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MINIMAL ANNULI IN $\mathbb{R}^3$ BOUNDED BY NON-COMPACT COMPLETE CONVEX CURVES IN PARALLEL PLANES

YI FANG

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Abstract

In this paper we consider the Plateau problem for surfaces of annular type bounded by a pair of convex, non-compact curves in parallel planes. We prove that for certain symmetric boundaries there are solutions to the non-compact Plateau problems (Theorem B). Except for boundaries consisting of a pair of parallel straight lines, these are the first known examples.

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1. Introduction

In this paper we consider the Plateau problem for surfaces of annular type bounded by a pair of convex, non-compact curves in parallel planes. We will prove that for certain symmetric boundaries there are solutions to the non-compact Plateau problems (Theorem B). Except for boundaries consisting of a pair of parallel straight lines, these are the first known examples.

We now fix some notation in this paper. Let $P_t = \{(x, y, z) \in \mathbb{R}^3 | z = t\}$ be the plane at height t parallel to the xy-plane, and let $S(t_1, t_2) = \{(x, y, z) \in \mathbb{R}^3 \mid t_1 \leq z \leq t_2\}$ be the slab with boundary equal to $P_{t_1} \cup P_{t_2}$.

We briefly review the known results for a pair of Jordan curves. Let $\Gamma \subset \mathbb{R}^3$ be a pair of rectifiable Jordan curves, $\Gamma = \Gamma_1 \cup \Gamma_2$, $\Gamma_1 \cap \Gamma_2 = \emptyset$. Douglas [2] considered the Plateau problem for Γ. He proved that if

$$\inf\{\text{Area}(S)\} < \text{Area } (S_1) + \text{Area}(S_2),$$

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where S denotes any continuous annulus such that $\partial S = \Gamma$, and S_1 and S_2 are area minimizing disks such that $\partial S_1 = \Gamma_1$, $\partial S_2 = \Gamma_2$, then there is an area minimizing annulus A such that $\partial A = \Gamma$.

D. Hoffman, W. Meeks, and B. White, also considered this kind of Plateau's problem. A combined result of Hoffman and Meeks, and Meeks and White, is as follows.

THEOREM A. (Theorems 1.1, 1.2 of [5], and Theorem 1.1, Lemma 2.1 of [6]) *Suppose D_1 and D_2 are two open disks lying on parallel planes, and suppose their boundaries C_1 and C_2 are smooth convex Jordan curves. If A' is a connected non-planar compact branched minimal surface such that $\partial A' \subset D_1 \cup D_2$, then there exist exactly two embedded compact minimal annuli A and B, $\partial A = \partial B = C_1 \cup C_2$. The annulus A is stable and has the property that for any disks $D' \subset D_1$ and $D'' \subset D_2$ with continuous boundaries, if there is a connected compact branched minimal surface N such that $\partial N = \partial D' \cup \partial D''$, then N is contained in the solid bounded by $A \cup D_1 \cup D_2$. In particular, if $A \neq N$, then* $\mathrm{int}(A) \cap \mathrm{int}(N) = \emptyset$. *On the other hand, B is unstable and* $\mathrm{int}(B) \cap \mathrm{int}(N) \neq \emptyset$.

If merely $\partial A' \subset \overline{D_1} \cup \overline{D_2}$, then there exists at least one embedded minimal annulus A such that $\partial A = C_1 \cup C_2$. Such an A is almost stable in the sense that the first eigenvalue of the second variation of A is larger than or equal to zero. Let N be a connected compact branched minimal surface such that $\partial N = \partial D' \cup \partial D''$, then N is contained in the solid bounded by $A \cup D_1 \cup D_2$. In particular, if $A \neq N$, then $\mathrm{int}(A) \cap \mathrm{int}(N) = \emptyset$.

Furthermore, the symmetry group of A and B are the same as the symmetry group of $C_1 \cup C_2$.

A very useful fact (which we will use) about minimal annulus is a result of Shiffman [11]. He proved that if C_1 and C_2 are *continuous* convex Jordan curves lying on planes parallel to the xy-plane, say on $P_{-1/2}$ and $P_{1/2}$, and A is a minimal annulus such that $\partial A = C_1 \cup C_2$, then each level set of $A \cap P_t$ is a strictly convex Jordan curve for $-1/2 < t < 1/2$. This is called Shiffman's first theorem.

If C_1 and C_2 are circles, Shiffman's second theorem states that each $A \cap P_t$ is a circle for $-1/2 \leq t \leq 1/2$.

From Shiffman's first theorem, it is clear that if A is a non-planar minimal annulus and ∂A consists of convex Jordan curves lying on planes parallel to the xy-plane, then A does not have vertical normal directions in its interior, as otherwise some level set would not be a Jordan curve.

We now state our existence theorem. Let $r > 0$ and $0 < b \leq \infty$ be fixed. Let K be the rotation of angle π about the y-axis and R be the reflection through the xz-plane.

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Define a convex curve C in $P_{-1/2}$ by

$$C = \{(x, y, -1/2) \mid x = f(y)\},$$

where f satisfies :

(i) $f : (-b, b) \to \mathbb{R}$ is a C^∞ function such that $f(-y) = f(y)$;
(ii) $f(0) = -r$, $f'' \geq 0$, and $\lim_{y\to b} f(y) = \infty$.

Define the planar domain

$$X = \{(x, y, -1/2) \mid x > f(y)\}. \tag{1}$$

Now consider the non-compact Plateau problem with the boundary

$$\Gamma = C \cup K(C) = \partial X \cup K(\partial X). \tag{2}$$

We have

THEOREM B. *If there exists a compact non-planar minimal annulus A' such that $\partial A' \subset X \cup K(X)$, then there are two embedded non-compact minimal annuli $\mathscr{A}$ and $\mathscr{B}$ in $S(-1/2, 1/2)$, which are solutions to the non-compact Plateau problem with the boundary Γ given in* (2).

For any $-1/2 < t < 1/2$, $\mathscr{A} \cap P_t$ and $\mathscr{B} \cap P_t$ are strictly convex Jordan curves.

Furthermore, $\operatorname{int}(\mathscr{A}) \cap \operatorname{int}(\mathscr{B}) = \emptyset$. *Let N be any connected compact non-planar branched minimal surface such that $\partial N \subset \overline{X} \cup K(\overline{X})$. Then $\mathscr{A}$ and $\mathscr{B}$ have the properties*

$$\operatorname{int}(\mathscr{A}) \cap \operatorname{int}(N) = \emptyset \quad \textit{and} \quad \mathscr{B} \cap N \neq \emptyset.$$

REMARK 1. Let C'_R be the circle of radius R in $P_{-1/2}$, centered at $(0, 0, -1/2)$. It is well known that there is a constant $h_2 > 0$, such that if $R \geq 1/h_2 \simeq 0.754439698$ then the coaxial circles C'_R and $K(C'_R)$ bound a piece of a catenoid. Hence by Theorem A, if $C'_R \subset X$, then there will be two non-compact minimal annuli $\mathscr{A}$ and $\mathscr{B}$ which solve the Plateau problem with the boundary Γ given in (2).

The only previously known example of a non-planar non-compact embedded minimal annulus in a slab $S(t_1, t_2)$ is an embedded minimal annulus $\mathscr{A}$ such that $\partial\mathscr{A}$ consists of a pair of parallel straight lines, and $\mathscr{A} \cap P_t$ is a circle for every $t_1 < t < t_2$. Repeatedly rotating about the straight-line boundaries produces a singly-periodic complete minimal surface which is called a Riemann's example. There is a one-parameter family of Riemann's examples. It was Riemann who discovered these minimal surfaces. See [7, pp. 85–90].

A basic piece of a Riemann's example is the portion bounded by two consecutive parallel straight lines. Such a piece is an annulus, which we will denote by $\mathscr{R}$. In [4],

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it is proved that any embedded minimal annulus bounded by a pair of parallel straight lines must be a basic piece $\mathscr{R}$ of some Riemann's example. See also [12]. For a more general result, see [3].

Since the proof of Theorem B is quite long, we give a sketch here to give the ideas and also the difficulties encountered when one tries to simplify the proof. The basic idea is to approximate the non-compact boundary with compact ones. Then by using Theorem A, we get a sequence of approximating minimal annuli $\{A_n\}$ and $\{B_n\}$. We use the symmetry conditions of the boundary to divide the approximating annuli into two graphs, each of which is stable and simply connected. Then we estimate the boundary arc-length of compact pieces of the graphs to prove the existence of a limit surface. The trouble is to prove that the limit surface is an annulus with the claimed properties. To accomplish this, we use the properties stated in Theorem A of these approximating surfaces, and the estimates of $A_n \cap P_0$ and $B_n \cap P_0$ in Lemmas 1 – 8 to prove that the limit surface intersecting P_t in a convex Jordan curve for $|t|$ small enough. The last difficulty is to prove that the limit surface is not only an annulus, but is also a compact annulus in any proper subslab contained in the original slab. We use the Enneper-Weierstrass representation of the approximating surfaces to establish the needed estimate. Together with a result of Osserman and Schiffer, we are able to prove the desired fact.

