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A proof of the estimation from below in Pesin's entropy formula

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Abstract. We give a proof of Pesin entropy formula in a very general setting.

1. *Introduction, formulation of the results*

(1.1). The celebrated Pesin entropy formula asserts that when M is a compact Riemannian manifold, when $\phi \in \mathrm{Diff}^{1+\varepsilon}(M)$ and when μ is an absolutely continuous (i.e. absolutely continuous with respect to the measure on M induced by the Riemannian metric) invariant probability measure, then

$$h_\mu(\phi) = \int_M \Big(\sum_{\chi_i(x)>0} \chi_i(x) \Big) \, d\mu(x) \tag{1.1}$$

when, as usual, $\chi_1(x) \geq \cdots \geq \chi_{\dim M}(x)$ denote all Lyapunov characteristic exponents of ϕ at x and $h_\mu(\phi)$ the metric entropy of the system (M, μ, ϕ).

The aim of this paper is to give a proof of formula (1.1) which remains valid if one consider maps with singularities as defined in [**6**] as well as some class of measures somewhat larger than the class of absolutely continuous measures (see § 1.3).

More precisely we prove only the estimation of the entropy through Lyapunov characteristic exponents from below. Indeed, the estimation from above is completely independent and in the case of smooth maps of a smooth compact manifold it holds for all invariant probability measures (see [**17**]). In the case of smooth maps with singularities (see [**6**], [**8**], [**23**]) this estimation of entropy from above is proved for a very large class of invariant probability measures in [**8**]. Let us note that the classes of spaces M considered in [**6**], [**7**] and in [**8**] are not exactly the same, but for the proof of the estimation from below, we need only the assumptions from [**6**]. Thus we suppose that all assumptions from [**6**] are satisfied.

The fact that the Pesin entropy formula holds for the class of measures considered here was well known to Pesin himself, although not clearly stated. Furthermore, R. Mañé [**11**] recently gave a very ingenious simple proof of the estimation of the entropy from below in formula (1.1). As noted by A. Katok (University of Maryland, USA), it seems that Mañé's proof can also be extended to the more general framework we are considering. Nevertheless, the original proof given in [**13**], whose idea goes back to Ja. G. Sinaï (see [**21**]) is also beautiful, natural and, if one admits only the Pesin theory of invariant manifolds, this proof is in fact rather simple. It allows us also to identify the Pinsker algebra of the system, that is the σ-algebra of sets A such that the mean entropy $h(\phi, \gamma_A)$ of the partition $\gamma_A = \{A, M\backslash A\}$ is zero. The proof given here follows Pesin. One of our aims is to make it understandable by a large readership. Except for the knowledge of results stated in [**6**] (propositions 1.1 and 3.3 below) and for elementary entropy theory (see [**16**]), this paper is self contained. The notation used is the same as in [**6**] and is not explained here. If the reader is interested only in $C^{1+\varepsilon}$ diffeomorphisms of a compact manifold M, he may, in what follows, put

$$\Omega_{\alpha,\gamma} = \Omega_\alpha = N = M.$$

(1.2). The subset $\tilde{\Lambda} \subset M$ is defined in § 3 of [**6**]. For $x \in \tilde{\Lambda}$,

$$0 \neq v \in T_x N,$$

let us denote by

$$\chi^+(x, v) = \lim_{n\to\infty} \frac{1}{n} \log \|d\phi_x^n(v)\|$$

the Lyapunov characteristic exponent of vector v.

We call $\tilde{\Lambda}^+$ the following subset of $\tilde{\Lambda}$:

$$\tilde{\Lambda}^+ = \{x \in \tilde{\Lambda};\, \chi^+(x, v) > 0 \text{ for some } 0 \neq v \in T_x N\}.$$

Let $\tilde{\Lambda}_{s,r,\alpha} = \tilde{\Lambda}_{s,r} \cap \Omega_\alpha$ (see §§ 3 and 4 of [**6**]), where $0 < \alpha < 1$, $\alpha = \alpha(s, r)$, is so big that all results of [**6**] are applicable. Let us denote also

$$\tilde{\Lambda}^+_{s,r,\alpha} = \tilde{\Lambda}^+ \cap \tilde{\Lambda}_{s,r,\alpha} \text{ and } \tilde{P}^+ = \bigcup_{s\geq 1, r\geq 1} \tilde{\Lambda}^+_{s,r,\alpha}.$$

The local unstable manifold of ϕ (i.e. the local stable manifold for ϕ^{-1}) passing through $x \in \tilde{P}^+$ and constructed in [**6**] will be denoted here by $V_{\text{loc}}(x)$. Let us denote

$$V(x) = \begin{cases} \bigcup_{-\infty}^{\infty} \phi^n(V_{\text{loc}}(\phi^{-n}x)) & \text{for } x \in \tilde{P}^+, \\ \{x\} & \text{for } x \text{ elsewhere.} \end{cases}$$

We call $V(x)$ the global unstable manifold passing through x. Let us also note that $\mu(\tilde{\Lambda}^+) = \mu(\tilde{P}^+)$ and that $\tilde{P}^+$ is ϕ invariant.

Theorem 1.1 is proved in our framework exactly in the same way as in [**14**] or in [**18**].

THEOREM 1.1. *Let $x, y \in \tilde{P}^+$. Then the following are true*:

(1.1.1) *if $y \notin V(x)$ then $V(x) \cap V(y) = \emptyset$*;

(1.1.2) *if $y \in V(x)$ then $V(x) = V(y)$*;

(1.1.3) *if $K(x) = \dim N - \dim \{v \in T_x N,\ \chi^+(x, v) \leq 0\}$ then $V(x)$ is a $K(x)$ dimensional open submanifold of N which is generally not connected*;

(1.1.4) $\Phi(V(x)) = V(\Phi(x))$;

(1.1.5) *if $y \in V(x)$ then* $\lim_{n\to\infty} \rho_{V(\Phi^{-n}x)}(\Phi^{-n}x, \Phi^{-n}y) = 0$;

where $\rho_{V(z)}$ denotes the distance induced on the connected components of the sub-manifold $V(z)$ by the Riemannian metric on N.

