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SOME BASIC RESULTS FOR PROPER FREE G-MANIFOLDS, WHERE G IS A DISCRETE GROUP

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Abstract Let G be a countable discrete group and let M be a proper free C^r G-manifold and N a C^r G-manifold, where $1 \leqslant r \leqslant \omega$. We prove that if G acts properly and freely also on N and if $\dim(N) \geqslant 2\dim(M)$, then equivariant immersions form an open dense subset in the space $C^r_G(M,N)$ of all equivariant C^r maps from M to N. The space $C^r_G(M,N)$ is equipped with a topology, which coincides with the Whitney C^r topology if G is finite and is suited to studying equivariant maps. We also prove an equivariant version of Thom's transversality theorem and show that $C^\omega_G(M,N)$ is dense in $C^r_G(M,N)$, for $1 \leqslant r \leqslant \infty$.

Keywords: immersion; transverse; real analytic; dense; equivariant

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1. Introduction

Let G be a Lie group and let M and N be proper C^r G-manifolds, $1 \leqslant r \leqslant \omega$ (as usual, C^ω denotes real analytic). We denote the space of all C^r G-maps from M to N by $C^r_G(M,N)$ and equip it with the strong–weak C^r topology defined in [**5**], which is well suited to studying group actions. For compact G, this topology coincides with the Whitney C^r topology.

Let $\mathrm{Imm}^r_G(M,N)$ and $\mathrm{Prop}^r_G(M,N)$ denote the sets of C^r G-equivariant immersions and C^r G-equivariant proper maps from M to N, respectively. We prove the following equivariant version of Whitney's immersion theorem.

Theorem 1.1. *Let G be a countable discrete group and let M and N be proper free C^r G-manifolds, where $1 \leqslant r \leqslant \omega$ and $\dim(N) \geqslant 2\dim(M)$. Then*

(1) $\mathrm{Imm}^r_G(M,N)$ *is open and dense in* $C^r_G(M,N)$, *and*

(2) $\mathrm{Imm}^r_G(M,N) \cap \mathrm{Prop}^r_G(M,N)$ *is open and dense in* $\mathrm{Prop}^r_G(M,N)$.

We also prove an equivariant version of Thom's transversality theorem.

Theorem 1.2. *Let G be a countable discrete group and let M be a proper free C^r G-manifold and N a C^r G-manifold, where $1 \leqslant r \leqslant \omega$. Let N' be a closed C^r G-submanifold*

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of N, and L a closed G-invariant subset of M. Then the set of C^r G-maps from M to N which are transverse to N' along L is open and dense in $C^r_G(M,N)$.

Theorem 1.2 does not always hold if the action of G on M is not free (see [**1**, § 2] or [**2**, § 2]). In both cases, easy counterexamples are constructed, where a finite group acts non-freely.

Finally, we obtain the following density result for real analytic G-equivariant maps.

Theorem 1.3. *Let G be a countable discrete group and let M be a proper free C^ω G-manifold and N a C^ω G-manifold. Then $C^\omega_G(M,N)$ is dense in $C^r_G(M,N)$, where $1 \leqslant r \leqslant \infty$.*

Since the set of all G-equivariant diffeomorphisms is open in $C^r_G(M,N)$ [**5**, Theorem 7.5], we obtain the following corollary.

Corollary 1.4. *Let G be a countable discrete group and let M and N be proper free C^ω G-manifolds. If M and N are C^r G-diffeomorphic, where $1 \leqslant r \leqslant \infty$, then they are C^ω G-diffeomorphic.*

In [**6**, Theorem II] it was proved that $C^\omega_G(M,N)$ is dense in $C^r_G(M,N)$, where $1 \leqslant r \leqslant \infty$, if G is a closed subgroup of a virtually connected Lie group and M and N are proper C^ω G-manifolds. Using the same simple idea as in the proofs of Theorems 1.2 and 1.3, one can also drop the assumption there that the action on N is proper.

2. Proofs of the theorems

Throughout the paper, let G be a countable discrete group. We call M a proper free C^r G-manifold, where $1 \leqslant r \leqslant \omega$, if the action $G \times M \to M$ is properly discontinuous, free and C^r differentiable. Then the map $G \times M \to M \times M$, $(g,x) \mapsto (x,gx)$, is a proper map, i.e. the inverse image of every compact set is compact for it. By [**9**, Corollary I 3.24], the orbit map $\pi_M \colon M \to M/G$ is a covering, i.e. a locally trivial map with fibre G. Unless otherwise stated, M and N will be proper free C^r G-manifolds, where $1 \leqslant r \leqslant \omega$. All manifolds are assumed to be finite dimensional, second countable and without boundary.

