Bull. Aust. Math. Soc. **106** (2022), 102–112
doi:10.1017/S0004972721001258

AUTOMORPHISM AND OUTER AUTOMORPHISM GROUPS OF RIGHT-ANGLED ARTIN GROUPS ARE NOT RELATIVELY HYPERBOLIC

JUNSEOK KIM, SANGROK OH ✉ and PHILIPPE TRANCHIDA

(Received 14 September 2021; accepted 14 November 2021; first published online 24 January 2022)

Abstract

We show that the automorphism groups of right-angled Artin groups whose defining graphs have at least three vertices are not relatively hyperbolic. We then show that the outer automorphism groups are also not relatively hyperbolic, except for a few exceptional cases. In these cases, the outer automorphism groups are virtually isomorphic to either a finite group, an infinite cyclic group or $\mathrm{GL}_2(\mathbb{Z})$.

2020 *Mathematics subject classification*: primary 20F67; secondary 20F65.

Keywords and phrases: automorphism group, right-angled Artin group, relative hyperbolicity.

1. Introduction

Associated with a finite simplicial graph Γ whose vertex set and edge set are V and E, respectively, is the right-angled Artin group (RAAG) A_Γ which is defined by the group presentation:

$$A_\Gamma = \langle v \in V \mid [u, v] = 1 \text{ for } \{u, v\} \in E \rangle.$$

In these settings, Γ is said to be the defining graph of A_Γ. As extreme examples, RAAGs can be free abelian groups $\mathbb{Z}^n$, when the defining graphs are complete, or free groups $\mathbb{F}_n$, when the defining graphs have no edges. In contrast, generic RAAGs have interesting behaviours; for example, some of their subgroups may not be isomorphic to RAAGs. Subgroups of RAAGs, such as Bestvina–Brady groups [3], are actually quite wild and have been used to construct examples of groups with peculiar properties. For a brief introduction to RAAGs, we refer to Charney's note [5].

In this paper, we look at the automorphism and outer automorphism groups of A_Γ denoted by $\mathrm{Aut}(A_\Gamma)$ and $\mathrm{Out}(A_\Gamma)$, respectively (the inner automorphism group is denoted by $\mathrm{Inn}(A_\Gamma)$). Here, $\mathrm{Out}(\mathbb{Z}^n)$ will usually be identified with $\mathrm{GL}_n(\mathbb{Z})$. Even

The first and third authors were partially supported by Samsung Science and Technology Foundation grant No. SSTF-BA1702-01 and the National Research Foundation of Korea (NRF) grant funded by the Korean government (MSIT) (No. 2020R1C1C1A01006912).

though $\mathbb{Z}^n$ and $\mathbb{F}_n$ have a lot of opposite properties in the algebraic sense, $\mathrm{GL}_n(\mathbb{Z})$ and $\mathrm{Out}(\mathbb{F}_n)$ share many common properties: for example, both are virtually torsion-free, residually finite and have finite virtual cohomological dimension. Charney and Vogtmann extended these results to every $\mathrm{Out}(A_\Gamma)$ in their papers [6, 7].

Another interesting common feature of $\mathrm{GL}_n(\mathbb{Z})$ and $\mathrm{Out}(\mathbb{F}_n)$ is that they are not relatively hyperbolic, except when n is small enough, in which case they are actually hyperbolic. Anderson *et al.* [1] established a simple criterion for showing nonrelative hyperbolicity of groups generated by infinite order elements. Using this criterion, they proved that, as long as $n \geq 3$, $\mathrm{GL}_n(\mathbb{Z})$ and $\mathrm{Out}(\mathbb{F}_n)$, and even $\mathrm{Aut}(\mathbb{F}_n)$, are not relatively hyperbolic; see also Behrstock *et al.* [2]. It is then quite natural to ask whether $\mathrm{Aut}(A_\Gamma)$ and $\mathrm{Out}(A_\Gamma)$ are always never hyperbolic. This turns out to be true, except for a few cases.

Here are the two main theorems of this paper.

THEOREM 3.1. *If a finite simplicial graph Γ contains at least three vertices, then the automorphism group of the right-angled Artin group of Γ is not relatively hyperbolic.*

THEOREM 4.1. *If the outer automorphism group of a right-angled Artin group is infinite and relatively hyperbolic, then it is virtually cyclic or virtually isomorphic to* $\mathrm{GL}_2(\mathbb{Z})$.

We remark that even though $\mathrm{Aut}(A_\Gamma)$ is almost never relatively hyperbolic, Genevois proved in [9] that $\mathrm{Aut}(A_\Gamma)$ is acylindrically hyperbolic if and only if Γ is not a join and contains at least two vertices.

The definition and study of relatively hyperbolic groups come from the following observation: even when a group G fails to be hyperbolic, it might still exhibit hyperbolic behaviours if we look only 'outside' some proper subgroups, called parabolic subgroups. With this observation in mind, one obstruction for being relatively hyperbolic is the existence of a specific collection $\mathcal{A}$ of proper subgroups which are far from being hyperbolic (for example, free abelian subgroups) and are well-networked. The term 'well-networked' means that (1) the union of all the subgroups in $\mathcal{A}$ generates a finite index subgroup of G and (2) for any $A, A' \in \mathcal{A}$, there exists a sequence $A_1 = A, \ldots, A_n = A'$ such that $A_i \cap A_{i+1}$ is infinite. If we find such a collection of subgroups of G, then G is *never* relatively hyperbolic regardless of the choice of parabolic subgroups.