Let us note that the proof of (1.1.5) is easy if derived from the estimates obtained in the proof of theorem 6.1 from [**6**].

Let us remark also that the partition of N in global unstable manifolds can also be defined (mod 0) as the classes of the following equivalence relation between the points x and y

$$\limsup_{n\to\infty} \frac{1}{n} \log \rho(\Phi^{-n}(x), \Phi^{-n}(y)) < 0$$

(see theorem 6.1. from [**6**] and the very definition of unstable manifolds). We will call this partition the global unstable foliation and we will denote it by V.

(1.3). In what follows, we shall use the notions of measurable partitions and conditional measures (see [**16**]).

Let ξ be a measurable partition of M. For $x \in M$, we denote $C_\xi(x)$ the element of the partition ξ which contains the point x and $\{\mu_{C_\xi(x)}\}_{x\in M}$ the set of conditional measures defined by μ on the elements $\{C_\xi(x)\}_{x\in M}$ of the partition ξ. If μ is another measurable partition, then $\xi \le \mu$ means that for μ almost every $x \in M$ one has $C_\mu(x) \subset C_\xi(x)$. By $\mathcal{M}$ we denote the σ-algebra of all measurable subsets of M.

The global unstable foliation defined in § 1.2 is generally not a measurable partition. But we may consider the σ-algebra $\mathcal{M}_V$ consisting of the measurable subsets of M which are the union of unstable manifolds ($A \in \mathcal{M}_V$ iff $A \in \mathcal{M}$ and $x \in A$ implies $V(x) \subset A$).

We shall also consider measures which are in some sense well disintegrated by the global unstable foliation. For if W is a manifold imbedded in N, W inherits from N a Riemannian structure and hence a Riemannian measure that we shall call the induced measure on W. We denote μ_x the induced measure on $V(x)$.

Definition 1. A measure μ on M will be called absolutely continuous with respect to the global unstable foliation of Φ iff for any measurable partition ξ of M such that $C_\xi(x) \subset V(x)$ and $\mu_x(C_\xi(x)) > 0$ for μ almost every $x \in \tilde{P}^+$, the conditional measures $\mu_{C_\xi(x)}$ are absolutely continuous with respect to the induced measure μ_x.

This property could also be called quasi-invariance by the global unstable foliation (think of the foliation by the orbits of a flow, for instance).

A short name ought to be Sinaï measures. In fact Sinaï did stress the importance of that property in [**21**] and in a lot of examples, the existence of such a measure is the key property for explaining 'stochastic' properties of the system (see [**1**], [**2**], [**3**], [**4**], [**5**], [**6**], [**7**], [**13**], [**14**], [**19**]). One is led to believe (see [**20**]) that measures absolutely continuous with respect to the unstable foliation play the same role and have the same importance for multidimensional systems as absolutely continuous invariant measures do in the theory of maps of an interval. The result proved in this paper is the easiest step in that direction.

Let us now recall conditions (2.1) and (2.2) from [**6**] concerning the Φ-invariant measure μ.

Condition 2.1. There exist positive constants C_1 and a such that for every positive ε,

$$\mu(U_\varepsilon(A)) \leq C_1 \varepsilon^a.$$

Here A denotes the singular set of mapping Φ and U_ε its ε neighbourhood.

Condition 2.2. $\int_M \log^+ \|d\Phi_x^{\pm 1}\| \, d\mu(x) < +\infty$ where $\log^+ x = \max(\log x, 0)$.

Let us note that when Φ is a diffeomorphism of a smooth compact manifold these conditions are automatically satisfied.

THEOREM 1.2. *If the Φ-invariant probability measure μ satisfying conditions* (2.1) *and* (2.2) *from* [**6**] *is absolutely continuous with respect to the global unstable foliation of Φ, then*

$$h_\mu(\Phi) = \int_M \Big(\sum_{\chi_i(x)>0} \chi_i(x) \Big) \, d\mu(x) \tag{1.2}$$

and the Pinsker σ-algebra of the system coincides (mod 0) *with $\mathcal{M}_V$* (i.e. *for any subset A in the Pinsker σ-algebra there is a $B \in \mathcal{M}_V$ with*

$$\mu(A \backslash B) = \mu(B \backslash A) = 0).$$

The above characterization of Pinsker σ-algebra was given by Ia. B. Pesin in [**14**].

Before proving theorem 1.2, we want to emphasize that you can exchange the role of Φ and Φ^{-1} in all that is written here, thus replacing in every place unstable by stable and vice versa. The characterization we give is more 'natural' because it relates entropy and the coefficient of expansion of the volume.

If A is a linear operator between two Euclidean spaces of the same finite dimension m, we denote $A^{\wedge k}$ the k'th exterior power of A. Let us denote

$$\|A^\wedge\| = 1 + \sum_{K=1}^{m} \|A^{\wedge k}\|.$$

With this notation, it follows from [**17**] (see also [**8**]) that μ almost everywhere

$$\sum_{\chi_i(x)>0} \chi_i(x) = \lim_{n\to\infty} \frac{1}{n} \log \|(d\Phi_x^n)^\wedge\| \tag{1.3}$$

and

$$\int_M \Big(\sum_{\chi_i(x)>0} \chi_i(x) \Big) \, d\mu(x) = \lim_{n\to\infty} \frac{1}{n} \int_M \log \|(d\Phi_x^n)^\wedge\| \, d\mu(x).$$

Thus, the Pesin entropy formula may be also written as follows:

$$h_\mu(\Phi) = \lim_{n\to\infty} \frac{1}{n} \int_M \log \|(d\Phi_x^n)^\wedge\| \, d\mu(x). \tag{1.4}$$

We even conjecture that (1.4) or equivalently (1.2) is a characteristic property of Sinaï measures, thus following the analogy with the one-dimensional case [**9**].

