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Motives of uniruled 3-folds

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Abstract. J. Murre has conjectured that every smooth projective variety X of dimension d admits a decomposition of the diagonal $\Delta = p_0 + \cdots + p_{2d} \in CH^d(X \times X) \otimes \mathbb{Q}$ such that the cycles p_i are orthogonal projectors which lift the Künneth components of the identity map in étale cohomology. If this decomposition induces an intrinsic filtration on the Chow groups of X, we call it a Murre decomposition. In this paper we propose candidates for such projectors on 3-folds by using fiber structures. Using Mori theory, we prove that every smooth uniruled complex 3-fold admits a Murre decomposition.

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1. Introduction

Let F be a subfield of $\mathbb{C}$. We denote by $V(F)$ the category of smooth, projective varieties over F with the usual morphisms. Let $CV(F)$ be the category with the same underlying object, but where the morphisms are replaced by correspondences of degree zero, i.e. for two irreducible varieties X, Y we have $\mathrm{Mor}(X, Y) := CH^{\dim(X)}(X \times Y)$. If $f \in \mathrm{Mor}(X, Y)$ we view it as a homomorphism $f_* : CH^*(X) \to CH^*(Y)$, by defining $f_*(W) = (pr_2)_*((W \times X) \cap f)$. Given $X_1, X_2, X_3 \in V(F)$ the composition of correspondences $f \in \mathrm{Mor}(X_1, X_2)$ and $g \in \mathrm{Mor}(X_2, X_3)$ is defined by

$$g \circ f = (pr_{13})_*\{(pr_{12})^* f \cap (pr_{23})^* g\}$$

An element $p \in \mathrm{Mor}(X, X)$ is called a **projector** if $p \circ p = p$. A special example is the diagonal, denoted by Δ. Finally denote by $M(F)$ the category of **effective Chow motives**, where objects are pairs (X, p) with $X \in V(F)$ and $p \in \mathrm{Mor}(X, X)$ a projector. The morphisms are described by $\mathrm{Mor}((X, p), (Y, q)) := q \circ \mathrm{Mor}(X, Y) \circ p$.

DEFINITION 1.1. Let $M = (X, p) \in M(F)$. Define

$$CH^i(M) := p_* CH^i(X) \otimes \mathbb{Q}$$

DEFINITION 1.2. Let $X \in V(F)$ be a smooth projective variety of dimension d. We say that X has a **Murre decomposition**, if there exist projectors $p_0, p_1, \ldots, p_{2d}$ in $CH^d(X \times X) \otimes \mathbb{Q}$ such that the following properties hold (modulo rational equivalence for (1) and (2)):

(1) $p_j \circ p_i = \delta_{i,j} \cdot p_i$
(2) $\Delta = \sum p_i$
(3) In cohomology the p_i induce the $(2d-i, i)$-th Künneth component of the diagonal.
(4) $p_0, \ldots, p_{j-1}$ and $p_{2j+1}, \ldots, p_{2d}$ act trivially on $CH^j(X) \otimes \mathbb{Q}$.
(5) If we put $F^0 CH^j(X) \otimes \mathbb{Q} = CH^j(X) \otimes \mathbb{Q}$ and inductively $F^k CH^j(X) \otimes \mathbb{Q} :=$ Ker$(p_{2j+1-k} \mid_{F^{k-1}})$, then this descending filtration is intrinsic, i.e. does not depend on the particular choice of the p_i.
(6) Always $F^1 CH^j(X) \otimes \mathbb{Q} = CH^j_{\text{hom}}(X) \otimes \mathbb{Q}$.

The motives (X, p_i) are traditionally denoted by $h^i(X)$ and we write $h(X) = h^0(X) + \cdots + h^{2d}(X)$. In (6) one also wants to have that $F^2 CH^j(X) \otimes \mathbb{Q}$ is the kernel of the cycle class map in rational Deligne cohomology, but this is very hard to verify in general.

(1)–(6) have been proved for curves, surfaces ([11]), products of a curve and a surface ([10]), abelian varieties ([2]) and certain varieties close to projective varieties. Recently B. Gordon and J. Murre [4] computed the Chow motive of elliptic modular varieties using work of A. Scholl [13].

S. Saito has proposed a filtration in [12] which has property (6). Manin ([8]) and Murre ([11]) have quite generally defined $p_0, p_1, p_{2d-1}, p_{2d}$ for every X. A. Scholl has refined this in [13] to have also the property that $p_i = p^{tr}_{2d-i}$, where p^{tr} denotes a transpose of a projector p. Murre has formulated the following

CONJECTURE: *Every smooth projective F-variety X admits a Murre decomposition.*

J. Murre ([10]) has studied the case of a product of a curve with a surface where one in fact has a Murre decomposition. Inspired by this, we have tried to construct projectors in the following situation: Let $f \colon Y \to S$ be a morphism from a smooth 3-fold Y to a smooth surface S with connected fibers. Choose a smooth hyperplane section $i \colon Z \hookrightarrow Y$ and let $h = f|_Z$. Look the following cycles

$$\pi_{i0} := \frac{1}{m}(i \times 1)_*(h \times f)^* \pi_i(S),$$

$$\pi_{i2} := \frac{1}{m}(1 \times i)_*(f \times h)^* \pi_i(S),$$

in $CH^3(Y \times Y) \otimes \mathbb{Q}$. Here the $\pi_i(S)$ are orthogonal projectors of a Murre decomposition of S as constructed by Murre ([11]) and m is the number of points on a