Following the above idea, the notion of the *commutativity graph* of a group G is recalled in Section 2.2 as a tool to show that G is not relatively hyperbolic. One of the main assumptions to define the commutativity graph is the existence of a (possibly infinite) generating set of G which consists of infinite order elements. However, there are finite order elements in the usual generating sets of $\mathrm{Aut}(A_\Gamma)$ and $\mathrm{Out}(A_\Gamma)$. To handle this problem, in Section 2.1, we find a finite index subgroup which is generated by a finite collection of infinite order elements. In Sections 3 and 4, we prove that $\mathrm{Aut}(A_\Gamma)$ and $\mathrm{Out}(A_\Gamma)$, respectively, are in general not relatively hyperbolic, by using those finite index subgroups and the fact that being (or not being) relatively hyperbolic is a quasi-isometry invariant.

2. Preliminaries

We always assume Γ is a finite simplicial graph with vertex set V. For a vertex $v \in V$, the *link* of v, denoted by $\mathrm{lk}(v)$, is the the full subgraph of Γ spanned by vertices adjacent to v. Similarly, the *star* of v, denoted by $\mathrm{st}(v)$, is the full subgraph of Γ spanned by vertices adjacent to v and v itself. We then say that $v \leq w$ if $\mathrm{lk}(v) \subset \mathrm{st}(w)$. This partial order induces an equivalence relation on V by setting $v \sim w$ if $v \leq w$ and $w \leq v$. The partial order then descends to a partial order on the collection of the equivalence classes of vertices by setting $[v] \leq [w]$ if for some, and thus all, representatives $v' \in [v]$ and $w' \in [w]$, we have $v' \leq w'$. A vertex $v \in V$ is maximal if any vertex w such that $v \leq w$ is actually equivalent to v.

2.1. (Outer) automorphism groups of RAAGs. A theorem which was conjectured by Servatius [12] and proved by Laurence [10] says that $\mathrm{Aut}(A_\Gamma)$ is generated by the following four finite classes of automorphisms.

(1) *Graph automorphisms*. An automorphism of Γ induces an automorphism of A_Γ because it preserves the edges of Γ and thus the relations of A_Γ. The automorphism obtained this way is called a *graph automorphism*.
(2) *Inversions*. An automorphism of A_Γ sending one generator v to its inverse v^{-1} is called an *inversion*.
(3) *Transvections*. Take two vertices v and w in Γ such that $v \leq w$. Then the automorphism sending v to vw and fixing all the other vertices is called a *right transvection* and is denoted by R_{vw}. We can similarly define a *left transvection* L_{vw} by sending v to wv and still fixing all the other vertices.
(4) *Partial conjugations*. Let C be a connected component of $\Gamma - \mathrm{st}(v)$, for a vertex $v \in \Gamma$. The automorphism defined by conjugating every vertex in C by v is called a *partial conjugation* and is denoted by P_v^C. If a component C of $\Gamma - \mathrm{st}(v)$ is composed of a single vertex w, we write P_v^w instead of $P_v^{\{w\}}$.

Transvections and partial conjugations have infinite order, in contrast to inversions and graph automorphisms which have finite order. The abelianisation map $A_\Gamma \to \mathbb{Z}^n$ for $n = |V|$ induces a homomorphism $\mathrm{Aut}(A_\Gamma) \to \mathrm{GL}_n(\mathbb{Z})$, whose restriction on the subgroup generated by inversions and graph automorphisms is injective; in particular, this subgroup is finite because its image in $\mathrm{GL}_n(\mathbb{Z})$ is finite. Let $\mathrm{Aut}^*(A_\Gamma)$ be the subgroup of $\mathrm{Aut}(A_\Gamma)$ generated only by transvections and partial conjugations.

LEMMA 2.1. $\mathrm{Aut}^*(A_\Gamma)$ *is a finite index normal subgroup of* $\mathrm{Aut}(A_\Gamma)$.

PROOF. The fact that $\mathrm{Aut}^*(A_\Gamma)$ is a normal subgroup can be shown by checking that the conjugate of a transvection or a partial conjugation by an inversion or a graph automorphism is still a transvection or a partial conjugation. Because the image of the quotient map $\mathrm{Aut}(A_\Gamma) \to \mathrm{Aut}(A_\Gamma)/\mathrm{Aut}^*(A_\Gamma)$ is generated by the images of graph automorphisms and inversions, its cardinality is smaller than or equal to that of the subgroup generated by inversions and graph automorphisms, which is finite. We can thus deduce that $\mathrm{Aut}^*(A_\Gamma)$ is of finite index. □

Under the quotient map $\mathrm{Aut}(A_\Gamma) \to \mathrm{Out}(A_\Gamma) := \mathrm{Aut}(A_\Gamma)/\mathrm{Inn}(A_\Gamma)$, $\mathrm{Out}(A_\Gamma)$ is generated by the images of graph automorphisms, inversions, transvections and partial conjugations. (Note that some images of partial conjugations may be trivial in $\mathrm{Out}(A_\Gamma)$.) Similarly, $\mathrm{Out}^*(A_\Gamma)$ is defined to be the subgroup generated by the images of transvections and partial conjugations in $\mathrm{Out}(A_\Gamma)$; it can be considered as the image of $\mathrm{Aut}^*(A_\Gamma)$ in $\mathrm{Out}(A_\Gamma)$. By the above lemma, $\mathrm{Out}^*(A_\Gamma)$ is also a finite index normal subgroup of $\mathrm{Out}(A_\Gamma)$.