(1.4). The absolute continuity theorem proved for diffeomorphisms of compact manifolds in [**15**] and for the mappings with singularities in [**7**] asserts that if μ is an absolutely continuous measure on N, then μ is absolutely continuous with respect to both global unstable and global stable foliation of Φ.

It is interesting to note that the measures which are absolutely continuous with respect to global unstable (or stable) foliation of Φ but which are not absolutely continuous occur very frequently in Anosov and related dynamical systems (see [**22**] and [**3**]).

2. *Preliminaries*

(2.1). If H is a finite dimensional Euclidean space, we denote Vol_H the volume on H. Let E and F be two Euclidean spaces of the same dimension, $E_1 \subset E$ be a linear subspace of E and $A : E \to F$ a linear mapping. Let us define

$$|A|_{E_1}| = \frac{\mathrm{Vol}_{F_1}(A(U))}{\mathrm{Vol}_{E_1}(U)}$$

where U is an arbitrary open and bounded subset of E_1, F_1 is an arbitrary linear subspace of F of the same dimension as E_1 and $A(U) \subset F_1$. We also denote

$$|\det A| = |A|_E|.$$

Let X and Y be two Riemannian manifolds of the same finite dimension and $T : X \to Y$ be a C^1 diffeomorphism. Riemannian measures (i.e. the measures induced by the Riemannian metrics) on X and Y will be denoted ν_X and ν_Y respectively. We recall the change of variables formula.

PROPOSITION 2.1. *If $f \in L^1(Y, \nu_Y)$, then*

$$\int_X (f \circ T)\, |\det dT|\, d\nu_X = \int_Y f\, d\nu_Y.$$

(2.2). Let $(M, \mathcal{M}, \mu)$ be a measure space of finite measure and $T : M \to M$ a measurable measure preserving map. We shall use the following well-known result.

PROPOSITION 2.2. *Let g be a positive finite measurable function defined on M such that*

$$\log^- \frac{g \circ T}{g} \in L^1(M, \mu), \text{ where } \log^- a = \min(\log a, 0).$$

Then

$$\lim_{n\to\infty} \frac{1}{n} \log g(T^n x) = 0 \quad \mu \text{ almost everywhere}, \tag{2.1}$$

$$\int_M \log \frac{g \circ T}{g}\, d\mu = 0. \tag{2.2}$$

Proof. Let us first note that when $\log g \in L^1(M, \mu)$ (2.2) is immediate and (2.1) follows from the Birkhoff ergodic theorem applied to the function

$$\log \frac{g \circ T}{g}.$$

Let us also note that the Birkhoff ergodic theorem is still true when applied to a function h, $h = h_+ - h_-$, with $h_+ \geq 0$, $h_- \leq 0$ and $h_- \in L^1(M, \mu)$, but in general the

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limit can be infinite. As

$$\log^{-}\frac{g\circ T}{g}\in L^1(M,\mu),$$

this shows that the following limit exists μ almost everywhere

$$\lim_{n\to\infty}\frac{1}{n}\sum_{i=0}^{n-1}\log\frac{g\circ T^{i+1}}{g\circ T^i}=\lim_{n\to\infty}\frac{1}{n}\log\frac{g\circ T^n}{g}\overset{\text{def}}{=}K$$

and moreover

$$\int_M K d\mu=\int_M \log\frac{g\circ T}{g}d\mu,$$

where both sides may be equal $+\infty$.

As

$$\frac{1}{n}\log g\to 0\ \mu \text{ almost everywhere,}$$

we have therefore

$$K=\lim_{n\to\infty}\frac{1}{n}\log(g\circ T^n)\ \mu\text{ almost everywhere.}$$

On the other hand, we know, as $0<g<\infty$ μ almost everywhere, that

$$\frac{1}{n}\log(g\circ T^n)$$

converges to 0 in measure, because T preserves the measure μ. Thus there is a sequence $n_i\nearrow+\infty$ such that

$$\lim_{i\to\infty}\frac{1}{n_i}\log(g\circ T^{n_i})=0\ \mu\text{ almost everywhere.}$$

This implies $K(x)=0$ μ almost everywhere and proves (2.1) and (2.2). □

Let $\mathcal{M}_1$ be some sub σ-algebra of $\mathcal{M}$. We will denote $L^2(M,\mathcal{M}_1,\mu)$ the space of all $\mathcal{M}_1$ measurable functions in $L^2(M,\mu)$. If α is a measurable partition, we denote $\mathcal{M}_\alpha$ the σ-algebra generated by α and call a function f α-measurable iff f is $\mathcal{M}_\alpha$ measurable.

(2.3). Let Φ be an invertible measurable and measure preserving map on a probability space $(M,\mathcal{M},\mu)$. We recall two results of entropy theory.

PROPOSITION 2.3. *Let α be a measurable partition of M such that $\alpha\le\Phi^{-1}\alpha$. Then*

$$h_\mu(\Phi)\ge h(\Phi^{-1},\alpha)\ge H(\Phi^{-1}\alpha|\alpha).$$

Proof. Proposition 2.3 follows immediately from the very definition of entropy of Φ^{-1} (see [**16**, § 7.1]). □

PROPOSITION 2.4. *Let α be a measurable partition of M such that $\alpha\le\Phi^{-1}\alpha$,*

$$h_\mu(\Phi)=H(\Phi^{-1}\alpha|\alpha)<+\infty$$

and that the partitions $\{\Phi^n\alpha\}_{n\in Z}$ generate $\mathcal{M}$. Then the Pinsker σ-algebra of the system coincides (mod 0) *with* $\bigwedge_{n\in Z}\mathcal{M}_{\Phi^n\alpha}$.