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general fiber of h. We are able to construct orthogonal projectors $\pi_0, \dots, \pi_6$ in the following way:

$$\pi_0 := \pi_{00}$$

$$\pi_1 := \pi_{10} - \frac{\pi_{10} \circ \pi_{02}}{2} - \frac{\pi_{10} \circ \pi_{22}}{2} - \frac{\pi_{10} \circ \pi_{32}}{2}$$

$$\pi_2 := \pi_{20} + \pi_{02} - \pi_{20} \circ \pi_{02} - \frac{\pi_{10} \circ \pi_{02}}{2} - \frac{\pi_{20} \circ \pi_{22}}{2} - \frac{\pi_{20} \circ \pi_{32}}{2}$$

$$\pi_4 := \pi_{40} + \pi_{22} - \pi_{40} \circ \pi_{22} - \frac{\pi_{10} \circ \pi_{22}}{2} - \frac{\pi_{20} \circ \pi_{22}}{2} - \frac{\pi_{40} \circ \pi_{32}}{2}$$

$$\pi_5 := \pi_{32} - \frac{\pi_{10} \circ \pi_{32}}{2} - \frac{\pi_{20} \circ \pi_{32}}{2} - \frac{\pi_{40} \circ \pi_{32}}{2}$$

$$\pi_6 := \pi_{42}$$

$$\pi_3 := \Delta - \sum_{i \neq 3} \pi_i.$$

The π_j do not operate in the right way on cohomology, but if all higher direct images sheaves $R^i f_* \mathcal{O}_Y$ vanish for $i \geqslant 1$, they can be modified to form a Murre decomposition. In particular a suitable blow up Y of any smooth **uniruled** 3-fold X over a subfield of the complex numbers has this property. Recall that a 3-fold X is called uniruled, if there exists a dominant rational map $\varphi\colon S \times \mathbb{P}^1 \dashrightarrow X$ for some smooth projective surface S. By a theorem of Mori and Miyaoka ([9]), this is equivalent to saying that X has Kodaira dimension $-\infty$. There is no structure theorem for these varieties which is as simple as in the case of ruled surfaces, but there is a version in the category of 3-folds with $\mathbb{Q}$-factorial and terminal singularities ([7]) stating that X is birationally equivalent to a 3-fold Y which has a fiber structure with rationally connected fibers over a base variety which can be a point, a smooth curve or a normal surface. Using this and suitable modifications of the projectors above we can therefore prove:

THEOREM 4.4. *Let X be a smooth uniruled complex projective 3-fold. Then X admits a Murre decomposition.*

We verify property (5) of a Murre decomposition in the sense that the induced filtration on $CH^*(X) \otimes \mathbb{Q}$ depends only on the geometry of the birational mapping $r\colon X \dashrightarrow Y$. In the proof of this theorem, which makes heavy use of Fulton's machinery of intersection theory, the Murre decomposition suggests the following description of the **Chow motive** of a complex uniruled 3-fold X (ignoring torsion):

Motive M	$h^0(X)$	$h^1(X)$	$h^2(X)$	$h^3(X)$	$h^4(X)$	$h^5(X)$	$h^6(X)$
$CH^0(M)$	$\mathbb{Z}$	0	0	0	0	0	0
$CH^1(M)$	0	$\mathrm{Pic}^0(X)$	$\mathrm{NS}(X)$	0	0	0	0
$CH^2(M)$	0	0	$\mathrm{Ker}(\psi)$	$\mathrm{Im}(\psi)$	$H^{2,2}(X,\mathbb{Z})$	0	0
$CH^3(M)$	0	0	0	0	$\mathrm{Ker}(\mathrm{alb}_X)$	$\mathrm{Alb}(X)$	$\mathbb{Z}$

However it remains to prove that $CH^2(h^2(X)) = \mathrm{Ker}(\psi)$ and $CH^2(h^3(X)) = \mathrm{Im}(\psi)$, where $\psi: CH^2_{\mathrm{hom}}(X) \to J^2(X)$ is the Abel-Jacobi map. We hope that our approach may also be used to construct projectors in other situations.

2. Projectors for special varieties

The easiest case in which one has a Murre decomposition is the case of projective space, because there $H^{2k+1}(X,\mathbb{C}) = 0$ for all $k \geqslant 0$ and the other groups admit a basis represented by algebraic cycles. One has a more general theorem:

THEOREM 2.1. *Let X be a smooth variety of dimension n and assume that for certain $1 \leqslant q \leqslant n-1$ there is a basis $\{E_1,\dots,E_t\}$ of $H^{2q}(X,\mathbb{Q})$ and a basis $\{\ell_1,\cdots,\ell_t\}$ of $H^{2(n-q)}(X,\mathbb{Q})$ represented by classes of algebraic cycles. Then:*

(a) *There exists a matrix $B = (b_{ij}) \in \mathbf{GL_n}(\mathbb{Q})$ such that the cycle $p = \sum b_{ij}(\ell_i \times E_j) \in CH^n(X \times X) \otimes \mathbb{Q}$ operates as the identity on $H^{2q}(X,\mathbb{Q})$.*

(b) *For the same choice of b_{ij}, $p^{tr} = \sum b_{ij}(E_j \times \ell_i) \in CH^n(X \times X) \otimes \mathbb{Q}$ operates as the identity on $H^{2(n-q)}(X,\mathbb{Q})$.*

(c) *Both cycles, p and p^{tr} are idempotent and therefore projectors.*

Proof. Let $A = (E_i \cdot \ell_j)$ be the intersection matrix, then take $B = A^{-1}$. □

Moreover, one can explicitely say how these projectors operate on cycles, namely:

PROPOSITION 2.2. *Let p be as before and let $k \neq q$. Then, for all $Z \in CH^k(X) \otimes \mathbb{Q}$ one has $p(Z) = 0$ as an element of $CH^k(X) \otimes \mathbb{Q}$.*

Proof. By dimension reasons, as $p(Z) \in \langle E_i \rangle \subset CH^q(X) \otimes \mathbb{Q}$. □

LEMMA 2.3. *Let p be as before and $Z \in CH^q(X) \otimes \mathbb{Q}$. If $[Z]$ denotes the homology class of Z on $H^{2q}(X,\mathbb{Q})$, then $[p(Z)] = p([Z]) = [Z]$.*

Proof. p operates as the identity on $H^{2q}(X,\mathbb{Q})$. □

COROLLARY 2.4. *Let p be as before, then $(\mathrm{Ker}\ p) \cap CH^q(X) \otimes \mathbb{Q} = CH^q_{\mathrm{hom}}(X) \otimes \mathbb{Q}$.*