2.2. Nonrelative hyperbolicity. We first recall the definition of relative hyperbolicity by Bowditch [4]. Let G be a finitely generated group and $\mathcal{H}$ a finite collection of proper finitely generated subgroups of G. Choose a finite generating set S of G and consider the Cayley graph $\Lambda = \Lambda(G, S)$. The *coned-off Cayley graph*$\hat{\Lambda}(G, \mathcal{H})$ is defined as follows: starting with the Cayley graph Λ, for each coset gH_i with $g \in G$, $H_i \in \mathcal{H}$, we add a vertex $v(gH_i)$ to Λ and connect $v(gH_i)$ by an edge to each vertex in gH_i. We then say that G is *relatively hyperbolic with respect to*$\mathcal{H}$ if:

(1) the coned-off Cayley graph $\hat{\Lambda}(G, \mathcal{H})$ is δ-hyperbolic; and
(2) $\hat{\Lambda}(G, \mathcal{H})$ is *fine* (this means that for each integer k, all edges e of $\hat{\Lambda}(G, \mathcal{H})$ are contained in finitely many simple cycles of length k).

Whenever $\hat{\Lambda}(G, \mathcal{H})$ satisfies the first of the above conditions, an element in $\mathcal{H}$ is said to be a *parabolic subgroup* of G and G is said to be *weakly relatively hyperbolic* (with respect to $\mathcal{H}$). If G is not relatively hyperbolic with respect to any choice of a finite collection of proper finitely generated subgroups $\mathcal{H}$, then it is said to be *not relatively hyperbolic*.

There are two necessary conditions for parabolic subgroups of relatively hyperbolic groups. Let G be a finitely generated group which is relatively hyperbolic with respect to a finite collection $\mathcal{H} = \{H_i\}$ of parabolic subgroups. The first is about virtual malnormality of parabolic subgroups and the second slightly generalises [1, Lemma 5].

THEOREM 2.2 [11, Theorem 1.4]. *For $H_i, H_j \in \mathcal{H}$ and $g_1, g_2 \in G$, $g_1 H_i g_1^{-1} \cap g_2 H_j g_2^{-1}$ is finite if either H_i and H_j are distinct or $H_i = H_j$ and $g_1^{-1} g_2 \notin H_i$.*

In particular, this implies that parabolic subgroups are almost malnormal.

LEMMA 2.3. *Suppose H is a subgroup of G isomorphic to a RAAG A_Γ whose defining graph Γ is connected. Then H is contained in a conjugate of a parabolic subgroup $H_i \in \mathcal{H}$.*

PROOF. This is a direct consequence of Theorems 4.16 and 4.19 in [11], stating that a free abelian subgroup of rank 2 has to be contained in a conjugate of $H_i \in \mathcal{H}$. Because the subgroup generated by the end points of each edge of Γ is a free abelian subgroup of rank two, we can deduce that H is contained in a conjugate of a parabolic subgroup. □

From these necessary conditions for parabolic subgroups, a simple criterion for detecting nonrelative hyperbolicity is developed as in [1]. Let G be a group and S be a (possibly infinite) generating set consisting only of infinite order elements. The *commutativity graph* $K(G, S)$ of G with respect to S is the simplicial graph with vertex set S in which two distinct vertices s and s' are connected by an edge if there exist integers n_s and $n_{s'}$ such that $\langle s^{n_s}, (s')^{n_{s'}} \rangle$ is abelian. The main theorem of [1] is then the following result.

THEOREM 2.4 [1]. *Let G be a finitely generated group and S be a (possibly infinite) generating set of G which consists of infinite order elements and contains at least two elements. Suppose that $K(G, S)$ is connected and that there are at least two vertices s and s' in $K(G, S)$ such that $\langle s^{n_s}, (s')^{n_{s'}} \rangle$ is a rank two free abelian group for some integers n_s and $n_{s'}$. Then, G is not relatively hyperbolic.*

Finally, owing to the result of Druţu, to know whether a finitely generated group G is relatively hyperbolic or not, we may look at other groups quasi-isometric to G (for example, finite index subgroups).