As mentioned in § 1, the topology in the set $C^r_G(M,N)$, $1 \leqslant r \leqslant \omega$, of all C^r G-maps from M to N is the strong–weak C^r topology defined in [**5**, §§ 1, 4]. This topology depends on the action of G and coincides with the Whitney C^r topology (see, for example, [**4**, Chapter 2]) if G is finite. In particular, whenever we consider spaces of maps between manifolds without a group action, the topology will be the Whitney C^r topology. In the strong–weak C^r topology the basic neighbourhoods can be defined in the same way as in the Whitney C^r topology, but by using only families of charts in M whose images in the orbit space M/G form a locally finite family. For the notation of basic neighbourhoods etc., see [**5**].

Lemma 2.1. *Let N' be a closed C^r G-submanifold of N and let $x \in M$. Let $f \colon M \to N$ be a G-equivariant map and $\tilde{f} \colon M/G \to N/G$ the map induced by f. Then*

(1) *f is C^k differentiable, if and only if $\tilde{f}$ is C^k differentiable, $1 \leqslant k \leqslant r$;*

(2) *f is an immersion, if and only if $\tilde{f}$ is an immersion;*

(3) *f is a submersion, if and only if $\tilde{f}$ is a submersion;*

(4) *f is proper, if and only if $\tilde{f}$ is proper;*

(5) *f is transverse to N' at x, if and only if $\tilde{f}$ is transverse to N'/G at $\pi_M(x)$;*

(6) *f is a (closed) C^k embedding, if and only if $\tilde{f}$ is a (closed) C^k embedding; and*

(7) *f is a C^k diffeomorphism, if and only if $\tilde{f}$ is a C^k diffeomorphism.*

Proof. The first three claims follow at once from the fact that the orbit maps are coverings. The fourth claim follows from [**5**, Lemmas 3.7 and 3.9]. The last three claims are easy to verify. □

Let $f_0, f_1 \colon M \to N$ be C^r maps. By a C^r homotopy between f_0 and f_1 we mean a homotopy $M \times I \to N$ between f_0 and f_1 which can be extended to be C^r on $M \times J$, where J is some open interval containing the unit interval I.

The following version of the covering homotopy theorem of Palais (see [**7**, Theorem 2.4.1]) holds for properly discontinuous free C^r actions. Although first proved for actions of compact groups, the covering homotopy theorem holds for proper actions as well, as pointed out by Palais in [**8**, § 4.5]. The lift is of class C^k by part (1) of Lemma 2.1.

Theorem 2.2. *Let G be a countable discrete group and let M and N be proper free C^r G-manifolds, $1 \leqslant r \leqslant \omega$. Let $f \colon M \to N$ be a C^k G-map, $0 \leqslant k \leqslant r$. If $\tilde{H} \colon M/G \times I \to N/G$ is any C^k homotopy of the induced map $\tilde{f}$, then there exists a C^k G-homotopy $H \colon M \times I \to N$ of f with induced map $\tilde{H}$.*

Lemma 2.3. *Let M and N be C^r manifolds, $1 \leqslant r \leqslant \omega$. Let $f \colon M \to N$ be a C^r map and let $\mathcal{N}$ be a neighbourhood of f in $C^r(M,N)$. Then f has a neighbourhood $\mathcal{M}$ in $C^r(M,N)$ such that every $h \in \mathcal{M}$ is homotopic to f by some C^r homotopy $H \colon M \times I \to N$ and $H_t \in \mathcal{N}$ for every $t \in I$.*

Proof. We can assume that $\mathcal{N}$ is a basic neighbourhood, i.e. of form

$$\bigcap_{i \in \Lambda} \mathcal{N}^r(f; K_i, (U_i, \varphi_i), (V_i, \psi_i), \varepsilon_i)$$