Proof. $p(Z) = \sum b_{ij}(\ell_i \cdot Z)E_j$. □

EXAMPLES: Smooth Fano 3-folds and Calabi-Yau 3-folds have the property that the Hodge numbers $h^{i,0}$ are always zero for $i = 1, 2$ and therefore theorem 2.1 applies. Another example is a del Pezzo fibration $f: X \to B$: to illustrate this, let ℓ be the extremal rational curve, F a general fiber, Y be a section of $|-mK_X|$, C a twofold intersection in the linear system $|Y|$ and hence a multisection of f over B, such that C is a smooth curve dominating B. $H^2(X, \mathbb{Q})$ is free of rank two. Then theorem 2.1 produces the following projector

$$p_2 := \frac{1}{r}(C \times F) + \frac{1}{m}(\ell \times Y) - \frac{d}{m \cdot r}(\ell \times F),$$

where $d = Y^3$ and $r := (C.F)$. Note that $(-K_X.\ell) = 1$. p_2 is unique as a cycle up to the choices of Y, C, F and ℓ.

3. Murre decompositions of birational conic bundles

Let $f: Y \to S$ be a morphism from a smooth projective 3-fold Y to a smooth projective surface S, such that every fiber of f is rationally connected and the general fiber of f is isomorphic to $\mathbb{P}^1$. Choose a smooth hyperplane section $i: Z \hookrightarrow Y$ such that $h := f_{|Z}: Z \to S$ is surjective and generically finite. Then define cycles

$$\pi_{i0} := \frac{1}{m}(i \times 1)_*(h \times f)^* \pi_i(S),$$

$$\pi_{i2} := \frac{1}{m}(1 \times i)_*(f \times h)^* \pi_i(S),$$

in $CH^3(Y \times Y) \otimes \mathbb{Q}$ for $0 \leqslant i \leqslant 4$. Here the $\pi_i(S)$ are the orthogonal projectors of a Murre decomposition of S as constructed by Murre ([11]) (and improved by A. Scholl in [13] to have also the property that $\pi_i = \pi_{4-i}^{tr}$) and m is the number of points on a general fiber of h. The following is our **key result** in some sense:

LEMMA 3.1.

(a) $\pi_{i0} \circ \pi_{j0} = \delta_{ij}\pi_{i0}$
(b) $\pi_{i2} \circ \pi_{j2} = \delta_{ij}\pi_{i2}$
(c) $\pi_{j2} \circ \pi_{i0} = 0$
(d) $\pi_{40} \circ \pi_{02} = \pi_{00} \circ \pi_{j2} = \pi_{i0} \circ \pi_{42} = 0$

Proof. (a) Using the projection formula and the theory of Gysin maps for l.c.i. morphisms from [3, prop. 6.6(c)] in the following diagram

$$\begin{array}{ccc} Y \times Y \times Y & \to & Y \times Y \\ \uparrow & & \uparrow \\ Z \times Y \times Y & \to & Z \times Y \\ \downarrow & & \downarrow \\ Z \times S \times Y & \to & Z \times Y \\ \downarrow & & \downarrow \\ S \times S \times S & \to & S \times S \end{array}$$

where the vertical maps are canonical l.c.i. morphisms, one obtains:

$$\begin{aligned}
\pi_{i0} \circ \pi_{j0} &= \frac{1}{m^2} (pr_{13}^{Y \times Y \times Y})_* ((i \times 1)_* ((h \times f)^* (\pi_j(S)) \times Y \cap Y \\
&\quad \times (i \times 1)_* ((h \times f)^* (\pi_i(S)))) \\
&= \frac{1}{m^2} (pr_{13}^{Y \times Y \times Y})_* ((i \times 1 \times 1)_* (h \times f \times f)^* (\pi_j(S) \times S) \\
&\quad \cap (1 \times i \times 1)_* (f \times h \times f)^* (S \times \pi_i(S))) \\
&= \frac{1}{m^2} (pr_{13}^{Y \times Y \times Y})_* (i \times 1 \times 1)_* [(h \times f \times f)^* (\pi_j(S) \times S) \\
&\quad \cap (i \times 1 \times 1)^* (1 \times i \times 1)_* (f \times h \times f)^* (S \times \pi_i(S))] \\
&= \frac{1}{m^2} (i \times 1)_* (pr_{13}^{Z \times Y \times Y})_* [(h \times f \times f)^* (\pi_j(S) \times S) \\
&\quad \cap (1 \times i \times 1)_* (i \times 1 \times 1)^* (f \times h \times f)^* (S \times \pi_i(S))] \\
&= \frac{1}{m^2} (i \times 1)_* (pr_{13}^{Z \times Y \times Y})_* [(h \times f \times f)^* (\pi_j(S) \times S) \\
&\quad \cap (1 \times i \times 1)_* (h \times h \times f)^* (S \times \pi_i(S))] \\
&= \frac{1}{m^2} (i \times 1)_* (pr_{13}^{Z \times Y \times Y})_* (1 \times i \times 1)_* [(1 \times i \times 1)^* \\
&\quad \times (h \times f \times f)^* (\pi_j(S) \times S) \cap (h \times h \times f)^* (S \times \pi_i(S))] \\
&= \frac{1}{m^2} (i \times 1)_* (pr_{13}^{Z \times S \times Y})_* (1 \times h \times 1)_* (h \times h \times f)^* \\
&\quad \times [(\pi_j(S) \times S) \cap (S \times \pi_i(S))]
\end{aligned}$$

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$$= \frac{1}{m}(i \times 1)_*(pr_{13}^{Z\times S\times Y})_*(h \times 1 \times f)^*[(\pi_j(S) \times S)$$

$$\cap(S \times \pi_i(S))]$$

$$= \frac{1}{m}(i \times 1)_*(h \times f)^*(pr_{13}^{S\times S\times S})_*(\pi_j(S) \times S) \cap (S \times \pi_i(S))$$

$$([3,\ \ \text{prop.6.6(c)}])$$

$$= \frac{1}{m}(i \times 1)_*(h \times f)^*(\pi_i(S) \circ \pi_j(S)) = \delta_{ij}\pi_{i0}.$$

Similarly one proves (b).