THEOREM 2.5 [8]. *In the class of finitely generated groups, being relatively hyperbolic is a quasi-isometry invariant.*

3. Automorphism group

The goal of this section is to prove that $\mathrm{Aut}(A_\Gamma)$ is in general not relatively hyperbolic. To use Theorem 2.4, we work with $\mathrm{Aut}^*(A_\Gamma)$ instead of $\mathrm{Aut}(A_\Gamma)$. Indeed, $\mathrm{Aut}^*(A_\Gamma)$ is generated by infinite order elements and is a finite index subgroup, by Lemma 2.1. By Theorem 2.5, it is then enough to show that $\mathrm{Aut}^*(A_\Gamma)$ is not relatively hyperbolic.

THEOREM 3.1. *Let Γ be a graph which has at least three vertices and S the set of all transvections and partial conjugations in $\mathrm{Aut}(A_\Gamma)$. Then the commutativity graph $K(\mathrm{Aut}^*(A_\Gamma), S)$ is connected. Hence, $\mathrm{Aut}(A_\Gamma)$ is not relatively hyperbolic.*

PROOF. The proof is divided into three steps. First, we show that, as long as they exist, any two transvections are joined by a path in $K = K(\mathrm{Aut}^*(A_\Gamma), S)$. Then, we show that the same holds for any two partial conjugations. Finally, we show that any partial conjugation and transvection are joined by a path, as long as they exist.

Claim 1. If there are at least two distinct transvections, then any two transvections are joined by a path in K.

Let a and b be vertices in Γ such that $a \leq b$. If a and b are adjacent, then $R_{ab} = L_{ab}$. Otherwise, R_{ab} and L_{ab} are distinct but $[R_{ab}, L_{ab}] = 1$, that is, R_{ab} and L_{ab} are joined by an edge in K. (In both cases, $[R_{ab}, L_{ab}] = 1$.) Thus, to prove the claim, we only need to show that there is a path in K from R_{ab} to either R_{cd} or L_{cd} for any two vertices $c, d \in \Gamma$ with $c \leq d$. There are five cases to handle.

Case 1. If $c = a$, then R_{ab} and L_{ad} are joined by an edge because $[R_{ab}, L_{ad}] = 1$.

Case 2. If $c \neq a, b$ and $d \neq a$, then R_{ab} and R_{cd} are joined by an edge because $[R_{ab}, R_{cd}] = 1$.

Case 3. If $c \neq a, b$ and $d = a$, then R_{ab} and L_{ca} are joined by a path. Indeed, $c \leq d = a \leq b$ and thus, $c \leq b$. This means that there is a transvection R_{cb}. Then there is a path joining R_{ab} and L_{ca} because $[R_{ab}, R_{cb}] = [R_{cb}, L_{ca}] = 1$.

Case 4. If $c = b$ and $d \neq a, b$, then R_{ab} and R_{bd} are joined by a path. Indeed, $a \leq b \leq d$ and thus, $a \leq d$ so that there is a transvection L_{ad}. Then there is a path joining R_{ab} and R_{bd} because $[R_{ab}, L_{ad}] = [L_{ad}, R_{bd}] = 1$.

Case 5. If $c = b$ and $d = a$, then $a \sim b$. There are two subcases.

Subcase 5-1: a and b are adjacent, that is, $\mathrm{st}(a) = \mathrm{st}(b)$. In this case, we show that R_{ab} and R_{ba} are joined by a path. If $\mathrm{st}(a)$ does not cover the whole graph Γ, choose $v \in \Gamma - \mathrm{st}(a)$ and let Γ_0 be the component of $\Gamma - \mathrm{st}(a)$ containing v. Note that Γ_0 does not contain a and b. Then

$$[R_{ab}, P_b^{\Gamma_0}] = [P_b^{\Gamma_0}, P_a^{\Gamma_0}] = [P_a^{\Gamma_0}, R_{ba}] = 1.$$

Otherwise, $\mathrm{st}(a)$ covers the whole graph so that $w \leq a \sim b$ for any vertex $w \neq a, b$ in Γ. Then

$$[R_{ab}, R_{wb}] = [R_{wb}, L_{wa}] = [L_{wa}, R_{wa}] = [R_{wa}, R_{ba}] = 1.$$

Subcase 5-2: a and b are not adjacent, that is, $\mathrm{lk}(a) = \mathrm{lk}(b)$. In this case, we show that R_{ab} and L_{ba} are joined by a path. If the link $\mathrm{lk}(a)$ is empty, then $a \sim b \leq w$ for any vertex $w \neq a, b$ in Γ. Then

$$[R_{ab}, L_{aw}] = [L_{aw}, R_{bw}] = [R_{bw}, L_{ba}] = 1.$$

Otherwise, choose $w \in \mathrm{lk}(a)$. If $\mathrm{st}(w)$ does not cover the whole graph, then

$$[R_{ab}, P_b^a] = [P_b^a, P_w^{\Gamma'}] = [P_w^{\Gamma'}, P_a^b] = [P_a^b, L_{ba}] = 1,$$

where Γ' is a component of $\Gamma - \mathrm{st}(w)$. If $\mathrm{st}(w) = \Gamma$, then $a \sim b \leq w$, and so

$$[R_{ab}, L_{aw}] = [L_{aw}, R_{bw}] = [R_{bw}, L_{ba}] = 1.$$

By these five cases, any two transvections (if they exist) are joined by a path in K.