It turns out that the argument for the existence of A (which is the limit of sequence of stable annuli) is much easier than the argument for the existence of B (which is the limit of sequence of unstable annuli). For the former, we can give a much shorter and simpler proof, without using Lemmas 5 to 8. For the latter, we have to establish those lemmas to be able to apply Theorem A. We prove those preparatory lemmas in Section 2. Section 3 is devoted to the proof of Theorem B.

2. Preparatory lemmas

We denote the xz-plane by P. Suppose that $C' \subset \overline{X}$ is a smooth convex Jordan curve symmetric with respect to P. Let $A \subset S(-1/2, 1/2)$ be a minimal annulus such that $\partial A = C' \cup K(C')$.

In Lemmas 1 to 4, we study the properties of such a minimal annulus A.

LEMMA 1. *The intersection $A \cap P$ consists of two curves σ_1 and σ_2 such that $K(\sigma_1) = \sigma_2$. Moreover, $\sigma_1 \subset \{(x, 0, z) \in P \mid x < 0\}$ and $\sigma_2 \subset \{(x, 0, z) \in P \mid x > 0\}$ are two convex graphs. Precisely, there are two smooth functions f_1 and f_2, $f_1(z) < 0$, $f_2(z) > 0$, for $-1/2 \le z \le 1/2$, and $f_1''(z) < 0$, $f_2''(z) > 0$, for $-1/2 < z < 1/2$, such that*

$$\sigma_1 = \{(x, 0, z) \mid x = f_1(z)\}, \qquad \sigma_2 = \{(x, 0, z) \mid x = f_2(z)\}.$$

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PROOF. By Theorem A and Shiffman's first theorem, A is invariant under both K and R, and $A \cap P_t$ is strictly convex for $-1/2 < t < 1/2$. Since each $A \cap P_t$ is a strictly convex curve and symmetric with respect to P, $A \cap P$ has exactly two components and they are graphs over the z-axis. Let them be σ_1 and σ_2. By $K(A \cap P) = A \cap P$ we have $K(\sigma_1) = \sigma_2$. If we write

$$\sigma_1 = \{(x, 0, z) \mid x = f_1(z)\}, \quad \sigma_2 = \{(x, 0, z) \mid x = f_2(z)\}, \quad -1/2 \le z \le 1/2,$$

then $f_2(z) = -f_1(-z)$. By the symmetry with respect to P, $(f_1(z), 0, z)$ and $(f_2(z), 0, z)$ are the extreme points of the strictly convex curve $A \cap P_z$ and we can assume that $f_1(z) < f_2(z)$ and $A \cap P_z \subset \{(x, y, z) \mid f_1(z) \le x \le f_2(z)\}$. As the fixed point sets of an isometry (the reflection R) on A, both σ_1 and σ_2 are geodesics, and their tangent directions are the principal directions on A. The tangent directions of each level set $A \cap P_z$ at $y = 0$ are also principal directions on A, since they are perpendicular to P by the invariance under R and hence perpendicular to the tangent direction of σ_1 or σ_2 respectively. Let $(\sin\theta, 0, \cos\theta)$ be the inward normal vector to A at the point $p \in \sigma_1$, where θ is the angle between the inward normal vector and the positive z-axis. Since A cannot have vertical normal vectors, $\sin\theta > 0$, and hence it must be the case that $0 < \theta < \pi$. Let k_1 and k_2 be the principal curvatures of A at $p \in \sigma_1 \cap P_z$ along the directions of σ_1 and $A \cap P_z$ respectively. Notice that k_1 is also the plane curvature of σ_1 with respect to the normal direction of positive x-coordinate. Letting k be the plane curvature of $A \cap P_z$ with respect to the inner normal, then $k > 0$ and $k_2 = k \sin\theta > 0$ on $\sigma_1 \cap P_z$. Since A is minimal, $k_1 = -k_2 < 0$ at $\sigma_1 \cap P_z$. By $k_1 = f''(z)/(1 + f_1'(z)^2)^{3/2}$, we know that $f_1''(z) < 0$. Since $f_2(z) = -f_1(-z)$, we have $f_2''(z) > 0$.

We need to prove that $f_1(z) < 0$ and $f_2(z) > 0$. If $f_2(z) \le 0$ for some z, then since $f_1(z) < f_2(z)$ and $A \cap P_z \subset \{(x, y, z) \mid f_1(z) \le x \le f_2(z)\}$, the convex curve $A \cap P_z$ is contained in the half plane $\{x \le 0\}$. Thus $A \cap P_{-z} = K(A \cap P_z) \subset P_{-z} \cap \{x \ge 0\}$ and the orthogonal projections of $A \cap P_z$ and $A \cap P_{-z}$ on P_0 have at most one common point $(0, 0, 0)$ and, in particular, $z \ne 0$. Without loss of generality we may assume that $z > 0$. Let C_1 and C_2 be two circles lying on P_{-z} and P_z respectively, such that $A \cap P_{-z}$ and $A \cap P_z$ are contained in the disks bounded by C_1 and C_2. We can arrange that $R(C_i) = C_i$ for $i = 1, 2$ and the orthogonal projections of C_1 and C_2 on the xy-plane have at most one common point $(0, 0, 0)$. This means that the horizontal distance between the centers of C_1 and C_2 is greater than or equal to the sum of their radii. By Theorem A there is a minimal annulus N in $S(-z, z)$ bounded by C_1 and C_2. By Shiffman's second theorem $N \cap P_t$ is a circle for $-z < t < z$. By a theorem of Nitsche, the horizontal distance between the two centres of C_1 and C_2 is less than the sum of their radii, see [7, pp. 88–89]. This contradiction proves that $f_1(z) < 0$ and $f_2(z) > 0$.

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Let $d(z) = f_2(z) - f_1(z)$ for $-1/2 \le z \le 1/2$. The function d is the distance function between $(f_1(z), 0, z)$ and $(f_2(z), 0, z)$.

LEMMA 2. *The function d satisfies*

$$d(z) > d(0) > 0 \quad \text{for} \quad -1/2 \le z \le 1/2,\ z \ne 0. \tag{3}$$

PROOF. We have $d(z) = f_2(z) - f_1(z)$, $d'(z) = f_2'(z) - f_1'(z) = f_2'(z) - f_2'(-z)$, and $d''(z) = f_2''(z) + f_2''(-z) > 0$. Since $d'(0) = f_2'(0) - f_2'(0) = 0$, $d(0)$ is the unique minimum value of d, and hence $d(z) > d(0) > 0$ for $-1/2 \le z \le 1/2, z \ne 0$.

Let H be the half space $\{y \ge 0\}$ and $D = A \cap H$. Let D' be the convex disk such that $\partial D' = C'$. Let $l_1 = P \cap D'$, $l_2 = K(l_1)$. Let Ω be the domain in P bounded by $\sigma_1 \cup \sigma_2 \cup l_1 \cup l_2$. Since $f_1(z) < f_2(z)$ for $-1/2 \le z \le 1/2$, Ω is a domain and obviously it is simply connected.

LEMMA 3. *The minimal surface $D = A \cap H$ is a minimal graph over Ω.*

PROOF. By Theorem A and Shiffman's first theorem, A is invariant under both K and R and each level set $A \cap P_t$ is a strictly convex Jordan curve. Let f_1 and f_2 be the functions that define σ_1 and σ_2 in Lemma 1. By symmetry with respect to P, each $D \cap P_t$ is a convex graph over the interval $f_1(t) \le x \le f_2(t)$, $-1/2 < t < 1/2$. Thus D is a minimal graph over the domain Ω.

Recall the convex function f that defines the boundary $C = \partial X$. We need to define its inverse for $y \ge 0$. Since $f'(0) = 0$ and $f''(y) \ge 0$ for any $-b < y < b$, so $f'(y) \ge 0$ for $b > y \ge 0$. Thus in H, f is nondecreasing. Because $\lim_{y \to b} f(y) = \infty$ for each $x > -r = f(0)$, $f^{-1}(x)$ is not empty. Since f is nondecreasing, if $f^{-1}(x) \cap H$ contains more than one point, it must contain an interval $[c, d]$ with $d > c > 0$ and hence on $[c, d]$ we would have $f'(y) = 0 = f'(0)$. Since $f'' \ge 0$, f' is nondecreasing, we would have $f'(0) = 0$ on $[0, d]$, thus $f(y) = -r$ on $[0, d]$, contradicting the fact that $x > -r$. Therefore we have proved that $f^{-1}(x) \cap H$ is a single point for $x > f(0) = -r$. Thus $h = f^{-1}$ is a well defined function on $(-r, \infty)$, and $h' > 0$, $h'' \le 0$. If we define $h(-r) = \sup\{f^{-1}(-r)\}$ then h is a well defined function on $[-r, \infty)$ and is strictly increasing.