(c) As before, one finds that

$$\pi_{j2} \cdot \pi_{i0} = \frac{1}{m^2}(i \times i)_*(pr_{13}^{Z\times S\times Z})_*(1 \times f \times 1)_*[(1 \times f \times 1)^*$$

$$\times(h \times 1 \times h)^*(\pi_i(S) \times S \cap S \times \pi_j(S)) \cap (Z \times Y \times Z)]$$

$$= \frac{1}{m^2}(i \times i)_*(pr_{13}^{Z\times S\times Z})_*[(h \times 1 \times h)^*(\pi_i(S) \times S \cap S$$

$$\times\pi_j(S)) \cap (1 \times f \times 1)_*(Z \times Y \times Z)] = 0$$

because $(1 \times f \times 1)_*(Z \times Y \times Z) = 0$ due to dimension reasons.

(d) In a similar way these 3 identities follow for dimension reasons. □

Define now a set of cycles $\pi_0, \ldots, \pi_6$ in the following way:

$$\pi_0 := \pi_{00}$$

$$\pi_1 := \pi_{10} - \frac{\pi_{10} \circ \pi_{02}}{2} - \frac{\pi_{10} \circ \pi_{22}}{2} - \frac{\pi_{10} \circ \pi_{32}}{2}$$

$$\pi_2 := \pi_{20} + \pi_{02} - \pi_{20} \circ \pi_{02} - \frac{\pi_{10} \circ \pi_{02}}{2} - \frac{\pi_{20} \circ \pi_{22}}{2} - \frac{\pi_{20} \circ \pi_{32}}{2}$$

$$\pi_4 := \pi_{40} + \pi_{22} - \pi_{40} \circ \pi_{22} - \frac{\pi_{10} \circ \pi_{22}}{2} - \frac{\pi_{20} \circ \pi_{22}}{2} - \frac{\pi_{40} \circ \pi_{32}}{2}$$

$$\pi_5 := \pi_{32} - \frac{\pi_{10} \circ \pi_{32}}{2} - \frac{\pi_{20} \circ \pi_{32}}{2} - \frac{\pi_{40} \circ \pi_{32}}{2}$$

$$\pi_6 := \pi_{42}$$

$$\pi_3 := \Delta - \sum_{i\neq 3} \pi_i.$$

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COROLLARY 3.2. *The π_j defined above form a set of orthogonal projectors such that $\pi_k = \pi_{6-k}^{tr}$.*

THEOREM 3.3.

$$\pi_i = \delta_{ij} on \begin{cases} f^*H^j(S,\mathbb{Q}) \quad \textit{if} \quad j=0,1 \\ f^*H^j(S,\mathbb{Q}) \oplus \mathbb{Q}\cdot[Z] \quad \textit{if} \quad j=2 \\ f^*H^j(S,\mathbb{Q}) \oplus [Z]\cdot f^*H^2(S,\mathbb{Q}) \quad \textit{if} \quad j=4 \\ [Z]\cdot f^*H^3(S,\mathbb{Q}) \quad \textit{if} \quad j=5 \\ [Z]\cdot f^*H^4(S,\mathbb{Q}) \quad \textit{if} \quad j=6. \end{cases}$$

Proof. First note that one has the equation:

$$\begin{aligned}
\pi_{i0}(f^*\alpha) &= \frac{1}{m}(i\times 1)_*(h\times f)^*\pi_i(S)(f^*\alpha) \\
&= \frac{1}{m}(pr_2^{Y\times Y})_*[(i\times 1)_*(h\times f)^*\pi_i(S)\cap(f^*\alpha\times Y)] \\
&= \frac{1}{m}(pr_2^{Y\times Y})_*(i\times 1)_*[(h\times f)^*\pi_i(S)\cap(i\times 1)^*(f^*\alpha\times Y)] \\
&= \frac{1}{m}(pr_2^{Y\times Y})_*(i\times 1)_*(h\times f)^*[\pi_i(S)\cap\alpha\times S] \\
&= \frac{1}{m}(pr_2^{Z\times Y})_*(h\times f)^*[\pi_i(S)\cap\alpha\times S] \\
&= \frac{1}{m}(pr_2^{S\times Y})_*(h\times 1)_*(h\times f)^*[\pi_i(S)\cap\alpha\times S)] \\
&= (pr_2^{S\times Y})_*(1\times f)^*[\pi_i(S)\cap\alpha\times S)] \\
&= f^*(pr_2^{S\times S})_*[\pi_i(S)\cap\alpha\times S] = f^*\pi_i(S)(\alpha).
\end{aligned}$$

Therefore π_{i0} operates as δ_{ij} on $f^*H^j(S)$, proving the assertion for π_0 and π_1. On the other hand, using projection formula, one gets

$$\begin{aligned}
\pi_{i2}(f^*\alpha) &= \frac{1}{m}(pr_2^{Y\times Y})_*[(1\times i)_*(f\times h)^*\pi_i(S)\cap(f^*\alpha\times Y) \\
&= \frac{1}{m}i_*(pr_2^{S\times Z})_*(f\times 1)_*[(f\times 1)^*(1\times h)^*(\pi_i(S) \\
&\quad \cap(\alpha\times S))\cap(Y\times Z)]
\end{aligned}$$

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$$= \frac{1}{m} i_*(pr_2^{S\times Z})_*[(1\times h)^*(\pi_i(S) \\ \cap(\alpha\times S))\cap(f\times 1)_*(Y\times Z)] = 0,$$

since $(f\times 1)_*(Y\times Z)=0$.