(see [**5**]). Let J be some bounded open interval containing I and let $H_f \colon M \times J \to N$ be the constant extension of the constant homotopy induced by f. Then

$$\tilde{\mathcal{N}} = \bigcap_{i \in \Lambda} \mathcal{N}^r(H_f; K_i \times I, (U_i \times J, \varphi_i \times \mathrm{id}), (V_i, \psi_i), \varepsilon_i)$$

is a neighbourhood of H_f. By the embedding theorems of Whitney ($1 \leqslant r \leqslant \infty$) and Grauert ($r = \omega$), there exists a closed C^r embedding $e \colon N \to \mathbb{R}^p$, for some p. Let $r \colon W \to e(N)$ be a C^r tubular neighbourhood of $e(N)$. Let $\mathcal{W} \subset C^r(M,N)$ be a neighbourhood of f such that if $h \in \mathcal{W}$, then

$$te \circ f(x) + (1-t)e \circ h(x) \in W,$$

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for every $t \in J$ and for every $x \in M$. Then the mapping

$$A\colon\ \mathcal{W} \to C^r(M \times J, N),$$
$$A(h)(x,t) = e^{-1} \circ r \circ (te \circ f(x) + (1-t)e \circ h(x)),$$

is continuous in the Whitney C^r topology and $A(f) = H_f$. Thus f has a neighbourhood $\mathcal{M}$ such that $A(\mathcal{M}) \subset \tilde{\mathcal{N}}$. Therefore $A(h)_t \in \mathcal{N}$, for every $h \in \mathcal{M}$ and for every $t \in I$. □

The following theorem will be crucial in proving Theorems 1.1, 1.2 and 1.3.

Theorem 2.4. *The map*

$$\tau\colon C^r_G(M,N) \to C^r(M/G, N/G),$$

taking f to the induced map $\tilde{f}$, is open and continuous.

Proof. We begin by proving the continuity. Let $f \in C^r_G(M,N)$ and let

$$\tilde{\mathcal{N}} = \bigcap_{i \in \Lambda} \mathcal{N}^r(\tilde{f}; \tilde{K}_i, (\tilde{U}_i, \tilde{\varphi}_i), (\tilde{V}_i, \tilde{\psi}_i), \varepsilon_i)$$

be a basic neighbourhood of $\tilde{f}$ such that $\tilde{K}_i$ is connected and the diagrams

$$\begin{array}{ccc} \tilde{U}_i \times G & \xrightarrow{\approx} & \pi_M^{-1}(\tilde{U}_i) \\ {\scriptstyle \mathrm{pr}_1}\downarrow & \swarrow {\scriptstyle \pi_M|} & \\ \tilde{U}_i & & \end{array} \qquad \begin{array}{ccc} \tilde{V}_i \times G & \xrightarrow{\approx} & \pi_N^{-1}(\tilde{V}_i) \\ {\scriptstyle \mathrm{pr}_1}\downarrow & \swarrow {\scriptstyle \pi_M|} & \\ \tilde{V}_i & & \end{array}$$

are commutative for every i. The restrictions $\pi_M|\colon U_i \to \tilde{U}_i$ and $\pi_N|\colon V_i \to \tilde{V}_i$ are diffeomorphisms, for some charts U_i of M and V_i of N, respectively, where $GU_i = \pi_M^{-1}(\tilde{U}_i)$, $GV_i = \pi_N^{-1}(\tilde{V}_i)$ and $f(\pi_M^{-1}(\tilde{K}_i) \cap U_i) \subset V_i$. Then

$$\mathcal{N} = \bigcap_{i \in \Lambda} \mathcal{N}^r(f; \pi_M^{-1}(\tilde{K}_i) \cap U_i, (U_i, \tilde{\varphi}_i \circ \pi_M|), (V_i, \tilde{\psi}_i \circ \pi_N|), \varepsilon_i)$$

is a basic neighbourhood of f in $C^r_G(M,N)$ and $\tau(\mathcal{N}) \subset \tilde{\mathcal{N}}$. Consequently, τ is continuous at f. Since f was arbitrary, it follows that τ is continuous.

It remains to prove that τ is open. It suffices to show that τ maps every basic neighbourhood onto an open set in $C^r(M/G, N/G)$. Let f, $\tilde{f}$, $\mathcal{N}$ and $\tilde{\mathcal{N}}$ be as above. By Lemma 2.3, $\tilde{f}$ has a neighbourhood $\tilde{\mathcal{M}}$ such that every $\tilde{h} \in \tilde{\mathcal{M}}$ is homotopic to $\tilde{f}$ by some C^r homotopy $\tilde{H}\colon M/G \times I \to N/G$ and $\tilde{H}_t \in \tilde{\mathcal{N}}$ for every $t \in I$. Since the sets of form $\mathcal{N} \cap \tau^{-1}(\tilde{\mathcal{M}})$ form a neighbourhood basis at f, it is enough to show that τ maps $\mathcal{N} \cap \tau^{-1}(\tilde{\mathcal{M}})$ onto an open set in $C^r(M/G, N/G)$. We will show that $\tau(\mathcal{N} \cap \tau^{-1}(\tilde{\mathcal{M}})) = \tilde{\mathcal{M}}$.