Since σ_2 is convex and $f_2(1/2) \le r$, for fixed $s > r$, $\{x = s\} \cap \Omega$ is an interval if it is nonempty. Also remember that $f_1(z) < 0$ and $f_2(z) > 0$, $\{x = 0\} \cap \Omega = \{(0, 0, z) \,\big|\, -1/2 < z < 1/2\}$.

Let $s > 0$ and $S(s)$ be the slab $\{(x, y, z) \mid -s \le x \le s\}$. Let $u : \Omega \to \mathbb{R}$ be the function that defines the minimal graph D in Lemma 3. We want to estimate u in $\Omega \cap S(s)$ and the boundary arc length of $D \cap S(s)$. We have the following lemma.

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LEMMA 4. *If $s > r$, then on the interval $I = \{x = s\} \cap \Omega$, $u(s, \cdot)$ is strictly decreasing and $0 \leq u(s,t) \leq h(s)$ for $t \in I$.*

We have $u(0,0) \leq u(0,z) \leq h(0)$, $-1/2 < z < 1/2$, and $u(0,\cdot)$ is strictly increasing in $(0, 1/2)$, strictly decreasing in $(-1/2, 0)$.

Moreover, if $s > r$, then the arc-length of $\partial(D \cap S(s))$ is less than or equal to $l(s) := 2(2 + 4s + 3h(s))$.

PROOF. We show that $u(s, \cdot)$ does not have local maxima in the interior of I. If $t_0 \in I$ is an interior critical point of $u(s, \cdot)$, then $\partial u/\partial z(s, t_0) = 0$, and by the minimal surface equation we have

$$\partial^2 u/\partial z^2(s, t_0) = \frac{-\partial^2 u/\partial x^2}{1 + (\partial u/\partial x)^2}(s, t_0).$$

Since $D \cap P_{t_0}$ is strictly convex at $(s, u(s, t_0), t_0) \in \text{int}(D)$, $\partial^2 u/\partial^2 x(s, t_0) < 0$, we have $\partial^2 u/\partial^2 z(s, t_0) > 0$. Thus $u(s, \cdot)$ can only achieve its maximum value on the boundary of Ω. Let $x \in f_2([-1/2, 1/2])$. Since f_2 is convex and $f_2(1/2) \leq r$, if $x > r$, then $f_2^{-1}(x)$ is well defined. Note that $(s, t) \in \partial\Omega$ if and only if $t = f_2^{-1}(s)$. Since u is zero along σ_2, and for $s > r$, by the condition $C' \subset \overline{X}$, the other boundary value of u along $x = s$ is $u(s, -1/2) \leq h(s)$, we have $0 = u(s, f_2^{-1}(s)) \leq u(s, z) \leq u(s, -1/2) \leq h(s)$ for $z \in I$. Since $u(s, \cdot)$ cannot achieve local maxima in the interior of I, it must be strictly decreasing.

Notice that by the symmetry $A = K(A)$, $u(0, z) = u(0, -z)$, we have $\partial u/\partial z(0, 0) = 0$. Similar argument proves that $\partial^2 u/\partial z^2(0, 0) > 0$, $u(0, 0)$ is a local minimum of $u(0, \cdot)$. A similar argument about local maxima proves that the statement about $u(0, \cdot)$ is true.

For each $s > r$, the boundary of $D \cap S(s)$ consists of $\sigma_1 \cap S(s)$, $\sigma_2 \cap S(s) = K(\sigma_1 \cap S(s))$, $C' \cap H \cap S(s)$, $K(C' \cap H \cap S(s))$, $D \cap \{x = s\}$, and $D \cap \{x = -s\} = K(D \cap \{x = s\})$. We only need to prove that the summation of the arc lengths of $\sigma_1 \cap S(s)$, $C' \cap H \cap S(s)$, and $D \cap \{x = s\}$ is less than or equal to $l(s)/2$.

Since $f_1''(z) < 0$ and $-s \leq f_1(z) < 0$, $\sigma_1 \cap S(s)$ is a convex graph over a subinterval of $-1/2 \leq z \leq 1/2$. An elementary estimate for convex graphs gives that the arc length of $\sigma_1 \cap S(s)$ is less than or equal to $1 + 2s$.

Note that $C' \subset \overline{X}$ and $h(t) \leq h(s)$ for $-r \leq t < s$. Then $H \cap C' \cap S(s) \subset \{0 \leq y \leq h(s)\} \cap S(s)$. Since C' is convex, an elementary estimate of convex curves gives that the arc length of $C' \cap H \cap S(s)$ is less than or equal to $2(s + h(s))$.

$D \cap \{x = s\}$ is a graph $\{(s, y, z) \mid y = u(s, z),\ -1/2 \leq z \leq f_2^{-1}(s)\}$, $0 \leq u(s, z) \leq u(s, -1/2) \leq h(s)$, and $u(s, \cdot)$ is strictly decreasing as just proved. By elementary arguments again, this time using the property of being strictly decreasing, the arc length of $D \cap \{x = s\}$ is less than or equal to $1 + h(s)$.

Thus the summation of the arc lengths of $\sigma_1 \cap S(s)$, $C' \cap H \cap S(s)$ and $D \cap \{x = s\}$ is less than or equal to $2 + 4s + 3h(s) = l(s)/2$, the lemma is proved.

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Since the following lemmas are not needed in the proof of the existence of $\mathscr{A}$, the reader can skip them and go to the proof of Theorem B.

To prove the existence of $\mathscr{B}$ mentioned in Theorem B, we have to clarify several facts about a basic piece $\mathscr{R}$ of a Riemann's example.

LEMMA 5. *Let L_1 and L_2 be parallel straight lines lying on $P_{-1/2}$ and $P_{1/2}$ respectively. Then there is an $E > 0$, such that whenever the horizontal distance between L_1 and L_2 is greater than E, there is a basic piece $\mathscr{R}$ of a Riemann's example such that $\partial\mathscr{R} = L_1 \cup L_2$.*

PROOF. It is well known that for any basic piece of Riemann's example in $S(-z_0, z_0)$, one half of the horizontal distance between the boundary straight lines $L_1 \subset P_{-z_0}$ and $L_2 \subset P_{z_0}$ is given by

$$R = \left| -b - \int_b^\infty \frac{(a^2 - b^2)t^2 - a^2b^2}{\Delta(t)(t^2 + \Delta(t))} dt \right|,$$

where $0 < b \le a$, $\Delta(t) = \sqrt{(t^2 + a^2)(t^2 - b^2)}$, and z_0 is given by

$$z_0 = ab \int_b^\infty \frac{dt}{\Delta(t)}.$$

See, [7, p. 89] and note the misprint in line 12.

Define $r := a/b \ge 1$. Substituting $s = t/b$, we can rewrite R and z_0 as

$$R = b \left| 1 + \int_1^\infty \frac{\left[(r^2 - 1)s^2 - r^2\right] ds}{\sqrt{(s^2 + r^2)(s^2 - 1)} \left(s^2 + \sqrt{(s^2 + r^2)(s^2 - 1)}\right)} \right|,$$

$$z_0 = rb \int_1^\infty \frac{ds}{\sqrt{(s^2 + r^2)(s^2 - 1)}}.$$

Thus $R' = R/2z_0$ is independent of a and b, and is a continuous function of r for $r \ge 1$. After a homothety, such that the surface is contained in $S(-1/2, 1/2)$, $L_1 \subset P_{-1/2}$ and $L_2 \subset P_{1/2}$, then R' is one half of the horizontal distance between L_1 and L_2. We only need to prove that $\lim_{r\to\infty} R' = \infty$. First we claim that for $r > \sqrt{2}$,

$$r \int_1^\infty \frac{dt}{\sqrt{(t^2 + r^2)(t^2 - 1)}} \le 1 + \cosh^{-1}(r).$$

In fact, for $t \ge r > \sqrt{2}$, $(r^2 - 1)t^2 - r^2 \ge 0$, and thus

$$(t^2 + r^2)(t^2 - 1) = t^4 + (r^2 - 1)t^2 - r^2 \ge t^4.$$

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We have

$$r\int_1^\infty \frac{dt}{\sqrt{(t^2+r^2)(t^2-1)}} = r\int_1^r \frac{dt}{\sqrt{(t^2+r^2)(t^2-1)}} + r\int_r^\infty \frac{dt}{\sqrt{(t^2+r^2)(t^2-1)}}$$
$$\leq \int_1^r \frac{dt}{\sqrt{t^2-1}} + r\int_r^\infty \frac{dt}{t^2} = 1+\cosh^{-1}(r).$$

The claim is true.

Next we will prove that for r large enough,

$$\left|\int_1^\infty \frac{[(r^2-1)t^2-r^2]dt}{\sqrt{(t^2+r^2)(t^2-1)}\left(t^2+\sqrt{(t^2+r^2)(t^2-1)}\right)}\right| \geq Cr,$$

for some $C > 0$.