If Γ is a complete graph, then there is no partial conjugation so that the theorem holds by the above claim. From now on, therefore, we assume that Γ is not complete. In particular, there are at least two partial conjugations.

Claim 2. Any two partial conjugations P_a^C and P_b^D are joined by a path in K for any choices of a, b, C, D.

Note that $[P_a^{C_1}, P_a^{C_2}] = 1$ whenever the partial conjugations are defined. Therefore, to see whether two partial conjugations P_a^C and P_b^D are joined by a path for any two distinct vertices a and b, it is enough to check only one particular choice of C and D.

Suppose Γ is connected. If there is no vertex in Γ whose star is the whole graph, then $\Gamma - \mathrm{st}(a) = \Gamma_1 \sqcup \cdots \sqcup \Gamma_m$ and $\Gamma - \mathrm{st}(b) = \Gamma'_1 \sqcup \cdots \sqcup \Gamma'_n$ for any two vertices $a, b \in \Gamma$, where each Γ_i and Γ'_j are components of $\Gamma - \mathrm{st}(a)$ and $\Gamma - \mathrm{st}(b)$, respectively. If $[a, b] = 1$, this implies that $[P_a^{\Gamma_i}, P_b^{\Gamma'_j}] = 1$ for any i and j so that the claim holds by the connectivity of Γ.

If $\Gamma = \mathrm{st}(b)$ for some vertex $b \in \Gamma$, then one can easily see that P_a^C and R_{ab} commute for any $a \in \mathrm{lk}(b)$ the complement of whose star has a nonempty component C. By Claim 1, we can deduce that any two partial conjugations are joined by a path.

Now, suppose Γ has at least two components Γ_1 and Γ_2, and there are two conjugations P_a^C and P_b^D for $a \in \Gamma_1$ and $b \in \Gamma_2$. We only need to show that there are some components C and D of $\Gamma - \mathrm{st}(a)$ and $\Gamma - \mathrm{st}(b)$, respectively, such that P_a^C and P_b^D are joined by a path in K. There are two cases depending on the number of vertices in Γ_2.

Case 1. Suppose Γ_2 has at least two vertices. There are three subcases.

Subcase 1-1: $\mathrm{st}(b) \subsetneq \Gamma_2$. Then $[P_a^{\Gamma_2}, P_b^D] = 1$, where D is a component of $\Gamma_2 - \mathrm{st}(b)$, because

$$P_a^{\Gamma_2}(P_b^D(s)) = P_a^{\Gamma_2}(bsb^{-1}) = aba^{-1} \cdot asa^{-1} \cdot ab^{-1}a^{-1} = absb^{-1}a^{-1},$$

and

$$P_b^D(P_a^{\Gamma_2}(s)) = P_b^D(asa^{-1}) = absb^{-1}a^{-1}$$

for any vertex $s \in D$.

Subcase 1-2: $\mathrm{st}(b) = \Gamma_2$ but Γ_2 is not a complete graph. There is a vertex $b_1 \in \Gamma_2$ and a component D_1 of $\Gamma_2 - \mathrm{st}(b_1)$. Because b and b_1 are adjacent,

$$[P_b^{\Gamma_1}, P_{b_1}^D] = [P_{b_1}^D, P_a^{\Gamma_2}] = 1.$$

Subcase 1-3: Γ_2 is a complete graph. Then

$$[P_b^{\Gamma_1}, R_{b_1 b}] = [R_{b_1 b}, P_a^{\Gamma_2}] = 1$$

for any vertex $b_1 \neq b$ in Γ_2 because $b_1 \leq b$.

Case 2. Suppose Γ_2 has only one vertex b. There are two subcases.

Subcase 2-1: Γ_1 is not a complete graph. Let a_1 be a vertex in Γ_1 such that $\mathrm{st}(a_1) \subsetneq \Gamma_1$. If $a = a_1$, then the claim holds because $[P_a^C, P_b^{\Gamma_1}] = 1$, where C is a component of

https://doi.org/10.1017/S0004972721001258 Published online by Cambridge University Press

$\Gamma_1 - \mathrm{st}(a)$. If $a \neq a_1$ but $\mathrm{st}(a) = \Gamma_1$, then we additionally have $[P_{a_1}^C, P_a^{\Gamma_2}] = 1$ so the claim holds.

Subcase 2-2: Γ_1 is a complete graph. We must divide into two cases again. If Γ_1 has at least two vertices, then for any vertex $a_1 \neq a$ in Γ_1, $a_1 \leq a$ so that

$$R_{a_1a}(P_b^{\Gamma_1}(a_1)) = R_{a_1a}(ba_1b^{-1}) = ba_1ab^{-1},$$

and

$$P_b^{\Gamma_1}(R_{a_1a}(a_1)) = P_b^{\Gamma_1}(a_1a) = ba_1b^{-1} \cdot bab^{-1} = ba_1ab^{-1}.$$

Thus,

$$[P_a^{\Gamma_2}, R_{a_1a}] = [R_{a_1a}, P_b^{\Gamma_1}] = 1.$$

If Γ_1 has only one vertex a, then

$$[P_a^{\Gamma_2}, R_{ba}] = [R_{ab}, P_b^{\Gamma_1}] = 1.$$

Because Γ has at least 3 vertices, there is a vertex c such that $a \sim b \leq c$. Since there is a path between R_{ab} and R_{ba} by *Claim 1*, $P_a^{\Gamma_2}$ and $P_b^{\Gamma_1}$ are joined by a path in K.

