcs terminating in V_1 defines the graph $G(A_{12}/A_{22})$. That is,

$$G(A_{12}/A_{22}) = (V, E^*), \quad \text{where} \quad E^* = E \backslash \{(i,j) \in E | j \in V_1\}.$$

Now, conditions (i)−(iii) imply that there can be only 1- and 2-cycles in G_A, and by definition, only 2-cycles are possible in $G(A_{12}/A_{22})$. Moreover, irreducibility of A implies that each node of $G(A_{12}/A_{22})$ is accessible from V_1. Condition (v) implies, under conditions (ii) and (iii), that G_A is covered by a set of disjoint 1- or 2-cycles. Then, V_2 is covered with a set of disjoint 2-cycles and V_1-rooted arcs in $G(A_{12}/A_{22})$. This and the accessibility of $G(A_{12}/A_{22})$ from V_1 implies that $G(A_{12}/A_{22})$ is spanned by a *cacti* [7], [10]. The following example explains the above arguments.

EXAMPLE 1. *Fig. 1 is the graph G_A corresponding to a sign matrix A which satisfies conditions* (i)−(v). *A complete matching [11] is shown by thick lines. Here, $V_1 = \{1,2\}$ and $V_2 = \{3,4,\ldots,11\}$. Fig. 2 depicts $G(A_{12}/A_{22})$, with thick lines indicating a covering of V_2. Fig. 3 shows a cacti which spans $G(A_{12}/A_{22})$. This is immediately obtained from the covering of Fig. 2 by adding some connecting stems.*

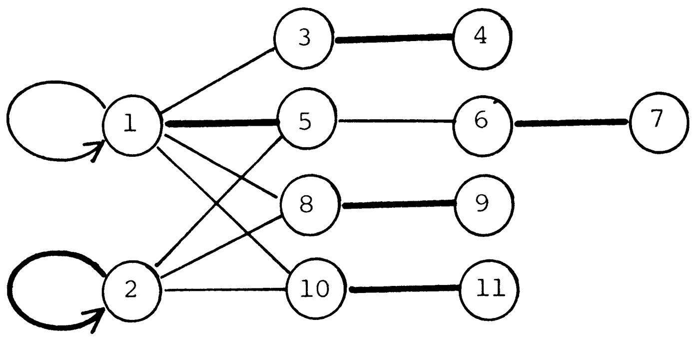

FIG. 1. Graph G_A (undirected arcs represent 2-cycles).

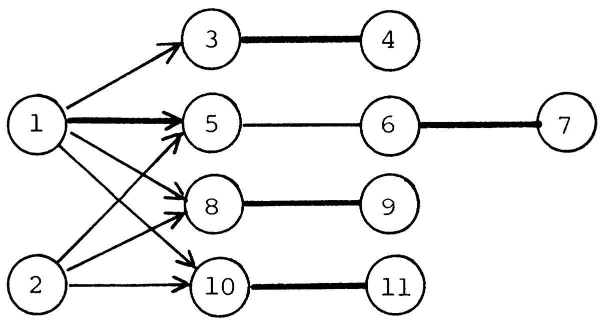

FIG. 2. Graph $G(A_{12}/A_{22})$.

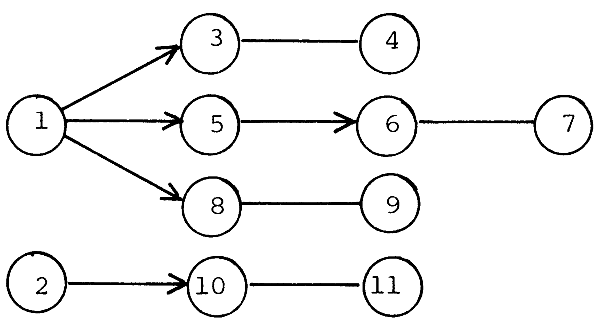

FIG. 3. A cacti spanning $G(A_{12}/A_{22})$.

By virtue of the well-known duality of controllability and observability in dynamical systems theory [8], the above result is exactly the condition for the pair (A_{22}, A_{12}) to be generically observable.[3]

[3]Using the notation of this paper, the structural controllability theorem says: (A, C) is generically observable *iff* $G(C'/A')$ is spanned by a cacti. See Lin [7] or Mayeda [10].