Take any $D\in H^k(S,\mathbb{Q})$ with $k=0,2,3,4$ and consider $C := i_*h^*(D)$. Observe that $[C]=f^*(D)\cdot[Z]$. The same computation as above in cohomology shows that

$$\pi_{i2}([C]) =: \frac{1}{m}(pr_2^{Y\times Y})_*[(1\times i)_*(f\times h)^*\pi_i(S)\cap[C]\times[Y]] \\ = i_*h^*(\pi_i(S)(D)).$$

As the $\pi_i(S)$ induce the Künneth decomposition of Δ_S on cohomology, it follows that $\pi_i(S)([D])=\delta_{ik}([D])$ and therefore one gets $\pi_{i2}([C])=\delta_{ik}[C]$.

Moreover, a similar argument together with Chow's moving lemma shows that

$$\begin{aligned}\pi_{i0}([C]) &= \frac{1}{m}(pr_2^{Y\times Y})_*[(i\times 1)_*(h\times f)^*\pi_i(S)\cap[C]\times[Y]]\\ &= \frac{1}{m}(pr_2^{Y\times Y})_*(i\times 1)_*[(h\times f)^*\pi_i(S)\cap(i\times 1)^*[C]\times[Y]]\\ &= \frac{1}{m}(pr_2^{Z\times Y})_*[(h\times f)^*\pi_i(S)\cap[C\cap Z]\times[Y]]\\ &= \frac{1}{m}(pr_2^{S\times Y})_*(h\times 1)_*[(h\times 1)^*(1\times f)^*\pi_i(S)\cap[C\cap Z]\times[Y]]\\ &= \frac{1}{m}(pr_2^{S\times Y})_*[(1\times f)^*\pi_i(S)\cap h_*[C\cap Z]\times[Y]]\\ &= \frac{1}{m}f^*(pr_2^{S\times S})_*[\pi_i(S)\cap h_*[C\cap Z]\times[S]]\\ &= \frac{1}{m}f^*\pi_i(S)(h_*[C\cap Z]) = 0,\end{aligned}$$

if $i\neq k+2$. As a consequence one also gets $\pi_{i0}\circ\pi_{j2}([C])=\delta_{jk}\pi_{i0}([C])$, which proves the assertion for π_2,π_4,π_5 and π_6 and the theorem. □

Now assume additionally that $f\colon Y\to S$ is a desingularization of a conic bundle morphism $f'\colon X'\to S'$ in the sense of [7], i.e. there is a commutative diagram

$$\begin{array}{ccc} Y & \xrightarrow{f} & S \\ \downarrow{\scriptstyle\sigma} & & \downarrow{\scriptstyle\tau} \\ X' & \xrightarrow{f'} & S', \end{array}$$

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with blow-up morphisms σ, τ. Also we assume $Z \subset Y$ is a sufficiently general smooth hyperplane section of Y that dominates S.

Then we can choose irreducible divisors $H_1, ..., H_r$ in Y such that $H_1 = Z$ and

$$H^{1,1}(Y, \mathbb{Q}) = \bigoplus_{i=1}^{r} \mathbb{Q}[H_i],$$

form a basis of $H^{1,1}(Y, \mathbb{Q})$ and such that $f_* H_i = 0$ in $CH^0(S)$ for $i \geqslant 2$, i.e. H_i is exceptional with respect to f for $i \geqslant 2$.

LEMMA 3.4. *For every cycle W one has $\pi_{20}(W) = \frac{1}{m} f^* \pi_2(S)(h_*(W \cap Z)) \in f^* CH^*(S) \otimes \mathbb{Q}$. Let W be a cycle with $f_*(W) = 0$. Then $\pi_{02}(W) = 0$ already in the Chow group of Y.*

Proof.

$$\pi_{02}(W) = \frac{1}{m}(pr_2^{Y \times Y})_* [(1 \times i)_* (f \times h)^* \pi_0(S) \cap (W \times Y)]$$

$$= \frac{1}{m} i_* (pr_2^{S \times Z})_* [(1 \times h)^* \pi_0(S) \cap (f \times 1)_* (W \times Z)] = 0$$

by [3, prop. 6.6(c)] and since $f_*(W) = 0 \in CH^*(S)$.
On the other hand

$$\pi_{20}(W) = \frac{1}{m}(pr_2^{Y \times Y})_* [(i \times 1)_* (h \times f)^* \pi_2(S) \cap (W \times Y)]$$

$$= \frac{1}{m}(pr_2^{Z \times Y})_* [(h \times f)^* \pi_2(S) \cap ((W \cap Z) \times Y)]$$

$$= \frac{1}{m}(pr_2^{S \times Y})_* [(1 \times f)^* \pi_2(S) \cap (h \times 1)_* ((W \cap Z) \times Y)]$$

$$= \frac{1}{m}(pr_2^{S \times Y})_* (1 \times f)^* [\pi_2(S) \cap h_*(W \cap Z) \times S)]$$

$$= \frac{1}{m} f^* (pr_2^{S \times S})_* [\pi_2(S) \cap h_*(W \cap Z) \times S]$$

$$= \frac{1}{m} f^* \pi_2(S)(h_*(W \cap Z)) \in f^* CH^*(S) \otimes \mathbb{Q}. \qquad \square$$

COROLLARY 3.5.

$$\pi_2(Y)(H_i) = \frac{1}{m} f^*(h_*(H_i \cap Z)) \in f^*CH^1(S) \otimes \mathbb{Q} \quad \textit{for } i \geqslant 2.$$

By theorem 3.3 $\pi_2(Y)$ operates as zero on $\mathrm{Pic}^0(Y)$, therefore the image of $\pi_2(Y)$ in $CH^1(Y) \otimes \mathbb{Q}$ is a finite dimensional vector space. By changing our generators H_i above modulo classes in $\mathrm{Pic}^0(Y) = f^*\mathrm{Pic}^0(S)$, we may assume that they generate $\mathrm{Im}(\pi_2) \subset CH^1(Y) \otimes \mathbb{Q}$. Then we write uniquely

$$\pi_2(Y)(H_i) = \sum_k a_{i,k} H_k \in CH^1(Y) \otimes \mathbb{Q},$$

with a matrix $A = (a_{i,k}) \in \mathrm{Mat}(r \times r, \mathbb{Q})$. $\pi_2(Y)$ being a projector implies that $A^2 = A$. Choose algebraic cycles $\ell_1, ..., \ell_r$ such that $\ell_1 = F$, a general fiber of f, and such that their cohomology classes form a basis of $H^{2,2}(Y, \mathbb{Q})$. By Poincaré duality the intersection matrix $M = (m_{i,j}) := (\ell_1, \dots, \ell_r)^T (H_1, \dots, H_r)$ has nonzero determinant.
We define

$$q_2 := \pi_2(Y) + \sum b_{i,j}(\ell_i \times H_j) - \sum b_{i,j}(\ell_i \times H_j) \circ \pi_2,$$

with some matrix $B = (b_{i,j}) \in \mathrm{Mat}(r \times r, \mathbb{Q})$.