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Proof. Proposition 2.4 is only a reformulation in terms of σ-algebra of theorems 12.1 and 12.3 of [**16**]. □

(2.4). From now on $\Phi: M \to M$ will be a map with singularities as in the statement of the theorem we want to prove.

For $x \in N$ let us recall that T_xN is the tangent space to N at the point x. For $x \in \tilde{\Lambda}$, T_xN decomposes in

$$T_xN = E_x^u \oplus E_x^0 \oplus E_x^s$$

where E_x^u, E_x^0, and E_x^s are linear subspaces corresponding respectively to positive, zero and negative Lyapunov characteristic exponents of Φ at x. This decomposition is invariant in the sense that

$$d\Phi_x(E_x^u) = E_{\Phi(x)}^u,\ d\Phi_x(E_x^0) = E_{\Phi(x)}^0 \quad \text{and} \quad d\Phi_x(E_x^s) = E_{\Phi(x)}^s.$$

Let us note that for $x \in \tilde{\Lambda}^+$, E_x^u is a non-trivial subspace of T_xN.

Let us consider (see § 2.1)

$$\mathcal{T}^u(x) = |d\Phi_x|_{E_x^u}|$$

PROPOSITION 2.5. $\log \mathcal{T}^u \in L^1(M, \mu)$ *and*

$$\int_M \Big(\sum_{\chi_i(x)>0} \chi_i(x) \Big)\, d\mu(x) = \int_M \log \mathcal{T}^u(x)\, d\mu(x). \tag{2.3}$$

Proof. The proof is based on the following fact implicitly contained in [**12**] but explicitly in [**17**] (compare with (1.3)). If $x \in \tilde{\Lambda}$ then

$$\sum_{\chi_i(x)>0} \chi_i(x) = \lim_{n\to\infty} \frac{1}{n} \log |d\Phi_x^n|_{E_x^u}|. \tag{2.4}$$

From the Hadamard inequality we have

$$\frac{1}{\|d\Phi_{\Phi(x)}^{-1}\|^m} \le \frac{1}{|d\Phi_{\Phi(x)}^{-1}|_{E_{\Phi(x)}^u}|} \le |d\Phi_x|_{E_x^u}| = \mathcal{T}^u(x) \le \|d\Phi_x\|^m,$$

where $m = \dim N$. Thus condition (2.2) from [**6**] and the Φ invariance of the measure μ, implies that $\log \mathcal{T}^u \in L^1(M, \mu)$. Now (2.3) follows from (2.4) and the Birkhoff ergodic theorem, because

$$\log |d\Phi_x^n|_{E_x^u}| = \sum_{i=0}^{n-1} \mathcal{T}^u(\Phi^i(x)). \qquad \square$$

(2.5). Let us now formulate a classical remark due to E. Hopf (see [**2**, theorem 4.4]). We write down the proof here for the sake of completeness.

PROPOSITION 2.6. *Let $\mathcal{M}_I$ be the σ-algebra of the Φ invariant subsets of $\mathcal{M}$. Then $\mathcal{M}_I \subset \mathcal{M}_V$.*

Proof. We shall prove an equivalent inclusion

$$L^2(M, \mathcal{M}_I, \mu) \subset L^2(M, \mathcal{M}_V, \mu).$$

We first consider a dense set F of continuous functions in $L^2(M, \mathcal{M}, \mu)$. By the Birkhoff ergodic theorem, for any $f \in F$, the limit

$$f^-(x) = \lim_{n\to\infty} \frac{1}{n} \sum_{i=0}^{n} f(\Phi^{-i}(x))$$

exists on a set M_f, $M_f \in \mathcal{M}_I$, $\mu(M_f) = 1$.

Let P be the orthogonal projection of $L^2(M, \mu)$ on $L^2(M, \mathcal{M}_I, \mu)$. By the mean ergodic theorem, f^- represents Pf. The set of Pf where $f \in F$ is dense in $L^2(M, \mathcal{M}_I, \mu)$. But for any two points $x, y \in M_f$ which are V equivalent, i.e. such that there exists some $z \in \tilde{P}^+$ with $x, y \in V(z)$, we have by (1.1.5) $\lim_{n\to\infty} \rho(\Phi_x^{-n}, \Phi_y^{-n}) = 0$ and thus $f^-(x) = f^-(y)$. Therefore any function in $\{Pf; f \in F\}$ is constant along the global unstable manifolds restricted to M_f and so coincides (mod 0) with an $\mathcal{M}_V$ measurable function. □

(2.6). Suppose now $\mu(\tilde{P}^+) = 1$. Using propositions 2.3, 2.4 and 2.5 we shall prove theorem 1.2 by constructing a measurable partition η of $\tilde{P}^+$ such that

$$\begin{cases} \eta \le \Phi^{-1}\eta, \ \bigwedge_{n\in\mathbb{Z}} \mathcal{M}_{\Phi^n\eta} = \mathcal{M}_V, \\ \{\Phi^n\eta\}_{n\in\mathbb{Z}} \text{ generate } \mathcal{M} \text{ (with respect to } \mu) \end{cases} \tag{2.5}$$

and

$$H(\Phi^{-1}\eta|\eta) = \int_M \log \mathcal{T}^u(x)\, d\mu(x). \tag{2.6}$$

This will at once prove the estimation of entropy from below and hence formula (1.2). The characterization of the Pinsker σ-algebra comes by applying proposition 2.4 also to the partition η.