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PROOF OF THEOREM 2 (Sufficiency part). Since (A_{22}, A_{12}) is generically observable, there exists a proper algebraic variety in $R^{\mu(A)}$ such that $(\bar{A}_{22}(\xi), \bar{A}_{12}(\xi))$ is observable for any $\xi \in R^{\mu(A)+} \setminus V$. Fix such a point $\xi \in R^{\mu(A)+} \setminus V$ and take a corresponding matrix $\bar{A} = \bar{A}(\xi)$. By (ii), (iii) and irreducibility of A, it is possible to choose $\nu_1, \nu_2, \ldots, \nu_n$, such that $\nu_i a_{ij} = -\nu_j a_{ji}$, $i, j \in \boldsymbol{n}$, $i \neq j$, and $\nu_i > 0$, $i \in \boldsymbol{n}$. Define a quadratic function $V(x)$ by

$$V(x) = \sum_{i=1}^{n} \nu_i x_i^2$$

The derivative of $V(x(t))$ along the trajectory of (1) is given by

$$dV(x(t))/dt = 2 \sum_{i \in \boldsymbol{n}} \nu_i x_i \dot{x}_i = 2 \sum_{i,j \in \boldsymbol{n}} \nu_i x_i a_{ij} x_j = 2 \sum_{i \in \boldsymbol{n}} \nu_i a_{ii} x_i^2 = 2 \sum_{i=1}^{p} \nu_i a_{ii} x_i^2 \leq 0.$$

Therefore, $V(x(t))$ does not increase with the evolution of (1). Assume that this does not decrease along the path of (1). Then, we must have

$$x(t) \in \theta_p \overset{\triangle}{=} \{x \in R^n | x_1 = x_2 = \ldots x_p = 0\}$$

However, since $(\bar{A}_{22}, \bar{A}_{12})$ is observable, this implies that

$$x_{p+1}(t) = x_{p+2}(t) = \ldots = x_n(t) = 0,$$

and therefore $x(t) = 0$. Thus, $V(\cdot)$ is a Lyapunov function for (1), and therefore $\bar{A}$ is stable. This argument holds true for any $\xi \in R^{\mu(A)+} \setminus V$, which completes the proof of sufficiency.[4] □

EXAMPLE 2. *Consider the following sign matrix. Note that the counterexample of Jeffries [4] had this sign pattern.*

$$A = \begin{bmatrix} 0 & -a & 0 & 0 & 0 \\ b & 0 & -c & 0 & 0 \\ 0 & d & -e & -f & 0 \\ 0 & 0 & g & 0 & -h \\ 0 & 0 & 0 & j & 0 \end{bmatrix}$$

By Theorem 1, we know that the real part of any eigenvalue of A is non-positive for any choice of parameters $= (a, b, \ldots, j) \in R^{8+}$. Therefore, a non-negative real part for an eigenvalue of A can occur only when A has a pure imaginary eigenvalue. By appropriate scaling, we can assume that this is i. Therefore, if A has such an eigenvalue, we must have $\det(iI - A) = 0$, or, equivalently,

$$e(1 - ab)(1 - hj) + i\{1 - (ab + cd + fg + hj) + abfg + abjh + cdjk\} = 0$$

It can be easily seen that the above equality holds *iff* $ab = 1$ and $hj = 1$. Therefore, $\bar{A} = \bar{A}(\xi)$ is stable, unless $\xi \in V = \{\xi = (a, b, \ldots, j) | ab = 1, hj = 1\}$.

[4]This part of proof is essentially the same to the proof in [3], where Ishida et al. proved that (i)–(iii) plus *sign-observability* of (A_{22}, A_{12}) are both necessary and sufficient for A to be sign-stable. However, they did not give any structural characterization for (A_{22}, A_{12}) to be sign observable.

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Appendix

PROOF OF LEMMA. Suppose that such an S and $\bar{A} = (\bar{a}_{ij})$ exist. Let $\bar{A}[\epsilon] = (\bar{a}_{ij}[\epsilon]) \in Q_A$ be defined by

$$\bar{a}_{ij}[\epsilon] = \begin{cases} \bar{a}_{ij} & \text{if} \quad (i,j) \in S \\ \epsilon \bar{a}_{ij} & \text{otherwise} \end{cases}$$

and let $f(\lambda, \epsilon) = \det(\lambda I - \bar{A}[\epsilon])$. Then, for $\bar{\epsilon} = 0$, $\bar{\lambda}$ is a non-repeated solution to $f(\lambda, \bar{\epsilon}) = 0$. By the Implicit Function Theorem (See, e.g., 10.2.2 of [2]), there exists a continuous function $\bar{\lambda}(\cdot)$ defined on a neighborhood U of $\bar{\epsilon} = 0$, such that $\bar{\lambda} = \bar{\lambda}(0)$ and $f(\bar{\lambda}(\epsilon), \epsilon) = 0$ for $\epsilon \in U$. By continuity of $\bar{\lambda}(\cdot)$, we have $\text{Re}[\bar{\lambda}(\epsilon^*)] > 0$ for sufficiently small $\epsilon^* > 0$, since $\text{Re}[\bar{\lambda}(0)] > 0$.

Let $\xi^* \in R^{\mu(A)+}$ be the point corresponding to $\bar{A}[\epsilon^*]$, i.e., $\bar{A}(\xi^*) = \bar{A}[\epsilon^*]$ and consider a sufficiently small neighborhood U^* of ξ^*. Again, by continuity of non-repeated eigenvalues to the change of matrix entries, we obtain $\text{Re}[\bar{\lambda}(\xi)] > 0$ for all $\xi \in U^*$. This implies that A is not generically sign-semistable. □

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