LEMMA 3.6. *If $B = M^{-1}(\mathbf{1} - A)$, then q_2 is a projector and operates as the identity on $H^2(Y, \mathbb{Q})$.*

Proof. π_2 acts as the identity on $f^*H^2(S, \mathbb{Q})$ by theorem 3.3. The higher direct images $R^i f_* \mathcal{O}_Y$ vanish for $i \geqslant 1$ by [7]. Therefore by the Leray spectral sequence $H^2(Y, \mathcal{O}_Y) = f^*H^2(S, \mathcal{O}_S)$ and it is enough to show that q_2 operates as the identity on $H^{1,1}(Y, \mathbb{Q})$ too. But q_2 acts via the matrix $MB + A + BA$ on $H^{1,1}(Y, \mathbb{Q})$ with respect to the basis $\{H_i\}$. Now $\pi_2^2 = \pi_2$ and we get $A^2 = A$ and therefore $BA = 0$. By definition of B, we obtain that $MB + A + BA = M(M^{-1}(\mathbf{1} - A)) + A = \mathbf{1}$.

To show that q_2 is a projector, let us write $q_2 = \pi_2 + \beta - \beta\pi_2$. Note that $\beta\beta = \beta$, since $BMB = B$. From $BA = 0$ we deduce that $\pi_2\beta = 0$. Therefore

$$\begin{aligned} q_2 \circ q_2 &= \pi_2^2 + \beta^2 + \beta\pi_2\beta\pi_2 + \pi_2\beta - \pi_2\beta\pi_2 \\ &\quad +\beta\pi_2 - \beta\beta\pi_2 - \beta\pi_2\pi_2 - \beta\pi_2\beta \\ &= \pi_2 + \beta - \beta\pi_2 = q_2 \end{aligned}$$

is a projector. □

THEOREM 3.7. *The following cycles* $p_0(Y) := \pi_0(Y), p_1(Y) := \pi_1(Y)$, $p_2(Y) := q_2 - \pi_1(Y) \circ \sum b_{i,j}(\ell_i \times H_j) - \pi_1(Y) \circ \sum b_{i,j}(\ell_i \times H_j) \circ \pi_2(Y) p_4 := p_2^{tr}(Y), p_5(Y) := \pi_5(Y), p_6(Y) := \pi_6(Y), p_3(Y) := \Delta - \sum_{i\neq 3} p_i$ *define projectors, which satisfy properties* (1), (3), (4) *and* (6) *of a Murre decomposition. Property* (5) *holds in the following sense:* $F^1CH^i(Y) \otimes \mathbb{Q} = CH^i_{\text{hom}}(Y) \otimes \mathbb{Q}$, $F^2CH^2(Y) \otimes \mathbb{Q} \cong f^*\text{Ker}(\text{alb}_S) \subset CH^2_{\text{AJ}}(Y) \otimes \mathbb{Q}$ *(kernel of Abel-Jacobi map) and* $F^2CH^3(Y) \otimes \mathbb{Q} \cong \text{Ker}(\text{alb}_Y)$. *Moreover* $p_0(Y), p_1(Y), p_2(Y)$ *are mutually orthogonal.*

Proof. By lemma 3.6 above, (1),(2) and (3) are straightforward.

To prove (4),(5) and (6) for $j = 1$, note that $\text{Pic}(Y) \otimes \mathbb{Q} = f^*\text{Pic}^0(S) \otimes \mathbb{Q} \oplus \bigoplus_i \mathbb{Q} \cdot H_i$. By theorem 3.3 above, p_1 operates on $\text{Pic}^0(Y) \otimes \mathbb{Q} = f^*\text{Pic}^0(S) \otimes \mathbb{Q}$ as the identity and trivially on $\bigoplus_i \mathbb{Q} \cdot H_i$. Vice versa p_2 is the identity on $\bigoplus_i \mathbb{Q} \cdot H_i$ and zero on $f^*\text{Pic}^0(S) \otimes \mathbb{Q}$, because it acts trivially on $f^*H^1(S, \mathbb{Q})$. All the other projectors are zero on $CH^1(Y) \otimes \mathbb{Q}$. Therefore we get (4)–(6) for $j = 1$ with $F^2CH^1(Y) \otimes \mathbb{Q} = 0$.

For $j = 2$, property (4) follows from the analogous assertion for S. By construction $F^1CH^2(Y) \otimes \mathbb{Q} = \text{Ker}(p_4) = CH^2_{\text{hom}}(Y) \otimes \mathbb{Q}$. Then $F^2CH^2(Y) \otimes \mathbb{Q} = \text{Ker}(p_3) \cap \text{Ker}(p_4) = \text{Im}(p_2) = \text{Im}(\pi_2(Y))$.

Now we show that $F^2CH^2(Y) \otimes \mathbb{Q} \cong f^*F^2CH^2(S) \otimes \mathbb{Q} \subset CH^2_{\text{AJ}}(Y) \otimes \mathbb{Q}$: π_{02} operates as zero on $CH^2(Y)$ by Chow's moving lemma and if C is any curve homologous to zero on Y, then by Lemma 3.4, $\pi_{20}(C) = f^*h_*(C \cap Z) \in f^*F^2CH^2(S) \otimes \mathbb{Q}$.