Let us remark also that there is nothing to prove in theorem 1.2 if $\mu(\tilde{P}^+) = 0$. The entropy of μ is zero (see [**8**]) and formula (1.2) is true. The Pinsker σ-algebra is thus $\mathcal{M}$ and as $V(x) = x$ μ almost everywhere, $\mathcal{M}_V$ also coincides (mod 0) with $\mathcal{M}$.

In general, when $0 < \mu(\tilde{P}^+) < 1$, theorem 1.2 follows clearly because μ can be written as

$$\mu = \mu(\tilde{P}^+)\mu_1 + (1 - \mu(\tilde{P}^+))\mu_2,$$

with $\mu_1(\tilde{P}^+) = 1$ and $\mu_2(\tilde{P}^+) = 0$.

This shows that to finish the proof we have only to construct a partition η satisfying (2.5) and (2.6) in the case when $\mu(\tilde{P}^+) = 1$, and this is what the rest of the paper is devoted to.

3. *Construction of the partition η*

(3.1). In this section we construct a measurable partition satisfying (2.5) for which we compute the entropy in § 4. More precisely we want to prove

PROPOSITION 3.1. *Let μ be a Φ-invariant probability measure such that $\mu(\tilde{P}^+) = 1$. Then there exists a measurable partition η of N such that*

(3.1.1) $\eta \le \Phi^{-1}\eta$;

(3.1.2) *for μ almost every point $x \in N$, $C_\eta(x) \subset V(x)$ contains a $V(x)$ neighbourhood of x and*

(3.1.3) $\bigcup_n \Phi^n C_\eta(\Phi^{-n}x) = V(x)$ *μ almost everywhere and* $\bigcap_n \mathcal{M}_{\Phi^n\eta} = \mathcal{M}_V$ (mod 0);

(3.1.4) *the partitions $\{\Phi^n\eta\}_{n\in Z}$ generate $\mathcal{M}$;*

(3.1.5) *for any Borel subset $B \subset M$ the function $\psi(x) = \mu_x(C_\eta(x) \cap B)$ is measurable and μ almost everywhere finite.*

Let us emphasize before proving proposition 3.1 that the absolute continuity of the measure μ with respect to the global unstable foliation does not play any role in this section.

We want also to point out one meaning of (3.1.3); we have already said that μ being absolutely continuous with respect to the global unstable foliation means that μ is quasi-invariant for the equivalence relation V: $x \sim y$ iff there exists z such that $x, y \in V(z)$. In that framework, it follows easily from (3.1.3) that the measured equivalence relation $(M, \mu; V)$ is hyperfinite.

The second property in (3.1.3) is a reformulation of the first one. Note that

$$\Phi^n C_\eta(\Phi^{-n}x) = C_{\Phi^n\eta}(x).$$

Therefore

$$\bigcup_n C_{\Phi^n\eta}(x) = V(x)$$

implies clearly that $\mathcal{M}_V \subset \mathcal{M}_{\Phi^n\eta}$ for all n and that any measurable set which is in all $\mathcal{M}_{\Phi^n\eta}$ belongs to $\mathcal{M}_V$.

(3.2). We first prove a general lemma from measure theory (see also [**9**, lemma 3.1]).

PROPOSITION 3.2. *Let $r_0 > 0$ and ν be a finite non-negative Borel measure on $\mathbb{R}$, concentrated on $[0, r_0]$, $0 < a < 1$. Then the Lebesgue measure of the set L_a,*

$$L_a = \left\{ r; 0 \le r \le r_0, \sum_{k=0}^{\infty} \nu([r - a^k, r + a^k]) < +\infty \right\}$$

is equal to r_0.

Proof. Let us define

$$N_{a,k} = \left\{ r; 0 \le r \le r_0, \nu([r - a^k, r + a^k]) > \frac{\nu([0, r_0])}{k^2} \right\}.$$

Let 'bad interval' denote an interval of length $2a^k$ with its centre in a point of $N_{a,k}$. It is easy to see the $N_{a,k}$ can be covered by bad intervals $C_{i,k}$, $1 \le i \le S(k)$, so that any point meets at most two bad intervals. We have

$$\frac{S(k)\nu([0, r_0])}{k^2} \le \sum_{i=1}^{S(k)} \nu(C_{i,k}) \le 2\nu([0, r_0])$$

and $|N_{a,k}| \le 2S(K)a^k$, where $|W|$ denotes the Lebesgue measure of the set W. We get $|N_{a,k}| \le 4a^k k^2$ and therefore, by the Borel–Cantelli lemma, Lebesgue almost any point r belongs only to a finite number of $N_{a,k}$ and thus the series

$$\sum_{k=0}^{\infty} \nu([r-a^k, r+a^k])$$

converges. □

(3.3). Let us recall some fact from Pesin theory (see [**6**, §§ 6, 7]). Let $B(x, r)$ denote the closed ball in M with centre x and radius r. Let us define the distance ρ_V:

$$\rho_V(x, y) = \begin{cases} \rho_{V(w)}(x, y) \text{ if } x, y \in V(w) & \text{for some } w \in \tilde{P}^+, \\ +\infty \text{ otherwise.} \end{cases}$$

The distance $\rho_V(x, y)$ does not depend on the choice of the point w.

