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AN EXAMPLE IN THE THEORY OF BILINEAR MAPS

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ABSTRACT. We give an example of a p-convex quasi-Banach space E with $0<p<1$ such that every bilinear map $B:E\times E\to F$ into a p-convex quasi-Banach space F is identically zero. This resolves a question of Waelbroeck.

An admissible topology on the tensor product $E\otimes F$ of two topological vector spaces E and F is any vector topology such that the natural bilinear form $E\times F\to E\otimes F$ is continuous. The question, raised by Waelbroeck (cf. [4], [5] and [6]), of whether there is a Hausdorff admissible vector topology on $E\otimes F$ for any pair of spaces E and F has recently been answered in the affirmative by Turpin ([1] and [2]). If E and F are p-convex quasi-Banach spaces, Turpin [2] shows that $E\otimes F$ may be given an r-convex quasi-norm topology where $r=\max(\frac{1}{2}p, p^2)$. In this note we show that it is not in general possible to give $E\otimes F$ a p-convex quasi-norm topology, thus answering a question raised by Waelbroeck [4] and Turpin [2]. In fact we produce a p-convex quasi-Banach space E such that every bilinear form $B:E\times E\to F$ into a p-convex quasi-Banach space is identically zero.

For the example, let Γ be the unit circle in the complex plane and denote by m normalized Haar measure on the circle, i.e. $dm=(2\pi)^{-1}\,d\theta$. We shall consider the space $L_p(\Gamma,m)$ (where $0<p<1$) of complex-valued m-measurable functions on Γ such that

$$\|f\|=\left(\int_\Gamma |f|^p\,dm\right)^{1/p}<\infty.$$

Suppose F is any p-convex quasi-Banach space; we may assume the quasi-norm on F p-subadditive i.e.

$$\|x_1+x_2\|^p\le\|x_1\|^p+\|x_2\|^p \qquad x_1, x_2\in F.$$

Let $B:L_p(\Gamma)\times L_p(\Gamma)\to F$ be any continuous bilinear map; for some $K<\infty$

$$\|B(f,g)\|\le K\,\|f\|\,\|g\| \qquad f,g\in L_p$$

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In [3] Vogt identifies the tensor product $L_p \hat{\otimes}_p L_p$ with $L_p(\Gamma \times \Gamma, m \times m)$; in our setting this implies that there is a continuous linear operator $T: L_p(\Gamma \times \Gamma, m \times m) \to F$ with $\|T\| \leq K$ and

$$T(f \otimes g) = B(f, g) \qquad f, g \in L_p(\Gamma)$$

where

$$f \otimes g(w, z) = f(w)g(z) \qquad w, z \in \Gamma.$$

Denote by H_p the usual Hardy subspace of L_p, i.e. the closure in $L_p(\Gamma, m)$ of the polynomials.

PROPOSITION. *Suppose* $0 < p < 1$ *and* $E = L_p/H_p$. *Let* F *be any p-convex quasi-Banach space. If* $B: E \times E \to F$ *is a continuous bilinear form, then* B *is identically zero.*

Proof. Let $\pi: L_p \to E$ be the quotient map and consider $B_0: L_p(\Gamma) \times L_p(\Gamma) \to F$ defined by $B_0(f, g) = B(\pi f, \pi g)$. As above there is a continuous linear operator $T: L_p(\Gamma \times \Gamma) \to F$ with $T(f \otimes g) = B_0(f, g)$. For $k \in Z$ let $e_k(z) = z^k$, $z \in \Gamma$. Then $e_k \otimes e_n \in L_p(\Gamma \times \Gamma)$ and $e_k \otimes e_n(w, z) = w^k z^n$, $w, z \in \Gamma$. The collection $(e_k \otimes e_n; k, n \in Z)$ has dense linear span in $L_p(\Gamma \times \Gamma)$. We shall show $T(e_k \otimes e_n) = 0$ for all k, n. If either k or n is non-negative then

$$T(e_k \otimes e_n) = B(\pi e_k, \pi e_n) = 0.$$

Otherwise suppose $k < 0$ and $n < 0$ and choose l so large that $l + k > 0$ and $l + n > 0$. As L_p has trivial dual for $p < 1$, given $\varepsilon > 0$ we can find N and $(c_j : -N \leq j \leq N)$ such that $c_0 = 1$ and

$$\left\| \sum_{j=-N}^{N} c_j e_j \right\| \leq \varepsilon$$

It is immediate that

$$\int_\Gamma \int_\Gamma \left| \sum_{j=-N}^{N} c_j w^{jl} z^{-jl} \right|^p dm(w)\, dm(z) \leq \varepsilon^p$$

and hence multiplying through by $w^k z^n$ inside the absolute value signs

$$\left\| \sum_{j=-N}^{N} c_j e_{k+jl} \otimes e_{n-jl} \right\| \leq \varepsilon$$

Now if $k + jl < 0$ and $n - jl < 0$ we have $n/l < j < -k/l$ i.e. $j = 0$. Hence $T(e_{k+jl} \otimes e_{n-jl}) = 0$ for $j \neq 0$ and so as $c_0 = 1$,

$$\|T(e_k \otimes e_n)\| \leq \|T\| \varepsilon.$$

As $\varepsilon > 0$ is arbitrary, $T(e_k \otimes e_n) = 0$ and we conclude that $T = 0$ and $B = 0$.

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