This proves that $F^2CH^2(Y) \otimes \mathbb{Q} \subset f^*F^2CH^2(S) \otimes \mathbb{Q}$, but since $\pi_2(Y)$ operates as the identity on every fiber of f, we get equality. This is then independent of all choices, because this is the case for $F^2CH^2(S)$ by [11]. Finally $F^3CH^2(Y) \otimes \mathbb{Q} = 0$, since p_2 acts as the identity on $F^2CH^2(Y) \otimes \mathbb{Q} = \text{Im}(p_2)$. Hence we get (5) and (6) for $j = 2$.

Finally consider $CH^3(Y)$: Clearly $F^1CH^3(Y) \otimes \mathbb{Q} = \text{Ker}(\pi_6) = CH^3_{\text{hom}}(Y) \otimes \mathbb{Q}$. Further $F^2CH^3(Y) \otimes \mathbb{Q} = \text{Ker}(\pi_5|_{F^1CH^3(Y)\otimes\mathbb{Q}})$ and we claim that $F^2CH^3(Y) \otimes$

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$\mathbb{Q} \cong \mathrm{Ker}(\mathrm{alb}_Y) \otimes \mathbb{Q}$, where $\mathrm{alb}_Y : CH^3(Y)_{\mathrm{hom}} \to \mathrm{Alb}(Y)$ is the Albanese map. But there is a commutative diagram

$$\begin{array}{ccc} CH^3(Y)_{\mathrm{hom}} & \longrightarrow & \mathrm{Alb}(Y) \\ {\scriptstyle f_*}\downarrow & & \downarrow{\scriptstyle f_*} \\ CH^2(S)_{\mathrm{hom}} & \longrightarrow & \mathrm{Alb}(S). \end{array}$$

Both vertical maps are isomorphisms. To compute $F^2CH^3(Y) \otimes \mathbb{Q}$ we take any closed point P in Y and compute that $f_*\pi_5(P) = f_*\frac{1}{m}i_*h^*(\pi_3(S)(P)) = \pi_3(S)(f_*(P))$.

This shows that $f_*F^2CH^3(Y) \otimes \mathbb{Q} \cong F^2CH^2(S) \otimes \mathbb{Q} \cong \mathrm{Ker}(\mathrm{alb}_S) \otimes \mathbb{Q}$ by [11]. Therefore $F^2CH^3(Y) \otimes \mathbb{Q} \cong \mathrm{Ker}(\mathrm{alb}_Y) \otimes \mathbb{Q}$, which is independent of all choices again by [11]. Finally $F^3CH^3(Y) \otimes \mathbb{Q} = 0$, since if $P = \sum a_iP_i$ is a zero cycle on Y with $\sum a_i = 0$, then $f_*\pi_4(P) = f_*\pi^t_{20}(P) + f_*\pi^t_{02}(P) = f_*\frac{1}{m}(1\times i)_*(f\times h)^*\pi_2(S)(P) + f_*\frac{1}{m}(i\times 1)_*(h\times f)^*\pi_4(S)(P)$. But $\pi_4(S) = S\times e$, hence the last term is zero and the first term becomes $\pi_2(S)(f_*P)$. But $\pi_2(S)$ acts as the identity on $F^2CH^2(S)\otimes\mathbb{Q}$. Thus $f_*F^3CH^3(Y)\otimes\mathbb{Q} \subset F^3CH^2(S)\otimes\mathbb{Q} = 0$.

This finishes the proof of the theorem. □

Remark. Using a non-commutative version of the Gram-Schmidt process ([11, remark 6.5.]), one can always modify $p_4(Y), p_5(Y), p_6(Y)$ such that $p_0(Y), \ldots, p_6(Y)$ are orthogonal.

4. Murre decompositions of uniruled 3-folds

Let $k = \mathbb{C}$. By a 3-fold we just mean a normal 3-dimensional complex variety.

DEFINITION 4.1. A 3-fold X is called **uniruled**, if there exists a dominant rational map $\varphi\colon S \times \mathbb{P}^1 \dashrightarrow X$ for some surface S.

THEOREM 4.2 (9). *A smooth projective 3-fold X is uniruled if and only if it has Kodaira dimension $-\infty$, i.e. no multiple of K_X has sections.*

THEOREM 4.3 (7). *Let X be a uniruled 3-fold with only $\mathbb{Q}$-factorial terminal singularities. Then there exists a birational mapping $r : X \dashrightarrow Y$ which is a composition of flips and divisorial contractions, such that Y has an extremal ray R whose extremal contraction map $f : Y \to Z$ satisfies one of the following cases:*

(a) $\dim(Z) = 0, Y$ *is a $\mathbb{Q}$-Fano 3-fold with $\rho(Y) = 1$, i.e. $-mK_Y$ is an ample Cartier divisor for some $m \geqslant 1$ and the divisor class group is free with one generator.*

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(b) *Z is a smooth curve and Y is a del Pezzo fibration over Z, i.e. the general fibre of f is a del Pezzo surface.*

(c) *Z is a surface with at most quotient singularities and Y is a conic bundle over Z. In cases (b) and (c) the reduced preimage of any irreducible divisor is again irreducible.*

THEOREM 4.4. *Let X be a smooth complex uniruled 3-fold. Then X admits a Murre decomposition.*

Remark. We verify property (5) of a Murre decomposition in the sense that the induced filtration on $CH^*(X) \otimes \mathbb{Q}$ depends only on the geometry of the birational mapping $r: X \dashrightarrow Y$.

Proof. Since X is uniruled, it is birational to one of the following varieties:

(a) A $\mathbb{Q}$-Fano 3-fold Y with $\rho(Y) = 1$, i.e. $-mK_Y$ is an ample Cartier divisor for some $m \geqslant 1$ and the divisor class group is free with one generator.

(b) A del Pezzo fibration over a smooth curve.

(c) A conic bundle over a normal surface with at most quotient singularities.