PROPOSITION 3.3. *There exists an increasing sequence* $\{\Lambda_l\}_{l\geq 1}$ *of closed subsets of* $\tilde{P}^+$, $\mu(\tilde{P}^+ \setminus \bigcup_l \Lambda_l) = 0$ *and positive numbers* r_l, A_l, B_l *and* C_l *such that*

(3.3.1) *for each* $y \in \Lambda_l$, $B(y, 3r_l) \subset N$;

(3.3.2) *for each* $x \in \Lambda_l$ *there exists* $\varepsilon(l)$, $0 < \varepsilon(l) < 1$, *such that for any*

$$y \in \Lambda_l \cap B(x, \varepsilon(l)r), 0 < r \leq r_l,$$

the local unstable manifold $V_{\mathrm{loc}}(y)$ *is such that* $V_{\mathrm{loc}}(y) \cap B(x, r)$ *is connected*;

(3.3.3) *the map* $y \to V_{\mathrm{loc}}(y) \cap B(x, r_l)$ *is continuous from* $B(x, \varepsilon(l)r_l) \cap \Lambda_l$ *into the space of subsets of* $B(x, r_l)$ *(endowed with the Hausdorff topology)*;

(3.3.4) *for each* $y \in \Lambda_l$ $V_{\mathrm{loc}}(y)$ *contains the closed ball of centre* y *and* ρ_V *radius* A_l *in* $V(y)$ *and*

(3.3.5) *if* $z \in V_{\mathrm{loc}}(y)$, *then for every* $n \geq 0$

$$\rho_V(\Phi^{-n}y, \Phi^{-n}z) \leq B_l e^{-nC_l} \rho_V(y, z);$$

(3.3.6) *for any* r, $0 < r \leq r_l$, *if two points* z_1 *and* z_2 *belong to*

$$S(x, r) = \bigcup_{y \in \Lambda_l \cap B(x, \varepsilon(l)r)} V_{\mathrm{loc}}(y) \cap B(x, r)$$

and are not in the same local leaf $V_{\mathrm{loc}}(y) \cap B(x, r)$ *for some* y *belonging to* $\Lambda_l \cap B(x, \varepsilon(l)r)$, *then* $\rho_V(z_1, z_2) > 2r$.

This proposition follows from results (and from the proofs) of §§ 6 and 7 of [**6**], if one takes as Λ_l the set (in the notation of [**6**])

$$\bigcup_{s,r=1}^{l} (\tilde{\Lambda}^l_{s,r} \cap \Omega_{\alpha,1/l} \cap \tilde{\Lambda}^+),$$

where $\alpha = \alpha(s, r)$ (see § 1.2).

(3.4). We will now prove proposition 3.1 in a particular case. We choose l and $x \in \Lambda_l$ such that $\mu(S(x, r)) > 0$ for all r, $0 < r \leq r_l$. This is possible by choosing first

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l such that $\mu(\Lambda_l)>0$ and then x in the support of the trace of the measure μ on Λ_l. For any r, $0<r\le r_l$, we consider the partition ξ_r of M defined by all the sets

$$V_{\text{loc}}(y)\cap B(x,r)$$

for $y\in\Lambda_l\cap B(x,\varepsilon(l)r)$ and the set $M\backslash S(x,r)$. From (3.3.3) it follows clearly that ξ_r is a measurable partition of M.

We define $\eta_r=\bigvee_{n=0}^{\infty}\Phi^n\xi_r$. The partition η of lemma 3.1 will be a partition η_r for some r, $0<r\le r_l$, that we choose later. Let us define

$$S_r=\bigcup_{n\ge 0}\Phi^n S(x,r).$$

We now prove properties (3.1.1)–(3.1.5) when $\mu(S_r)=1$.

(3.1.1) This property is clear from the definition of η_r.

(3.1.2) It is clear that for $z\in S_r$ and for some $n\ge 0$.

$$C_{\eta_r}(z)\subset\Phi^n V_{\text{loc}}(\Phi^{-n}z)\subset V(z).$$

On the other hand, we claim that there exists a function β_r, $\beta_r\ge 0$, such that $y\in V(z)$, $\rho_V(y,z)\le\beta_r(z)$ implies $y\in C_{\eta_r}(z)$.

The proof of (3.1.2) consists in constructing such a β_r and then choosing r such that $\beta_r>0$ μ almost everywhere.

We define β_r only on $\bigcup_l\Lambda_l$. For $z\in\bigcup_l\Lambda_l$, put

$$l(z)=\inf\{l';\, z\in\Lambda_{l'}\}$$

and

$$\beta_r(z)=\inf_{n\ge 0}\left\{A_{l(z)},\frac{1}{2B_{l(z)}}\rho(\Phi^{-n}z,\partial B(x,r))\,e^{nC_{l(z)}},\frac{r}{B_{l(z)}}\right\}.$$

Let us first prove our claim. Let $z\in\bigcup_l\Lambda_l$, $y\in V(z)$ and $\rho_V(y,z)\le\beta_r(z)$. We have to check that for any $n\ge 0$

$$C_{\xi_r}(\Phi^{-n}y)=C_{\xi_r}(\Phi^{-n}z). \tag{3.1}$$

First we know by (3.3.4) as $y\in V(z)$ and $\rho_V(y,z)\le A_{l(z)}$ that $y\in V_{\text{loc}}(z)$ and that (3.3.5) applies. Therefore we have for any $n\ge 0$

$$\rho_V(\Phi^{-n}y,\Phi^{-n}z)\le B_{l(z)}\,e^{-nC_{l(z)}}\rho_V(y,z)\le\tfrac{1}{2}\rho(\Phi^{-n}z,\partial B(x,r)) \tag{3.2}$$

and

$$\rho_V(\Phi^{-n}y,\Phi^{-n}z)\le B_{l(z)}\,e^{-nC_{l(z)}}\rho_V(y,z)\le r. \tag{3.3}$$

We have four cases to consider.

(i) If $\Phi^{-n}y$ and $\Phi^{-n}z$ both belong to $S(x,r)$, we have (3.1) by (3.3.6) and (3.3).

(ii) If neither $\Phi^{-n}y$ nor $\Phi^{-n}z$ belong to $S(x, r)$, we have (3.1) by the definition of ξ_r.
(iii)–(iv) If $\Phi^{-n}y$ belongs to $S(x, r)$ but not $\Phi^{-n}z$, or vice versa, we should have

$$\rho_V(\Phi^{-n}y, \partial(x, r)) \leq \rho_V(\Phi^{-n}y, \Phi^{-n}z)$$

which would contradict (3.2).
Thus only (i) and (ii) occur, which proves the claim.