In fact, for $1 < t \leq r^2/\sqrt{r^2-1}$,

$$\frac{|(r^2-1)t^2-r^2|}{\sqrt{(t^2+r^2)(t^2-1)}\left(t^2+\sqrt{(t^2+r^2)(t^2-1)}\right)} \leq \frac{r}{\sqrt{t^2-1}}.$$

For $t \geq r/\sqrt{r^2-1}$,

$$\frac{(r^2-1)t^2-r^2}{\sqrt{(t^2+r^2)(t^2-1)}\left(t^2+\sqrt{(t^2+r^2)(t^2-1)}\right)} \geq 0.$$

Since $\lim_{r\to\infty} r/\sqrt{r^2-1} = 1$, when r is large enough, $r/\sqrt{r^2-1} < 2$. We have

$$\int_1^\infty \frac{[(r^2-1)t^2-r^2]dt}{\sqrt{(t^2+r^2)(t^2-1)}\left(t^2+\sqrt{(t^2+r^2)(t^2-1)}\right)}$$
$$> \int_{r/\sqrt{r^2-1}}^r \frac{[(r^2-1)t^2-r^2]dt}{\sqrt{2}rt\,(t^2+\sqrt{2}rt)} - \int_1^{r/\sqrt{r^2-1}} \frac{|(r^2-1)t^2-r^2|dt}{\sqrt{(t^2+r^2)(t^2-1)}\left(t^2+\sqrt{(t^2+r^2)(t^2-1)}\right)}$$
$$> \frac{1}{\sqrt{2}r}\int_2^r \left(\frac{1}{2t} - \frac{r}{\sqrt{2}t^2} + \frac{r^2-3/2}{\sqrt{2}r+t}\right)dt - \int_1^{r/\sqrt{r^2-1}} \frac{r\,dt}{\sqrt{t^2-1}}$$
$$= \frac{1}{\sqrt{2}r}\left[\frac{1}{2}\log\frac{r}{2} + \frac{r}{\sqrt{2}}\left(\frac{1}{r}-\frac{1}{2}\right) + \left(r^2-\frac{3}{2}\right)\log\frac{r+\sqrt{2}r}{\sqrt{2}r+2}\right] - r\cosh^{-1}\left(\frac{r}{\sqrt{r^2-1}}\right)$$
$$= \frac{1}{\sqrt{2}}\left[\log\frac{r+\sqrt{2}r}{\sqrt{2}r+2} + \frac{1}{2r^2}\log\frac{r}{2} + \frac{1}{\sqrt{2}r}\left(\frac{1}{r}-\frac{1}{2}\right)\right.$$
$$\left. - \frac{3}{2r^2}\log\frac{r+\sqrt{2}r}{\sqrt{2}r+2} - \sqrt{2}\cosh^{-1}\left(\frac{r}{\sqrt{r^2-1}}\right)\right] r.$$

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Since

$$\begin{aligned}\lim_{r\to\infty}\Bigg[\log\frac{r+\sqrt{2}r}{\sqrt{2}r+2}+\frac{1}{2r^2}\log\frac{r}{2}+\frac{1}{\sqrt{2}r}\left(\frac{1}{r}-\frac{1}{2}\right)\\ -\frac{3}{2r^2}\log\frac{r+\sqrt{2}r}{\sqrt{2}r+2}-\sqrt{2}\cosh^{-1}\left(\frac{r}{\sqrt{r^2-1}}\right)\Bigg]\\ =\log\frac{1+\sqrt{2}}{\sqrt{2}},\end{aligned}$$

for r large enough, we can take $C=\frac{1}{4}\log[(1+\sqrt{2})/\sqrt{2}]$. Thus we have

$$\begin{aligned}\lim_{r\to\infty}R' &= \lim_{r\to\infty}\frac{R}{2z_0}\\ &= \lim_{r\to\infty}\frac{\left|1+\int_1^\infty((r^2-1)t^2-r^2)dt/\left(\sqrt{(t^2+r^2)(t^2-1)}\left(t^2+\sqrt{(t^2+r^2)(t^2-1)}\right)\right)\right|}{2r\int_1^\infty dt/\sqrt{(t^2+r^2)(t^2-1)}}\\ &\geq \lim_{r\to\infty}\frac{Cr-1}{2(1+\cosh^{-1}r)}=\infty.\end{aligned}$$

To be able to apply Theorem A, we need to know the stability of a basic piece $\mathscr{R}$ of a Riemann's example. First note that $K(\mathscr{R})=\mathscr{R}$, see for example, [7, p. 88, formula (55)]. In [4], the examples of Riemann are described in terms of their Enneper-Weierstrass Representation data g and η, where g is a meromorphic function and η is a meromorphic 1-form. Let N be the unit normal vector of the surface, and τ be the stereographic projection $S^2-\{(0,0,1)\}\to\mathbb{C}$. It is well known that $g=\tau\circ N$. Either g or N will be called the Gauss map.

Let $\lambda\geq 1$ and L be the lattice in $\mathbb{C}$ generated by $\{\lambda, i\}$. On the rectangular torus $T_\lambda=\mathbb{C}/L$, consider the elliptic function P which has a double pole at 0, a double zero at $\omega_2=(\lambda+i)/2$ and no other zeros or poles. The Weierstrass $\wp$-function $\wp$ has the property that $\wp-\wp(\omega_2)$ has exactly the same poles and zeros as P. To get a Riemann's example, take

$$g=P=\wp-\wp(\omega_2),\qquad \eta=idz/P.$$

It can be easily checked that P has the property that $P(\omega_2/2)=i$, and P is real precisely on the lines

$$\mathrm{Re}(z)=0,\ \mathrm{Re}(z)=\lambda/2,\ \mathrm{Im}(z)=0\ \text{and}\ \mathrm{Im}(z)=1/2.$$

By reflection, we have

$$P(x+iy)=\overline{P(\lambda-x,iy)},\qquad P(x+iy)=\overline{P(x,i(1-y))},$$

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and hence

$$P(\lambda - x, iy) = P(x, i(1-y)). \tag{4}$$

A basic piece $\mathscr{R}$ of a Riemann's example corresponds to the punctured rectangular $\{z = x + iy \mid 0 \le x < \lambda,\ 0 \le y \le 1/2\} - \{0, \omega_2\}$. Since $\deg \wp = 2$, by (4) we know that the Gauss map N of $\mathscr{R}$ maps onto $S^2 - \{(0,0,1), (0,0,-1)\}$ and is one-to-one in $\mathrm{int}(\mathscr{R})$.

LEMMA 6. *Let $\mathscr{R} \subset S(-1/2, 1/2)$ be a basic piece of a Riemann's example, then for small $\epsilon > 0$, $\mathscr{R} \cap S(-1/2+\epsilon, 1/2-\epsilon)$ is unstable.*

However, $\mathscr{R} \cap S(0, 1/2-\epsilon)$ is stable for any $0 < \epsilon < 1/2$, and by symmetry, so is $\mathscr{R} \cap S(-1/2+\epsilon, 0)$.

PROOF. Let g be the Gauss map of $\mathscr{R}$, we know that the image of N on $\mathscr{R}$ is $D := S^2 - \{(0,0,1), (0,0,-1)\}$, and N is one-to-one in $\mathrm{int}(\mathscr{R})$. Let Δ be the Laplace operator on S^2. Let U and V be open disks such that $(0,0,1) \in U$ and $(0,0,-1) \in V$ and $U \cap V = \emptyset$. It is well known that the first eigenvalue λ_1 of Δ on $S^2 - (U \cup V)$ is near zero if U and V are sufficiently small.

For $\epsilon > 0$ small enough, $N(\mathscr{R} \cap S(-1/2+\epsilon, 1/2-\epsilon) \supset S^2 - (U \cup V)$ for some small disks U and V, hence

$$\lambda_1(N(\mathscr{R} \cap S(-1/2+\epsilon, 1/2-\epsilon)) < 2.$$

Thus by a classical result, which says that if the first eigenvalue of the one-to-one image of the Gauss map of a piece of minimal surface is less than 2, then the piece of minimal surface is unstable, see, for example, [8, p. 215, Theorem 8.2]. Hence $S(-1/2+\epsilon, 1/2-\epsilon) \cap \mathscr{R}$ is unstable.

On the other hand, by a theorem of Barbosa and Do Carmo [1], or [8, p. 216, Corollary 8.5], a minimal surface M is stable if $\iint_M |K| dA < 2\pi$. Since N is one-to-one in $\mathrm{int}(\mathscr{R})$ and $N(\mathscr{R}) = S^2 - \{(0,0,1), (0,0,-1)\}$, $\iint_{\mathscr{R}} |K| dA = 4\pi$. Thus for any $0 < \epsilon < 1/2$, $\iint_{\mathscr{R} \cap S(-1/2+\epsilon, 1/2-\epsilon)} |K| dA < 4\pi$. By the symmetry

$$\mathscr{R} \cap S(-1/2+\epsilon, 1/2-\epsilon) = \mathscr{R} \cap S(-1/2+\epsilon, 0) \cup K(\mathscr{R} \cap S(-1/2+\epsilon, 0)),$$

it follows that

$$\iint_{\mathscr{R} \cap S(0, 1/2-\epsilon)} |K| dA = \frac{1}{2} \iint_{\mathscr{R} \cap S(-1/2+\epsilon, 1/2-\epsilon)} |K| dA < 2\pi,$$

and hence $\mathscr{R} \cap S(0, 1/2-\epsilon)$ is stable.