In cases (a), (b) $H^2(X, \mathbb{Q})$ and $H^4(X, \mathbb{Q})$ are generated by classes of algebraic cycles. Thus we define $p_0(X) = \{e\} \times X$ and $p_6(X) = X \times \{e\}$ for some rational point $e \in X$, $p_1(X)$ and $p_5(X)$ as in [11] and $p_2(X)$ and $p_4(X) = p_2(X)^{tr}$ as in theorem 2.1. Then it is immediate to verify all properties (2)-(6) similar to the proof of 3.7 while property (1) can be achieved like in [11, remark 6.5.], by the non-commutative Gram–Schmidt process.

In case (c) we may assume that after blowing up X along several smooth subvarieties, there is a situation as in the previous section:

Let $\varphi: Y \to X$ be the blow-up and assume that $f: Y \to S$ is a morphism to a smooth surface S with rationally connected fibers. Take the projectors $p_0(Y), \dots,$ $p_6(Y)$ as defined in the last section.

To define the projectors for X, consider the graph $\Gamma_\varphi \subset Y \times X$ of φ. Define

$$p_i(X) := \Gamma_\varphi \circ p_i(Y) \circ \Gamma_\varphi^{tr} = (\varphi \times \varphi)_*(p_i(Y)),$$

(by Liebermann's lemma [6]) for $0 \leqslant i \leqslant 2$. We claim that all $p_i(X)$ are orthogonal projectors.

By induction on the number of blow-ups we may assume that there is just one blow-up along a smooth subvariety $W \subset X$.

Consider the canonical diagram

$$\begin{array}{ccc} Y \times Y \times Y & \xrightarrow{pr_{13}} & Y \times Y \\ \big\downarrow & & \big\downarrow \\ X \times Y \times X & \xrightarrow{pr_{13}} & X \times X \end{array},$$

where the vertical maps are $\varphi \times 1 \times \varphi$ and $\varphi \times \varphi$. Let E be the exceptional divisor. Then we compute for $0 \leqslant i, j \leqslant 2$:

$$\begin{aligned}
&p_i(X) \circ p_j(X)\\
&\quad = (pr_{13})_*((\varphi \times \mathrm{id})_* p_j(Y) \times X \cap X \times (\mathrm{id} \times \varphi)_* p_i(Y))\\
&\quad = (\varphi \times \varphi)_*(pr_{13})_*(p_j(Y) \times Y \cap Y \times (\mathrm{id} \times \varphi)^*(\mathrm{id} \times \varphi)_* p_i(Y))\\
&\quad = (\varphi \times \varphi)_*(pr_{13})_*(p_j(Y) \times Y \cap Y \times (p_i(Y) + (\mathrm{id} \times j)_* Q_{i,j}))\\
&\quad = (\varphi \times \varphi)_*(pr_{13})_*(p_j(Y) \times Y \cap Y \times p_i(Y)) + (\varphi \times \varphi)_*(pr_{13})_*(p_j(Y)\\
&\qquad \times Y \cap Y \times (\mathrm{id} \times j)_* Q_{i,j})\\
&\quad = (\varphi \times \varphi)_*(p_i(Y) \circ p_j(Y) + (pr_{13})_*(p_j(Y) \times Y \cap Y \times (\mathrm{id} \times j)_* Q_{i,j})),
\end{aligned}$$

where $Q_{i,j} \in CH_3(Y \times E)$ and $j\colon E \hookrightarrow Y$ is the inclusion. Hence

$$\begin{aligned}
\mathcal{C}_i &:= p_i(X) \circ p_i(X) - p_i(X)\\
&\quad = (\varphi \times \mathrm{id})_*(pr_{13})_*(p_i(Y) \times X \cap Y \times (\mathrm{id} \times i)_*(\mathrm{id} \times \varphi^E)_* Q_{i,i})).
\end{aligned}$$

$p_i(Y) = \frac{1}{m}(i \times 1)_*(h \times f)^* \pi_i(S) + T_i$ with $T_0, T_1 = 0$ and $T_2 = \sum c_{ij}(\ell_i \times H_j) - \sum b_{i,j}(\ell_i \times H_j) \circ \pi_2(Y)$ for some integers $c_{i,j}, b_{i,j}$ which is supported on $(Z \times Y) \cup (\ell_i \times Y)$. Therefore $\mathcal{C}_i$ is supported on $\varphi(Z) \times W$. Here $i\colon W \to X$ is the inclusion and $\varphi^E\colon E \to W$ is the restriction of φ to E.

If W is a point, $\mathcal{C}_i = 0$ by dimension reasons. If W is a curve, $\mathcal{C}_i = a(\varphi(Z) \times W)$ with $a \in \mathbb{Z}$. But $\mathcal{C}_i = p_i(X) \circ p_i(X) - p_i(X)$ operates as zero on the cohomology class of every curve $T \in CH^2(X)$, since by Chow's moving lemma we can choose T to be disjoint from W and use that $p_i(Y)(T) = 0$ in cohomology for $i = 0, 1, 2$. Therefore $a = 0$ and $p_i(X)$ is a projector.

For $i \neq j$, $p_i(X) \circ p_j(X) = (\varphi \times \varphi)_*(pr_{13})_*(p_j(Y) \times Y \cap Y \times (\mathrm{id} \times j)_* Q_{i,j})$ since $p_i(Y)$ and $p_j(Y)$ are orthogonal. As above this implies that $p_i(X) \circ p_j(X)$ is supported on $\varphi(Z) \times W$ for all j. By the same argument with Chow's moving lemma for $CH^2(X)$ as before, $p_i(X) \circ p_j(X) = 0$.

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Now define

$$p_4(X) = p_2(X)^{tr}, p_5(X) = p_1(X)^{tr}, p_6(X) = p_0^{tr} \quad \text{and}$$

$$p_3(X) = \Delta - \sum_{i \neq 3} p_i(X)$$

Properties (3)–(6) follow from theorem 3.7 together with the split exact sequences ([3, prop. 6.7])

$$0 \to CH_k(W) \to CH_k(E) \oplus CH_k(X) \to CH_k(Y) \to 0$$

(1) and (2) can be obtained again via the Gram-Schmidt process. □

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