We will now choose r, $0 < r \leq r_l$ such that $\beta_r > 0$ μ almost everywhere. We will even show that for Lebesgue almost every choice of r, $0 < r \leq r_l$ $\beta_r > 0$ μ almost everywhere.

Let $x \in M$. Let ν be the finite non-negative measure on $[0, r_l]$ defined by

$$\nu(A) = \mu(\{y \in M; \rho(x, y) \in A\}),$$

and let p be an integer, $p \geq 1$. We get by proposition 3.2, applied to $a = e^{-C_p}$, that $|K_p| = r_l$, where

$$K_p = \left\{r; 0 \leq r \leq r_l, \sum_{k=0}^{\infty} \mu(\{y \in M; |\rho(x, y) - r| < e^{-kC_p}\}) < +\infty\right\}.$$

As Φ preserves the measure μ, we have also

$$K_p = \left\{r; 0 \leq r \leq r_l, \sum_{k=0}^{\infty} \mu(\{y \in M; |\rho(x, \Phi^{-k}y) - r| < e^{-kC_p}\}) < +\infty\right\}.$$

Note that from the uniformity of the Riemannian metric on $B(x, r_l)$ (see proposition 3.3.1) there is a constant $D > 0$ such that

$$\rho(z, \partial B(x, r_1)) \leq \tau$$

implies

$$|\rho(x, z) - r_1| \leq D\tau$$

for r_1 and τ such that $0 < \tau \leq r_1 \leq r_l$.

Thus we have for r in K_p

$$\sum_{k=0}^{\infty} \mu\left(\left\{y \in M; \rho(\Phi^{-k}y, \partial B(x, r)) \leq \frac{e^{-kC_p}}{D}\right\}\right) < +\infty.$$

This implies by the Borel–Cantelli lemma that for μ almost every y there exist only a finite number of k with

$$\rho(\Phi^{-k}y, \partial B(x, r)) \leq \frac{e^{-kC_p}}{D}.$$

The set

$$\mathscr{T} = \left\{r; \mu\left(\bigcup_n \Phi^n(\partial B(x, r))\right) > 0\right\}$$

is at most countable and for r in $(\bigcap_{p\geq 1} K_p) \setminus \mathscr{T}$ we have clearly $\beta_r > 0$ μ almost everywhere. This completes the proof of (3.1.2).

Fix $r \in (\bigcap_{p\geq 1} K_p) \setminus \mathscr{T}$ and omit the subscript r in β_r, ξ_r, y_r etc., except S_r.

(3.1.3). Firstly it is clear that for all n and z in S_r, $\Phi^n C_\eta(\Phi^{-n}z)$ are contained in $V(z)$.

On the other hand, let y be in $V(z)$. By (1.1.5) we have

$$\lim_{n\to\infty} \rho_V(\Phi^{-n}y, \Phi^{-n}z) = 0.$$

The invariance of μ implies

$$\lim_n \frac{1}{n} \sum_{i=1}^{n} \beta(\Phi^{-i}z) > 0 \ \mu \text{ almost everywhere}$$

and hence there will be some n as large as we want such that

$$\beta(\Phi^{-n}z) > \rho_V(\Phi^{-n}y, \Phi^{-n}z).$$

By the proof of (3.1.2), we have $\Phi^{-n}y \in C_\eta(\Phi^{-n}z)$. Therefore for any $y \in V(z)$ there exists some n such that

$$y \in \Phi^n C_\eta(\Phi^{-n}z).$$

(3.1.3) is proved.

(3.1.4). To show that the partitions $\{\Phi^n\eta\}_{n\in Z}$ generate $\mathcal{M}$, we will show that any two different points y and z of a set of full measure in S_r are separated by some partition $\Phi^{-n}\eta$, $n \geq 1$. Suppose y belongs to infinitely many $\Phi^{-n}S(x, r)$ and that

$$\Phi^{-n}C\eta(z) = \Phi^{-n}C_\mu(z) \text{ for all } n.$$

$\Phi^n z$ and $\Phi^n y$ are infinitely often in the same local unstable manifold and by (3.3.5)

$$\rho_V(y, z) \leq 2B_l r e^{-nC_l}.$$

Therefore $\rho_V(y, z) = 0$ and they coincide.

(3.1.5) Let B be a Borel subset of M. By (3.3.3) the function

$$y \to \mu_y(C_\xi(y) \cap B)$$

is measurable and finite on the set $S(x, r)$, possibly not in the complement.

It follows clearly that the function

$$f_n(y) = \mu_y(C_{\bigvee_{i=0}^{n-1} \Phi^i\xi}(y) \cap B)$$

is measurable and finite on the set $\bigcup_{i=0}^{n-1} \Phi^{-i}S(x, r)$, though possibly not on the complement. Moreover $f_n \geq f_{n+1}$. Therefore

$$\mu_y(C_\eta(y) \cap B) = \lim_{n\to\infty} f_n(y)$$

is measurable and finite μ almost everywhere on S_r and this completes the proof of proposition 3.1 on S_r.

(3.5). The proof of proposition 3.1 is completed if $\mu(S_r) = 1$ and in particular if μ is ergodic. If $\mu(S_r) < 1$, let us remark that by proposition 2.6 any invariant set A is a union of global unstable manifolds.

So if $\mu(A) < 1$, let us consider for some l a point x in the support of the restriction of the measure μ to $(M\backslash A) \cap \Lambda_l$. If we take the trace on $M\backslash A$ of the partition constructed in the same way as in § 3.4 starting from a small neighbourhood of x, it clearly satisfies the properties (3.1.1)–(3.1.5) on a new invariant subset of $M\backslash A$, of positive measure.