Claim 3. Any transvection is adjacent to a partial conjugation in K.

Suppose a and b are vertices in Γ such that $a \leq b$.

Case 1. If $\mathrm{st}(b) \subsetneq \Gamma$, then $[R_{ab}, P_b^C] = 1$ for any component C of $\Gamma - \mathrm{st}(b)$.

Case 2. Suppose $\mathrm{st}(b) = \Gamma$. Because Γ is not complete, there is a vertex c of Γ such that $\mathrm{st}(c) \subsetneq \Gamma$ (c may be equal to a). For any component C of $\Gamma - \mathrm{st}(c)$, we have $[R_{ab}, P_c^C] = 1$. Therefore, there is an edge joining a transvection and a partial conjugation if they exist.

In summary, if Γ is complete, by Claim 1, K is connected. Otherwise, $\mathrm{Aut}^*(A_\Gamma)$ contains at least two partial conjugations. If it has no transvections, by Claim 2, K is connected. If it has a transvection, by combining the three claims, we can show that K is connected. □

The only cases not covered by Theorem 3.1 are for RAAG's whose defining graphs have one or two vertices. If A_Γ is $\mathbb{Z}$, then $\mathrm{Aut}(A_\Gamma) = \mathbb{Z}_2$ is finite. In the two remaining cases, A_Γ is either $\mathbb{Z}^2$ or $\mathbb{F}_2$ so that $\mathrm{Aut}(A_\Gamma)$ is either $\mathrm{GL}_2(\mathbb{Z})$ or $\mathrm{Aut}(\mathbb{F}_2)$, respectively, and $\mathrm{GL}_2(\mathbb{Z})$ is hyperbolic because it is virtually free. For $\mathrm{Aut}(\mathbb{F}_2)$, consider the subgroup $\mathrm{Aut}^+(\mathbb{F}_2)$ which is the preimage of the special linear subgroup $\mathrm{SL}_2(\mathbb{Z}) \subset \mathrm{GL}_2(\mathbb{Z})$ under the homomorphism $\mathrm{Aut}(\mathbb{F}_2) \to \mathrm{GL}_2(\mathbb{Z})$ induced from the abelianisation map $\mathbb{F}_2 \to \mathbb{Z}^2$. Then $\mathrm{Aut}^+(\mathbb{F}_2)$ is a finite index subgroup of $\mathrm{Aut}(\mathbb{F}_2)$ and can be shown to be isomorphic to the pure mapping class group of a twice punctured torus, which is not relatively hyperbolic by (the proof of) Theorem 8.1 in [2].

We conclude this section with a remark that was pointed out to us by Anthony Genevois. If Γ is connected, there is a shorter argument to prove that $\mathrm{Aut}(A_\Gamma)$ is not relatively hyperbolic. Suppose that $\mathrm{Aut}(A_\Gamma)$ is relatively hyperbolic with respect

https://doi.org/10.1017/S0004972721001258 Published online by Cambridge University Press

to a finite collection $\mathcal{H} = \{H_i\}$ of parabolic subgroups. If there is a vertex $v \in \Gamma$ which is adjacent to all other vertices, then the subgroup of $\mathrm{Aut}(A_\Gamma)$ generated by all transvections induced by this central vertex v is an infinite normal free abelian subgroup contained in some $H_i \in \mathcal{H}$ by Lemma 2.3. Otherwise, $\mathrm{Inn}(A_\Gamma)$ is isomorphic to A_Γ and it is thus an infinite normal subgroup, contained in some $H_i \in \mathcal{H}$ by Lemma 2.3. In both cases, we find an infinite normal subgroup contained in an almost malnormal subgroup H_i. By Theorem 2.2, this implies that $H_i = \mathrm{Aut}(A_\Gamma)$, which is a contradiction.

4. Outer automorphism group

In this section, we look at the relative hyperbolicity of $\mathrm{Out}(A_\Gamma)$. In the same spirit as the proof of relative hyperbolicity of $\mathrm{Aut}(A_\Gamma)$, we work with the subgroup $\mathrm{Out}^*(A_\Gamma)$ instead of the whole group $\mathrm{Out}(A_\Gamma)$. Let S be the set of all transvections and partial conjugations as before and S' the set of all the (nontrivial) images of elements of S in $\mathrm{Out}(A_\Gamma)$. We want to investigate the connectivity of $K(\mathrm{Out}^*(A_\Gamma), S')$. Unfortunately, the proof of the connectivity of $K(\mathrm{Aut}^*(A_\Gamma), S)$ does not directly imply that $K(\mathrm{Out}^*(A_\Gamma), S')$ is connected because some partial conjugations in $\mathrm{Aut}(A_\Gamma)$ are sent to the identity element in $\mathrm{Out}(A_\Gamma)$. For the rest of this section, when we refer to a transvection or a partial conjugation, we mean its image in $\mathrm{Out}(A_\Gamma)$.