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Let $r' > r > 0$, where r is the number used in the definition of X, and $X' = \{(x, y, -1/2) \big| x \geq -r'\} \supset X$. Let $\mathscr{R}$ be a basic piece of a Riemann's example whose boundary $\partial\mathscr{R} = \partial X' \cup K(\partial X')$. Let $D_{\mathscr{R}}$ be the open plane disk such that $\partial D_{\mathscr{R}} = P_0 \cap \mathscr{R}$. Note that $\partial D_{\mathscr{R}}$ is a circle centered at $(0, 0)$. Let A be the minimal annulus which has been studied through Lemmas 1–4. Let D_A be the closed plane disk such that $\partial D_A = A \cap P_0$.

LEMMA 7. *$P_0 \cap A \not\subset D_{\mathscr{R}}$; in particular, $D_A \not\subset D_{\mathscr{R}}$.*

PROOF. Let D_t be the disk bounded by the circle $P_t \cap \mathscr{R}$. Since $\partial A \subset \overline{X} \cup K(\overline{X}) \subset X' \cup K(X')$ is compact, there is a $0 < d < 1/2$ such that whenever $0 < \epsilon < d$, $A \cap P_{1/2-\epsilon} \subset D_{1/2-\epsilon}$, and $A \cap P_{-1/2+\epsilon} \subset D_{-1/2+\epsilon}$. If $A \cap P_0 \subset D_{\mathscr{R}}$, then by Theorem A and the fact that $S(0, 1/2-\epsilon) \cap \mathscr{R}$ is stable, $A \cap S(0, 1/2-\epsilon) \cap \mathscr{R} = \emptyset$. Similarly, $\mathscr{R} \cap A \cap S(-1/2+\epsilon, 0) = \emptyset$. Thus $A \cap \mathscr{R} \cap S(-1/2+\epsilon, 1/2-\epsilon) = \emptyset$. However, since $\mathscr{R} \cap S(-1/2+\epsilon, 1/2-\epsilon)$ is unstable for small ϵ, by Theorem A, $A \cap \mathscr{R} \cap S(-1/2+\epsilon, 1/2-\epsilon) \neq \emptyset$. This contradiction proves the lemma.

Let $B_n \subset S(-1/2, 1/2)$ be a sequence of non-planar compact minimal annuli. Suppose that $K(B_n) = B_n$, $R(B_n) = B_n$, and $\partial B_n \subset \overline{X} \cup K(\overline{X}) \subset X' \cup K(X')$ is a pair of convex Jordan curves. If B_n converges to a minimal surface, we want to know the limit behaviour of $U_n := B_n \cap P_0$. By Lemma 7, the limit cannot shrink to a point inside $D_{\mathscr{R}}$. Since each U_n is a strictly convex Jordan curve and invariant under K and R, the limit is either a convex Jordan curve or a segment on the x or y-axis.

LEMMA 8. *U_n cannot converge to a segment on the y-axis.*

PROOF. Let d be the radius of the circle $P_0 \cap \mathscr{R}$. If U_n converges to a segment on the y-axis, by Lemma 4 the limit is a finite segment of length $2d' < \infty$, and $d' \geq d > 0$ by Lemma 7. Let p_n be one of the two fixed points of K on B_n which lies in the half space $H = \{(x, y, z) | y \geq 0\}$. A theorem of Meeks and White says that the Gauss map of B_n is one-to-one and $\iint_{B_n} |K| dA < 4\pi$, see [6, Lemma 2.2]. Since $B_n = (B_n \cap H) \cup R(B_n \cap H)$,

$$\iint_{B_n \cap H} |K| dA = \frac{1}{2} \iint_{B_n} |K| dA < 2\pi. \tag{5}$$

Hence $B_n \cap H$ is a stable minimal disk by a theorem of Barbosa and Do Carmo.

For n large enough, $p_n = (0, d_n, 0)$ with $d_n \geq d/2$. Since U_n is invariant under K and converges to a finite line segment on the y-axis, the plane curvature of U_n at p_n, $k(p_n)$, would go to infinity as $n \to \infty$. In fact, let u_n be the function that defines the minimal graph $B_n \cap H$ in Lemma 4, then $d_n = u_n(0, 0)$ and by symmetry,

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$\partial u_n/\partial x(0,0) = 0$. Thus if U_n converges to a line segment on the y-axis, then $u_n(0,0) \to d' \geq d > 0$. Since U_n is strictly convex and $R(U_n) = U_n$, for n large enough, there is a unique $x_n > 0$ such that $u_n(x_n, 0) = d'/2$, and

$$\frac{d'}{2} = u_n(x_n, 0) = u_n(0,0) + \frac{1}{2}x_n^2 \frac{\partial^2 u_n}{\partial x^2}(0,0) + o(x_n^2).$$

Since U_n converges to a line segment, $x_n \to 0$, it follows that

$$\left| \frac{1}{2}x_n^2 \frac{\partial^2 u_n}{\partial x^2}(0,0) \right| \geq \frac{d'}{3}.$$

Again $x_n \to 0$ forces that $|\partial^2 u_n/\partial x^2(0,0)| \to \infty$ as $n \to \infty$.

Since p_n is the only fixed point of $B_n \cap H$ under K, $k(p_n)$ is a principal curvature of B_n at p_n. Thus the Gauss curvature of B_n at p_n is $K(p_n) = -k^2(p_n)$. It would be

$$\lim_{n\to\infty} |K(p_n)| = \infty. \tag{6}$$

We have the Euclidean distance $\operatorname{dist}(p_n, \partial(B_n \cap H)) \geq d'' := \min\{d/2, 1/2\}$. We claim that the geodesic ball of $B_n \cap H$ centered at p_n has radius $r_n \geq d''$. If not, then $r_n < d''$. Since there are no conjugate points on a minimal surface, there is then an interior point q_n for which there are two length minimizing geodesics connecting p_n and q_n. Thus there is a loop γ_n such that $\gamma_n \cap \partial(B_n \cap H) = \emptyset$, $\gamma_n(0) = \gamma_n(1) = p_n$, $\gamma_n(1/2) = q_n$ and γ_n is a geodesic on $(0, 1/2)$ and $(1/2, 1)$. Let θ_1 and θ_2 be the exterior angles of γ_n at p_n and q_n, $-\pi < \theta_j < \pi$, $j = 1, 2$. Since $B_n \cap H$ is simply connected, γ_n bounds a disk $D_n \subset B_n \cap H$. By the Gauss-Bonnet Formula we have $\iint_{D_n} K dA + \theta_1 + \theta_2 = 2\pi$. Since $\iint_{D_n} K dA < 0$, we would have $\theta_1 + \theta_2 > 2\pi$, which is impossible. Hence we have proved that $r_n \geq d''$.

Since $B_n \cap H$ is a stable embedded minimal surface, by an estimate of Schoen, see [10], there is a constant $c > 0$ such that

$$|K(p_n)| \leq c r_n^{-2} \leq c d''^{-2},$$

contradicting (6). This contradiction proves the lemma.

3. The proof of Theorem B

We break the proof into several steps. In the following, we will not distinguish a sequence and its subsequence in notation.

Step 1: To establish two sequences of approximate minimal annuli.

Let $D_n \subset D_{n+1} \subset \overline{X}$ be open disks bounded by smooth convex Jordan curves $C_n \subset \overline{X}$, $R(C_n) = C_n$, and $\lim_{n\to\infty} D_n = X$. We can arrange that for each positive

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integer M, there is a positive integer $N(M)$ such that $X \cap \{x \le M\} = D_n \cap \{x \le M\}$ whenever $n \ge N(M)$.

Since there is a nonplanar compact minimal annulus A' such that $\partial A' \subset X \cup K(X)$, we can assume that $\partial A' \subset D_n \cup K(D_n)$. By Theorem A, there are exactly two nonplanar compact minimal annuli A_n and B_n in $S(-1/2, 1/2)$, such that $\partial A_n = \partial B_n = C_n \cup K(C_n)$. A_n is stable, B_n is unstable.

Step 2: To prove that there is a convergent subsequence of $\{A_n\}$ (resp. $\{B_n\}$).

The proof is the same for A_n and B_n.