Assume $\tilde{h} \in \tilde{\mathcal{M}}$ and let $\tilde{H}$ be a C^r homotopy between $\tilde{f}$ and $\tilde{h}$ with $\tilde{H}_t \in \tilde{\mathcal{N}}$ for every $t \in I$. By Theorem 2.2, $\tilde{H}$ has a C^r G-equivariant lift $H\colon M \times I \to N$ such that $H_0 = f$. It suffices to show that $H_1 \in \mathcal{N}$. Let $x \in \pi_M^{-1}(\tilde{K}_i) \cap U_i$, for some $i \in \Lambda$. Then $H_t(x) \in GV_i$, for every $t \in I$. Since $f(x) \in V_i$ and $V_i \cap gV_i = \emptyset$ unless g equals the identity element, it follows that $H(\{x\} \times I) \subset V_i$. In particular, $H_1(x) \in V_i$. Since this holds for every $x \in \pi_M^{-1}(\tilde{K}_i) \cap U_i$ and for every $i \in \Lambda$ and also the required inequalities for the norms of the differences of the partial derivatives of f and H_1 clearly hold, it follows that $H_1 \in \mathcal{N}$. □

Proof of Theorem 1.1. By [**4**, Theorems 2.1.1 and 2.1.5], $\mathrm{Imm}^r(M/G, N/G)$ and $\mathrm{Prop}^r(M/G, N/G)$ are open in $C^r(M/G, N/G)$. Thus the openness claims follow by using parts (2) and (4) of Lemma 2.1 and the fact that τ is continuous. By [**4**, Theorem 2.2.12], $\mathrm{Imm}^r(M/G, N/G)$ is dense in $C^r(M/G, N/G)$. The density claims follow by using parts (2) and (4) of Lemma 2.1 and the fact that τ is open. □

Notice that for the openness results one in fact does not need to assume that G is a discrete group acting freely and properly discontinuously. The strong–weak topology for $C^r_G(M, N)$ is defined when G is any Lie group and, by [**5**, Propositions 6.1 and 6.4], the sets $\mathrm{Imm}^r_G(M, N)$ and $\mathrm{Prop}^r_G(M, N)$ are open in $C^r_G(M, N)$, assuming that G acts properly on M and N.

Proof of Theorem 1.2. If the action of G on N is free and properly discontinuous, then the claim follows from Thom's transversality theorem (see, for example, [**4**, Theorem 3.2.1]), part (5) of Lemma 2.1 and Theorem 2.4. The proof is similar to that of Theorem 1.1.

Assume then that the C^r action of G on N is arbitrary. This case can be reduced to the case of a proper free action. Namely, the diagonal action of G on $M \times N$ is proper and free and of class C^r. Moreover, a C^r G-map $f\colon M \to N$ is transverse to N' along L, if and only if for every $h \in C^r_G(M, M)$, $(h, f)\colon M \to M \times N$ is transverse to $M \times N'$ along L. Both the density and openness results follow easily by using Proposition 4.6 in [**5**], according to which there is a canonical homeomorphism

$$C^r_G(M, M \times N) \approx C^r_G(M, M) \times C^r_G(M, N).$$

□

Proof of Theorem 1.3. If G acts freely and properly on N, then the claim follows as the proofs of Theorems 1.1 and 1.2, by using Whitney's approximation theorem, which implies that $C^\omega(M/G, N/G)$ is dense in $C^r(M/G, N/G)$, and part (1) of Lemma 2.1. The proof of the case of an arbitrary C^ω action of G on N can be reduced to the case of a proper free action, as in the proof of Theorem 1.2. □

Remark 2.5. In [**4**], Whitney's immersion theorem and Thom's transversality theorem are only stated for the cases $1 \leqslant r \leqslant \infty$. However, the C^ω case follows easily from Whitney's approximation theorem, according to which C^ω maps form a dense set in $C^\infty(M, N)$ when M and N are C^ω manifolds.

Remark 2.6. Studying ordinary transversality in the equivariant case only makes sense because the action of G on M is free and properly discontinuous. For smooth actions of a compact Lie group there exist the notions of general position (see [**1**]) and

G-transversality (see [**2**]). These concepts are equivalent by [**3**] and agree with ordinary transversality if G is a finite group acting freely. Notice that if one tries to generalize the results in [**1**] and [**2**] to the case of proper actions of non-compact Lie groups, one should not work with the Whitney C^r topology in $C^r_G(M,N)$, which is discrete [**5**, Proposition 4.7], but with the strong–weak C^r topology.

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