THEOREM 4.1. *Suppose Γ contains at least two vertices. If $\mathrm{Out}(A_\Gamma)$ is not finite, virtually $\mathbb{Z}$ nor virtually $\mathrm{GL}_2(\mathbb{Z})$, then $\mathrm{Out}(A_\Gamma)$ is not relatively hyperbolic.*

PROOF. Clearly, if $\mathrm{Out}^*(A_\Gamma)$ is finite or isomorphic to $\mathbb{Z}$, then $\mathrm{Out}(A_\Gamma)$ is hyperbolic. In particular, it is relatively hyperbolic. Now we examine the commutativity graph $K = K(\mathrm{Out}^*(A_\Gamma), S')$ for the case that S' has at least two distinct elements of $\mathrm{Out}^*(A_\Gamma)$.

Claim 1. If there are at least two nontrivial partial conjugations, then they are joined by a path in K.

Let P_a^C and $P_b^{C'}$ be two such partial conjugations. If a and b commute (a may be equal to b), then so do P_a^C and $P_b^{C'}$, so we can assume that a and b do not commute. Because neither P_a^C nor $P_b^{C'}$ is trivial, it means that $\Gamma - \mathrm{st}(a)$ and $\Gamma - \mathrm{st}(b)$ both consist of at least two components. Because a and b are not adjacent, there is a component D of $\Gamma - \mathrm{st}(a)$ and a component D' of $\Gamma - \mathrm{st}(b)$ which are disjoint. However, then $[P_a^C, P_a^D] = [P_a^D, P_b^{D'}] = [P_b^{D'}, P_b^{C'}] = 1$ and this defines a path in K from P_a^C to $P_b^{C'}$.

Claim 2. As long as they exist, any nontrivial partial conjugation and any transvection are joined by a path in K.

Let R_{ab} be a transvection and suppose that there is a nontrivial partial conjugation P_c^C. To show the existence of a path joining R_{ab} and P_c^C, we consider three cases according to the choice of the vertex c.

Case 1. If $c = b$, then $[R_{ab}, P_b^C] = 1$ for any component C.

https://doi.org/10.1017/S0004972721001258 Published online by Cambridge University Press

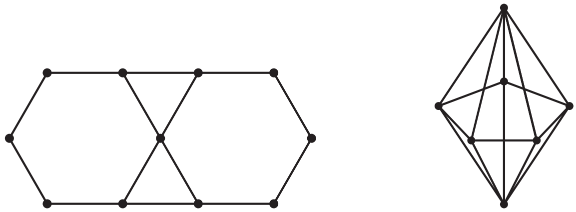

FIGURE 1. Typical examples of Γ with $\mathrm{Out}(A_\Gamma)$ relatively hyperbolic.

Case 2. Suppose $c = a$. If P_b^D represents a nontrivial element in $\mathrm{Out}(A_\Gamma)$ for some connected component D of $\Gamma - \mathrm{st}(b)$, then by Claim 1, there is a path from P_a^C to P_b^D and the claim follows from the fact that P_b^D and R_{ab} commute. If no such P_b^D exists, then $\Gamma - \mathrm{st}(b)$ has at most one connected component. Let $\Gamma_1, \ldots, \Gamma_n$ be the components of $\Gamma - \mathrm{st}(a)$. Note that $n \geq 2$. Because $a \leq b$, we have $\Gamma - \mathrm{st}(b) \subset \Gamma - \mathrm{lk}(a)$. This means that $\Gamma - \mathrm{st}(b)$ is either empty or contains only a or is entirely contained in some Γ_i. In all cases, there is a j such that $\Gamma_j \subset \mathrm{st}(b)$. Let w be any vertex in Γ_j. Because $\mathrm{lk}(w) \subset \Gamma_j \cup \mathrm{lk}(a)$, we have $\mathrm{lk}(w) \subset \mathrm{st}(b)$ and thus $w \leq b$. Choose $w_0 \in \Gamma_j$. Then $[R_{ab}, R_{w_0b}] = [R_{w_0b}, P_a^{\Gamma_k}] = 1$ for any $k \neq j$. Hence, $P_a^{\Gamma_k}$ and R_{ab} are joined by a path.

Case 3. Suppose that c is neither a nor b. If a or b is contained in $\mathrm{lk}(c)$, then P_c^C and R_{ab} can be joined by a path. If a and b are in the same component C of $\Gamma - \mathrm{st}(c)$, then

$$P_c^C(R_{ab}(a)) = P_c^C(ab) = cac^{-1} \cdot cbc^{-1} = cabc^{-1}$$

and

$$R_{ab}(P_c^C(a)) = R_{ab}(cac^{-1}) = cabc^{-1}.$$

If instead a and b are contained in different components of $\Gamma - \mathrm{st}(c)$, then $a \leq c$ because $a \leq b$. We have already checked that P_c^C and R_{ac} can be joined by a path for any component C of $\Gamma - \mathrm{st}(c)$. Because $[R_{ab}, L_{ac}] = [L_{ac}, R_{ac}] = 1$, P_c^C and R_{ab} also can be joined by a path.