Let $H = \{(x, y, z) \mid y \ge 0\}$, $H_n = A_n \cap H$ and let $S(s)$ be the slab $S(s) = \{(x, y, z) \mid -s \le x \le s\}$. By Lemma 1, the intersection of A_n and the xz-plane P consists of two graphs $\sigma_{n1} = \{(x, 0, z) \mid x = f_{n1}(z), -1/2 \le z \le 1/2\}$ and $\sigma_{n2} = \{(x, 0, z) \mid x = f_{n2}(z), -1/2 \le z \le 1/2\}$. By Lemma 3, H_n is a minimal graph over a domain Ω_n contained in P, where Ω_n is defined by

$$\Omega_n = \{(x, 0, z) \mid f_{n1}(z) < x < f_{n2}(z), \quad -1/2 < z < 1/2\}.$$

For $s > r$, $H_n \cap S(s)$ is topologically a disk and $\partial(H_n \cap S(s))$ is a piecewise smooth Jordan curve. Let $D := \{z \in \mathbb{C} \mid |z| < 1\}$ and let $X_n : D \to \mathbb{R}^3$ be the conformal embedding of $H_n \cap S(s)$. Since for n large enough, $C_n \cap S(s) = \partial X \cap S(s)$, we can arrange that each X_n maps three fixed points on ∂D to three fixed points on the arc $\partial X \cap S(s) \cap H$.

Let $l_n(s)$ be the arc length of $\partial(H_n \cap S(s))$. By Lemma 4, $l_n(s)$ is uniformly bounded by $2(2 + 4s + 3h(s))$. By the isoperimetric inequality for minimal disks, see [7, p. 280], Area$(H_n \cap S(s)) \le (l_n(s))^2/4\pi$. Since X_n is conformal, the X_n have uniformly bounded Dirichlet's integral. By the Courant-Lebesgue Lemma, the X_n are equicontinuous on ∂D, and hence on passing to a subsequence if necessary, $H_n \cap S(s)$ uniformly converges to a minimal surface $\mathscr{D}(s) \subset S(-1/2, 1/2) \cap S(s)$ parametrized by $Y_s = \lim_{n\to\infty} X_n : D \to \mathbb{R}^3$, $\mathscr{D}(s) = Y_s(D)$.

By a diagonal argument, in any compact subset of $S(-1/2, 1/2)$, H_n uniformly converges to a minimal surface $\mathscr{D}$, $\mathscr{D}(s) = \mathscr{D} \cap S(s)$. Since for each $-1/2 < t < 1/2$, $H_n \cap P_t$ is strictly convex, $\mathscr{D} \cap P_t$ is convex. Remember that $\sigma_{n1} \cup K(\sigma_{n1}) = H_n \cap P$ in Lemma 1.

$$\sigma_{n1} = \{(x, 0, z) \mid x = f_{n1}(z) < 0\}.$$

For n large, by our construction of C_n, $f_{n1}(-1/2) = -r$. Since σ_{n1} is convex, $|f_{n1}(z_1)| \le \max\{r, |f_{n1}(z_2)|\}$ for $-1/2 \le z_1 \le z_2 \le 1/2$. It follows that if $\lim_{n\to\infty} f_{n1}(z_2)$ exists and is finite, then $f_1(z) := \lim_{n\to\infty} f_{n1}(z)$ exists and is finite for $-1/2 \le z \le z_2$. Thus there is a d, $-1/2 \le d \le 1/2$, such that

$$\sigma_1 := \{(x, 0, z) \mid x = f_1(z) \quad -1/2 \le z < d\}$$

is a well defined graph over $I = [-1/2, d)$. Note that it may be the case that $d = -1/2$.

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Now let $\Omega \subset P \cap S(-1/2, 1/2)$ be such that $\partial\Omega = \sigma_1 \cup K(\sigma_1) \cup L_1 \cup L_2$, where $L_1 = \{(x, 0, -1/2) \mid x \geq -r\}$ and $L_2 = K(L_1)$. If $d = -1/2$, then $\Omega = \text{int}(P \cap S(-1/2, 1/2))$. By Lemmas 1 and 2, if $(0, 0, 0) \notin \sigma_1$, then Ω is a domain, that is, an open connected set.

Suppose $d > -1/2$. Since $\lim_{\sigma_{n1}} = \sigma_1$, $\Omega = \bigcup_{m=1}^{\infty} \bigcap_{n=m}^{\infty} \Omega_n$. Since $\overline{\Omega_n}$ is the orthogonal projection of H_n, $\overline{\Omega}$ is the orthogonal projection of $\mathscr{D}$. Taking any $(x, y, z) \in \mathscr{D}$, we have $(x, 0, z) \in \overline{\Omega}$. Suppose that $(x, 0, z) \in \text{int}(\Omega)$. Let $s > \max\{|x|, r\}$. Since $\sigma_{n1} \cap S(s)$ uniformly converges to $\sigma_1 \cap S(s)$, there is an $m > 0$ such that there is an open ball U such that $(x, 0, z) \in U \subset \bigcap_{n \geq m} \Omega_n$. Since H_n converges to $\mathscr{D}$, u_n converges to a function u on U and hence $\mathscr{D} \cap \{(x, y, z) \mid (x, 0, z) \in U\}$ is a minimal graph. In particular, $(x, y, z) = (x, u(x, z), z)$ is an interior point of $\mathscr{D}$. By the maximum principle, $y = u(x, z) > 0$. This also proves that $\sigma_1 \cup K(\sigma_1) = \mathscr{D} \cap P = \partial\mathscr{D} \cap P$.

Thus if we can prove that $(0, 0, 0) \notin \sigma_1$, then Ω is open and $\mathscr{D}$ is a minimal graph hence is embedded.

We now consider the surface $\mathscr{D} \cup R(\mathscr{D})$. If $d = -1/2$, then $\mathscr{D} \cup R(\mathscr{D})$ is a minimal surface. If $d > -1/2$, then I is an interval. By continuity of $\mathscr{D}$, for any closed subinterval $J \subset I$ and $z_0 \in J$, there is a $\delta(z_0) > 0$ such that the orthogonal projection of the convex curve $\mathscr{D} \cap P_z$ on the y-axis contains an interval $[0, \delta(z_0)]$ for $z \in (z_0 - \delta(z_0), z_0 + \delta(z_0)) \subset I$. Thus the orthogonal projection of $\mathscr{D} \cap (\cup_{z \in J} P_z)$ on the yz-plane contains a domain D' such that $\partial D' \cap P \supset J$. Since H_n uniformly converges to $\mathscr{D}$, for large n, the orthogonal projection of H_n on the yz-plane contains a common domain $D'' \subset D'$ such that $\partial D'' \cap P \supset J$. Now since $A_n \cap P_z$ is strictly convex for $z \in J$, a component of A_n containing $\sigma_{n1} \cap (\cup_{z \in J} P_z)$ is a minimal graph over $D'' \cup R(D'')$. Let v_n be the function that defines this graph. We have $0 > v_n(y, z) \geq f_{n1}(z) = v_n(0, z)$ since $(f_{n1}, 0, z)$ is the extreme point of $A_n \cap P_z$ in $\{x < 0\}$. Thus on $D'' \cup R(D'')$, v_n converges. This proves that $\mathscr{D} \cup R(\mathscr{D})$ is a minimal surface if $\sigma_1 \cap K(\sigma_1) = \emptyset$.

Thus in the case that $d = -1/2$ or that $d > -1/2$ and $\sigma_1 \cap K(\sigma_1) = \emptyset$, $\mathscr{D} \cup R(\mathscr{D})$ is a minimal surface.

We denote $\mathscr{D} \cup R(\mathscr{D})$ by $\mathscr{A}$ (resp. $\mathscr{B}$). We need to prove that σ_1 is a graph defined on $-1/2 \leq z < 1/2$. We first analyse $\mathscr{D} \cap P_0$. Note that so far we cannot say that $\mathscr{D} \cap P \cap P_0 \neq \emptyset$.

Step 3: To prove that $\mathscr{A} \cap P_0$ (resp. $\mathscr{B} \cap P_0$) is a convex Jordan curve.

First we claim that $\mathscr{D} \cap P \cap P_0 \neq \emptyset$. Let $(-a_n, 0, 0)$ and $(a_n, 0, 0)$ be the two extreme points of $A_n \cap P_0$ (resp. $B_n \cap P_0$). Let $(0, p_n, 0)$ be the middle point of $H_n \cap P_0$. By Lemma 4, $p_n \leq h(0)$, and thus $0 \leq p := \lim_{n\to\infty} p_n$ exists. In particular, $\mathscr{D} \cap P_0 \neq \emptyset$. If $\mathscr{D} \cap P \cap P_0 = \emptyset$, then $p > 0$ and $\lim_{n\to\infty} a_n = \infty$. By Step 2 and Lemma 2, $u_n(x, 0) \to u(x, 0)$ for $-\infty < x < \infty$. Thus $\mathscr{D} \cap P_0$ is a smooth convex graph $(x, u(x, z), z)$. Because $K(\mathscr{D} \cap P_0) = \mathscr{D} \cap P_0$, $u(x, 0) \leq p$ and $\partial u/\partial x(0, 0) = 0$,

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$\partial^2 u/\partial x^2(x, 0) \geq 0$. It follows that $\mathscr{D} \cap P_0$ is the straight line $\{y = p\} \cap \{z = 0\}$. It is well known that this would imply that $\mathscr{D}$ is invariant under a rotation of angle π about the straight line $\mathscr{D} \cap P_0$, but it is impossible since the boundary of $\mathscr{D}$ is not symmetric with respect to this kind of rotation. Hence we have proved the claim.

We know that $H_n \cap P_0$ is strictly convex and $K(H_n \cap P_0) = H_n \cap P_0$, hence $H_n \cap P_0 \subset P_0 \cap \{0 \leq y \leq p_n\}$. Thus $\mathscr{D} \cap P_0 \subset P_0 \cap \{0 \leq y \leq p\}$. If $p = 0$, then by Step 2, $(0, 0, 0) \in \partial\Omega$, and thus $(0, 0, 0) \in \sigma_1 \cap K(\sigma_1)$, $\mathscr{D} \cap P_0 = \{(0, 0, 0)\}$.

Hence $\mathscr{A} \cap P_0$ (resp. $\mathscr{B} \cap P_0$) must be either a convex curve, or a line segment on the y-axis, or the point $(0, 0, 0)$.

Let V_n be the compact solid bounded by $A_n \cup D_n \cup K(D_n)$. Since $A' \cap P_0 \subset V_n$, we know that $\mathscr{A} \cap P_0$ must be a convex Jordan curve.

By Lemma 5, 7, and 8, $\mathscr{B} \cap P_0$ can neither be a segment on the y-axis, nor be the point $(0, 0, 0)$. Hence $\mathscr{B} \cap P_0$ must be a convex Jordan curve.

Since $\mathscr{A} \cap P_0$ (resp. $\mathscr{B}$) is a convex curve, by Lemma 2, $\sigma_1 \cap K(\sigma_1) = \emptyset$ and Ω is open. Thus as pointed out in Step 2, $\mathscr{A}$ (resp. $\mathscr{B}$) is an embedded minimal surface. In Steps 4 to 6, the proofs for $\mathscr{A}$ and $\mathscr{B}$ are the same.

Step 4: To prove that there is an $\epsilon > 0$, such that $\mathscr{A} \cap P_t$ (resp. $\mathscr{B} \cap P_t$) is a convex Jordan curve for $-\epsilon \leq t \leq \epsilon$.

Let $(-a, 0, 0)$ and $(a, 0, 0)$, $a > 0$, be the two extreme points of $\mathscr{A} \cap P_0$. Then the number d defined in Step 2 must be greater than or equal to 0. Let $s > \max\{a, r\}$, where r is the number that defines the domain X. Consider the strictly convex boundary curve $\sigma_{n1}(s) \subset \partial(\Omega_n \cap S(s))$ in Lemma 1,

$$\sigma_{n1}(s) = \{(x, 0, z) \mid x = f_{n1}(z),\ -1/2 \leq z \leq 1/2,\ -s \leq x < 0\}.$$

We denote $(f_{n1}(-1/2), 0, -1/2)$, $(f_{n1}(0), 0, 0)$, and $(-s, 0, f_{n1}^{-1}(-s))$ by $q_n(1)$, $q_n(2)$, and $q_n(3)$. Note that for $x > r$, $f_{n1}^{-1}(-x)$ is well defined. Since $s > \max\{r, a\}$ and $-1/2 < f_{n1}^{-1}(-s) < 1/2$, we may assume that $\epsilon(s) = \lim_{n\to\infty} f_{n1}^{-1}(-s)$ exists. Thus f_1 is defined at $z = \epsilon(s)$ and $f_1(\epsilon(s)) = -s < -a = f_1(0)$. Now since σ_{n1} is convex, $|f_{n1}(z)| \leq \max\{r, |f_{n1}(0)|\}$ for $-1/2 \leq z \leq 0$. Were $\epsilon(s) \in [-1/2, 0]$, then $|f_1(\epsilon(s))| \leq \max\{r, a\}$. Since $f_1(\epsilon(s)) = -s$, $\epsilon(s) > 0$. Thus $d \geq \epsilon(s) > 0$. Select $0 < \epsilon < d$, then it follows that $\mathscr{A} \cap P \cap \{-\epsilon \leq z \leq \epsilon\} = (\sigma_1 \cup K(\sigma_1)) \cap \{-\epsilon \leq z \leq \epsilon\}$. This implies that the convex curve $\mathscr{D} \cap P_z$ has two extreme points $(f_1(z), 0, z)$ and $(-f_1(-z), 0, z)$, for $-\epsilon(s) \leq z \leq \epsilon(s)$. Thus $\mathscr{A} \cap P_z$ is a convex Jordan curve for $-\epsilon(s) \leq z \leq \epsilon(s)$.

One way to prove that $\mathscr{A} \cap P_t$ (resp. $\mathscr{B} \cap P_t$) is a convex Jordan curve for $-1/2 < t < 1/2$ is to prove that $d = 1/2$ or $\lim_{s\to\infty} \epsilon(s) = 1/2$. It requires a detailed study of the behaviours of the functions f_{n1}. Instead of doing that, we argue in an indirect manner. The benefit of our argument is that we can get an Enneper-Weierstrass representation of $\mathscr{A}$ and $\mathscr{B}$.

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Step 5: To prove that the Gauss map $N : \mathscr{A} \to S^2$ (resp. $N : \mathscr{B} \to S^2$) is not vertical along $\mathscr{A} \cap S(-\epsilon/2, \epsilon/2)$ (resp. $\mathscr{B} \cap S(-\epsilon/2, \epsilon/2)$).

Since the compact minimal annulus $\mathscr{A} \cap S(-\epsilon/2, \epsilon/2)$ is contained in the interior of $\mathscr{A}$, N is well defined. By Shiffman's first theorem N is not vertical on $\mathscr{A} \cap S(-\epsilon/2, \epsilon/2)$.

Step 6: To prove that $\mathscr{A} \cap P_t$ (resp. $\mathscr{B} \cap P_t$) is a convex Jordan curve, for $-1/2 < t < 1/2$.

Each A_n (resp. B_n) is an annulus with one-dimensional boundary, hence there is a unique R_n, $1 < R_n < \infty$, such that A_n is conformally equivalent to the annulus $A(R_n) = \{z \in \mathbb{C} \mid 1/R_n < |z| < R_n\}$. Let $X_n : A(R_n) \to \mathbb{R}^3$ be the conformal embedding of A_n. The third coordinate function X_{n3} is harmonic and maps $|z| = 1/R_n$ and $|z| = R_n$ to $-1/2$ and $1/2$ respectively. Thus it must be the case that

$$X_{n3}(z) = \frac{1}{2\log(R_n)} \log(|z|). \tag{7}$$

Let g_n be the Gauss map of the embedding X_n. It is a holomorphic map and $g_n = \tau \circ N_n \circ X_n$, where τ is the stereographic projection and $N_n : A_n \to S^2$ is the Gauss map of A_n.

By the Enneper-Weierstrass representation,

$$X_n(z) = \operatorname{Re} \int_1^z \left(\frac{1}{2}(1 - g_n^2)\eta_n,\ \frac{i}{2}(1 + g_n^2)\eta_n,\ g_n\eta_n \right) + V_n, \tag{8}$$

where η_n is a holomorphic 1-form and V_n is a constant vector in P_0. Comparing (7) with (8), we have $\eta_n = dz/(2\log(R_n) z g_n)$. The metric of A_n (resp. B_n) is given by

$$ds_n = \frac{1}{2}(1 + |g_n|^2)|\eta_n| = \left(\frac{1}{|g_n|} + |g_n| \right) \frac{|dz|}{4\log(R_n)|z|},$$

see, for example, [7, p. 147].

Since $N_n \to N$ on $A_n \cap S(-\epsilon/2, \epsilon/2)$ uniformly as $n \to \infty$, by Step 5 we know that there is a $B > 0$, such that for n large enough,

$$\frac{1}{B} < |g_n(z)| < B, \qquad R_n^{-\epsilon} \le |z| \le R_n^{\epsilon}. \tag{9}$$

Let $L_n(t)$ be the arc length of $A_n \cap P_t$. By a theorem of Osserman and Schiffer [9], L_n satisfies $L_n''(t) > 0$, for $-1/2 < t < 1/2$. (Note that what Osserman and Schiffer proved in Lemma 1 of [9] is that $d^2 L_n / d(\log r)^2 > 0$, $r = |z|$. In our case, $t = \log r / 2 \log R_n$.) Since A_n is invariant under K, we have $L_n(t) = L_n(-t)$. Thus $L_n'(t) = -L_n'(-t)$, and in particular $L_n'(0) = 0$. Hence $L_n(0)$ is the only minimum value of L_n, and L_n is strictly increasing for $0 < t < 1/2$ and strictly decreasing

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for $-1/2 < t < 0$. If $\mathscr{A} \cap P_t$ is not compact, then $\lim_{n\to\infty} L_n(t) = \infty$. For any $|t| < |s| < 1/2$, $\lim_{n\to\infty} L_n(s) \geq \lim_{n\to\infty} L_n(t) = \infty$ implies that $\mathscr{A} \cap P_s$ is not compact. Hence to prove that $\mathscr{A} \cap P_t$ is a Jordan curve for $-1/2 < t < 1/2$, it is enough to prove that there is a sequence $0 < t_j \uparrow 1/2$ and $M(j) > 0$ such that for each sequence $\{L_n(t_j)\}$ there is a subsequence $\{L_m(t_j)\} \subset \{L_n(t_j)\}$ such that $L_m(t_j) \leq M(j)$.