By these two claims, as long as $|S'| > 1$ and there is at least one nontrivial partial conjugation, K is connected. The last remaining case is where S' only consists of transvections. By examining the paths between transvections in $K(\mathrm{Aut}^*(A_\Gamma), S)$, there may exist a path in $K(\mathrm{Out}^*(A_\Gamma), S')$ between two transvections, except possibly between R_{ab} and R_{ba} (the paths joining them in $K(\mathrm{Aut}^*(A_\Gamma), S)$ may use partial conjugations which have trivial images in $\mathrm{Out}(A_\Gamma)$). So the only case still to be examined is the case with only four transvections R_{ab}, R_{ba}, L_{ab} and L_{ba}. In this case, we see that R_{ab} and L_{ab} are equal in $\mathrm{Out}(A_\Gamma)$ and so are R_{ba} and L_{ba}. Therefore, $\mathrm{Out}^*(A_\Gamma)$ is isomorphic to $\mathrm{Out}^*(\mathbb{F}_2)$ and $\mathrm{Out}(A_\Gamma)$ is virtually isomorphic to $\mathrm{GL}_2(\mathbb{Z})$. □

We finish by giving some examples of RAAGs whose outer automorphism groups are relatively hyperbolic. If the graph Γ is a cycle with n vertices and n edges, then $\mathrm{Out}(A_\Gamma)$ is a finite group, as long as $n \geq 5$. If Γ is the graph on the left in Figure 1, then there are no transvections, and the central vertex induces the unique partial conjugation

https://doi.org/10.1017/S0004972721001258 Published online by Cambridge University Press

in $\mathrm{Out}(A_\Gamma)$, which is thus virtually cyclic. If Γ is the graph on the right in Figure 1, then there is no partial conjugation in $\mathrm{Out}(A_\Gamma)$ and the two equivalent vertices on the top and the bottom induce two transvections in $\mathrm{Out}(A_\Gamma)$. By the argument in the last paragraph of the proof of Theorem 4.1, $\mathrm{Out}(A_\Gamma)$ is virtually isomorphic to $\mathrm{GL}_2(\mathbb{Z})$.

Acknowledgements

We thank our advisor Hyungryul Baik for useful comments, Donggyun Seo for useful conversations and Anthony Genevois for giving us helpful feedback.

References

[1] J. W. Anderson, J. Aramayona and K. J. Shackleton, 'An obstruction to the strong relative hyperbolicity of a group', *J. Group Theory* **10**(6) (2007), 749–756.
[2] J. Behrstock, C. Druu and L. Mosher, 'Thick metric spaces, relative hyperbolicity, and quasi-isometric rigidity', *Math. Ann.* **344** (2009), Article no. 543.
[3] M. Bestvina and N. Brady, 'Morse theory and finiteness properties of groups', *Invent. Math.* **129** (1997), 445–470.
[4] B. H. Bowditch, 'Relatively hyperbolic groups', *Internat. J. Algebra Comput.* **22**(03) (2012), Article no. 1250016.
[5] R. Charney, 'An introduction to right-angled Artin groups', *Geom. Dedicata* **125** (2007), 141–158.
[6] R. Charney and K. Vogtmann, 'Finiteness properties of automorphism groups of right-angled Artin groups', *Bull. Lond. Math. Soc.* **41**(1) (2009), 94–102.
[7] R. Charney and K. Vogtmann, 'Subgroups and quotients of automorphism groups of RAAGs', in: *Low-Dimensional and Symplectic Topology*, Proc. Symp. Pure Math., 82 (American Mathematical Society, Providence, RI, 2011), 9–27.
[8] C. Druţu, 'Relatively hyperbolic groups: geometry and quasi-isometric invariance', *Comment. Math. Helv.* **84**(3) (2009), 503–546.
[9] A. Genevois, 'Automorphisms of graph products of groups and acylindrical hyperbolicity', Preprint, 2020, arXiv:1807.00622.
[10] M. R. Laurence, 'A generating set for the automorphism group of a graph group', *J. Lond. Math. Soc. (2)* **52**(2) (1995), 318–334.
[11] D. Osin, *Relatively Hyperbolic Groups: Intrinsic Geometry, Algebraic Properties and Algorithmic Problems*, Memoirs of the American Mathematical Society, 179 (American Mathematical Society, Providence, RI, 2006), 85 pages.
[12] H. Servatius, 'Automorphisms of graph groups', *J. Algebra* **126**(1) (1989), 34–60.

JUNSEOK KIM, Department of Mathematical Sciences,
KAIST, 291 Daehak-ro Yuseong-gu, Daejeon 34141, South Korea
e-mail: jsk8818@kaist.ac.kr

SANGROK OH, Department of Mathematical Sciences,
KAIST, 291 Daehak-ro Yuseong-gu, Daejeon 34141, South Korea
e-mail: sangrokoh@kaist.ac.kr

PHILIPPE TRANCHIDA, Department of Mathematical Sciences,
KAIST, 291 Daehak-ro Yuseong-gu, Daejeon 34141, South Korea
e-mail: ptranchi@kaist.ac.kr

https://doi.org/10.1017/S0004972721001258 Published online by Cambridge University Press