Since $\mathscr{A} \cap P_0$ is a convex Jordan curve, we know that $\lim_{n\to\infty} L_n(0) = L(0) > 0$, where $L(0)$ is the arc length of $\mathscr{A} \cap P_0$. Let C'' be the curve $\{|z| = 1\}$ in $A(R_n)$,

$$L_n(0) = \frac{1}{4\log(R_n)} \int_{C''} \left(\frac{1}{|g_n(z)|} + |g_n(z)| \right) |dz|.$$

Because of (9) we have

$$\frac{2\pi}{4L_n(0)} \frac{2}{B} \leq \log(R_n) = \frac{1}{4L_n(0)} \int_{C''} \left(\frac{1}{|g_n(z)|} + |g_n(z)| \right) |dz| \leq \frac{2\pi}{4L_n(0)} 2B,$$

for n large enough. Hence there is a subsequence of $\{R_n\}$ and $1 < R < \infty$, such that $\lim_{n\to\infty} R_n = R$.

Let $A(R) = \lim_{n\to\infty} A(R_n)$ be the limit annulus in $\mathbb{C}$. Since the Gauss map g_n is one-to-one from $A(R_n)$ to $\mathbb{C} - \{0\}$, see [6, Lemma 2.2], $g_n(C'')$ is a Jordan curve in $\mathbb{C} - \{0\}$. By (9) and the one-to-one property of g_n, a subsequence of $\{|g_n|\}$ is uniformly bounded on either $\{1/R_n < |z| \leq R_n^{\epsilon}\}$ or $\{R_n^{-\epsilon} \leq |z| < R_n\}$, say on the latter. Any compact sub-annulus A'' in $\{R^{-\epsilon/2} \leq |z| < R\}$ is eventually contained in $A(R_n)$. Since the g_n's are uniformly bounded on A'', there is a subsequence of $\{g_n\}$ converging to a holomorphic function g on A''. Thus we may assume that g_n converges to g uniformly on compact sets of $\{R^{-\epsilon/2} < |z| < R\}$. Again by (9), $|g_n| > 1/B$ on $C'' \subset \{R^{-\epsilon/2} < |z| < R\}$, $g \not\equiv 0$. Since $g_n \neq 0$ in $A(R_n)$, by Rouché's theorem, $g \neq 0$ in $\{R^{-\epsilon/2} < |z| < R\}$. Hence there are sequences $r_j \uparrow R$, $\epsilon_j \downarrow 0$, $r_j + \epsilon_j < R$, such that on the compact annuli $A(j) = \{z \mid r_j - \epsilon_j \leq |z| \leq r_j + \epsilon_j\}$, g satisfies $0 < d_j \leq |g| \leq D_j$. For large n, $A(j) \cap A(R_n) = A(j)$, and g_n uniformly converges to g on $A(j)$. Thus for large n, on $A(j)$ we have $d_j/2 \leq |g_n| \leq 2D_j$, and

$$ds_n = \left(\frac{1}{|g_n|} + |g_n| \right) \frac{|dz|}{4|z|\log(R_n)} \leq \left(\frac{1}{d_j} + D_j \right) \frac{|dz|}{R\log(R)} := \frac{M(j)}{R}|dz|.$$

Let $t_j = \log(r_j)/2\log(R)$, then

$$t_j = \lim_{n\to\infty} \frac{\log(r_j)}{2\log(R_n)}.$$

Choose s_n such that $\{|z| = s_n\} \subset A(j)$ and $t_j = \log(s_n)/2\log(R_n)$.

$$L_n(t_j) = \int_{|z|=s_n} ds_n \leq \int_0^{2\pi} M(j)d\theta = 2\pi M(j).$$

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We have proved that $\mathscr{A} \cap P_t$ is a Jordan curve for $-1/2 < t < 1/2$. It follows that $\mathscr{A}$ is an embedded minimal annulus.

By symmetry we know that g_n converges to g uniformly on compact sets of $A(R)$, thus the Enneper-Weierstrass representation of $\mathscr{A}$ is

$$X(z) = \frac{1}{2\log R}\operatorname{Re}\int_1^z \left(\frac{1}{2}\left(1-g^2\right)(\zeta), \frac{i}{2}, \left(1+g^2\right)(\zeta), g(\zeta)\right)\frac{d\zeta}{\zeta g(\zeta)} + V, \ \frac{1}{R} < |z| < R.$$

Step 7: To prove the remaining claims in Theorem B.

For any $0 < \delta < 1/2$, since $\mathscr{A} \cap P_{1/2-\delta}$ and $\mathscr{A} \cap P_{-1/2+\delta}$ (resp. $\mathscr{B} \cap P_{1/2-\delta}$ and $\mathscr{B} \cap P_{-1/2+\delta}$) are convex, by Shiffman's first theorem, $\mathscr{A} \cap P_t$ (resp. $\mathscr{B} \cap P_t$) is strictly convex for $-1/2+\delta < t < 1/2-\delta$. Let δ go to zero, then $\mathscr{A} \cap P_t$ (resp. $\mathscr{B} \cap P_t$) is strictly convex for $-1/2 < t < 1/2$.

Let N be any connected non-planar compact branched minimal surface such that $\partial N \subset \overline{X} \cup K(\overline{X})$. If $N \cap \mathscr{B} = \emptyset$, then since N is compact and $\mathscr{B}$ is closed in $\mathbb{R}^3$, $\operatorname{dist}(N, \mathscr{B}) > 0$, which contradicts the facts that $N \cap B_n \neq \emptyset$ and $\mathscr{B} = \lim_{n\to\infty} B_n$. Thus it must be the case that $\mathscr{B} \cap N \neq \emptyset$. In particular, if $A' \subset X \cup K(X)$, then $\operatorname{int}(\mathscr{B}) \cap \operatorname{int}(A') \neq \emptyset$ since $\partial A' \cap \partial \mathscr{B} = \emptyset$.

Let V_n be the solid bounded by $A_n \cup D_n \cup K(D_n)$ and V be the solid bounded by $\mathscr{A} \cup \overline{X} \cup K(\overline{X})$. Then $B_n \subset V_n \subset V$ and hence $\mathscr{B} = \lim B_n \subset V$. By the comparison principle, $\mathscr{A} = \mathscr{B}$ or $\operatorname{int}(\mathscr{A}) \cap \operatorname{int}(\mathscr{B}) = \emptyset$. By the same argument we see that $\operatorname{int}(N) \cap \operatorname{int}(\mathscr{A}) = \emptyset$, for any connected nonplanar compact branched minimal surface N such that $\partial N \subset \overline{X} \cup K(\overline{X})$. In particular, $\operatorname{int}(A') \cap \operatorname{int}(\mathscr{A}) = \emptyset$ and hence if $A' \subset X \cup K(X)$ then $\mathscr{A} \neq \mathscr{B}$ and $\operatorname{int}(\mathscr{A}) \cap \operatorname{int}(\mathscr{B}) = \emptyset$.

The proof of Theorem B is complete.

REMARK 2. By Theorem A and the proof of Theorem B, we see that if merely $\partial A' \subset \overline{X} \cup K(\overline{X})$, then there is at least one minimal annulus $\mathscr{A}$ such that $\partial \mathscr{A} = \Gamma$ and $\mathscr{A} \cap P_t$ is strictly convex for $-1/2 < t < 1/2$.

Let N be any connected compact nonplanar branched minimal surface such that $\partial N \subset \overline{X} \cup K(\overline{X})$. Then $\mathscr{A}$ satisfies

$$\operatorname{int}(\mathscr{A}) \cap \operatorname{int}(N) = \emptyset.$$

REMARK 3. Checking the proof of Theorem B, we find that in the definition of the boundary $C \subset P_{-1/2}$, the relevant part is the existence of the inverse $h(s)$ of the C^∞ function f for $s > -r$. Hence even if for some $x > -r$, $h(s)$ is a constant for $s > x$, the proof of Theorem B is still valid. Thus we may assume that the boundary curve C is C^∞ convex, $R(C) = C$, and C contains two rays parallel to the x-axis.

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Centre for Mathematics and its Applications
School of Mathematical Sciences
The Australian National University
Canberra ACT 0200
Australia
e-mail: yi@maths.anu.edu.au

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