Therefore we can construct inductively a partition satisfying (3.1.1)–(3.1.5) almost everywhere. As this will be done in a countable number of steps, all measurability properties are preserved and we thus complete the proof of proposition 3.1. □

4. *Computation of the entropy*

(4.1). We have now only to prove (2.6), i.e.

$$H(\Phi^{-1}\eta|\eta)=\int_M \log \mathcal{T}^u(x)\,d\mu(x). \tag{2.6}$$

Let us define the Borel measure ν on M by

$$\nu(K)=\int_M \mu_y(C_\eta(y)\cap K)\,d\mu(y) \tag{4.1}$$

for all Borel subsets K of M. By (3.1.5) ν is a σ-finite measure on M.

Let us recall that by definition of the conditional measures $\{\mu_{C_\eta(y)}\}_{y\in M}$ we have

$$\mu(K)=\int_M \mu_{C_\eta(y)}(C_\eta(y)\cap K)\,d\mu(y) \tag{4.2}$$

for all Borel subsets K of M. In future we will use the short form $\mu_{C_\eta(y)}(K)$ instead of $\mu_{C_\eta(y)}(C_\eta(y)\cap K)$.

As the measure μ is absolutely continuous with respect to the global unstable foliation (we now use this assumption for the first time) we have that μ is absolutely continuous with respect to ν. We call g the Radon–Nikodym derivative $d\mu/d\nu$; $g(r)\geq 0$ μ almost everywhere on M. The next result is a well-known measure theoretic statement.

PROPOSITION 4.1. *For μ almost every $y\in M$*

$$g=\frac{d\mu_{C_\eta(y)}}{d\mu_y}$$

μ_y almost everywhere on $C_\eta(y)$.

Proof. Let $A\in\mathcal{M}_\eta$, $B\in\mathcal{M}$ be two arbitrary sets. As

$$\int_{A\cap B} g\,d\nu=\int_{A\cap B} d\mu,$$

from (4.1) and (4.2) one obtains that

$$\begin{aligned}\int_{A\cap B} g\,d\nu&=\int_A\left(\int_{B\cap C_\eta(y)} g(z)\,d\mu_y(z)\right)d\mu(y)\\&=\int_{A\cap B} d\mu=\int_A \mu_{C_\eta(y)}(B)\,d\mu(y).\end{aligned} \tag{4.3}$$

The measure space $(M,\mathcal{M},\mu)$ is separable as a Lebesgue space (see [**16**]). Let $\{B_i\}_{j=1}^{\infty}\subset\mathcal{M}$ be a dense subset in $\mathcal{M}$ (with respect to the standard metric

$$d(C,D)=\mu(C\backslash D)+\mu(D\backslash C)).$$

Fix j, $1\leq j<\infty$, and apply (4.3) to an arbitrary set $A\subset\mathcal{M}_\eta$ and to $B=B_j$. As A is arbitrary, (4.3) implies that there exists a measurable subset Z_j of $\mathcal{M}$, $\mu(Z_j)=1$

such that for every $y \in Z_j$ one has

$$\int_{B_j \cap C_\eta(y)} g(z)\, d\mu_y(z) = \mu_{C_\eta(y)}(B_j).$$

This implies our assertion. □

We may therefore write for any Borel subset B of M

$$\int_{B \cap C_\eta(y)} g(z)\, d\mu_y(z) = \mu_{C_\eta(y)}(B) \tag{4.4}$$

μ almost everywhere. In particular

$$\int_{C_\eta(y)} g(z)\, d\mu_\eta(z) = 1 \tag{4.5}$$

μ almost everywhere.

(4.2). We now compute the entropy. We denote

$$I(\Phi^{-1}\eta|\eta)(y) = -\log \mu_{C_\eta(y)}(C_{\Phi^{-1}\eta}(y)).$$

By (4.4) as $C_{\Phi^{-1}\eta}(y) \subset C_\eta(y)$, we have

$$I(\Phi^{-1}\eta|\eta)(y) = -\log \int_{C_{\Phi^{-1}\eta}(y)} g(z)\, d\mu_y(z).$$

Using now that $C_{\Phi^{-1}\eta}(y) = \Phi^{-1}(C_\eta(\Phi y))$ and proposition 2.1, we have

$$\int_{C_{\Phi^{-1}\eta}(y)} g(z)\, d\mu_y(z) = \int_{\Phi^{-1}[C_\eta(\Phi y)]} g(z)\, d\mu_y(z)$$

$$= \int_{C_\eta(\Phi y)} g(\Phi^{-1}z) \frac{1}{\mathcal{T}^u(\Phi^{-1}z)}\, d\mu_{\Phi(y)}(z).$$

The following proposition is the key point in our computation. We will prove it in § 4.3.

PROPOSITION 4.2. *The function*

$$L(z) = \frac{g(z)\mathcal{T}^u(\Phi^{-1}z)}{g(\Phi^{-1}z)}$$

is η measurable.

We have now

$$I(\Phi^{-1}\eta|\eta)(y) = -\log \int_{C_\eta(\Phi y)} \frac{g(z)}{L(z)}\, d\mu_{\Phi(y)}(z).$$

But by proposition 4.2, the function L is constant on μ almost all elements of the partition η. Therefore on $C_\eta(\Phi y)$ we have $L(z) = L(\Phi y)$ for μ almost every y.

Consequently by (4.5)

$$\int_{C_\eta(\Phi y)} \frac{g(z)}{L(z)}\, d\mu_{\Phi(y)}(z) = \frac{1}{L(\Phi y)} \int_{C_\eta(\Phi y)} g(z)\, d\mu_{\Phi(y)}(z) = \frac{1}{L(\Phi y)}$$

and finally

$$I(\Phi^{-1}\eta|\eta)(y) = \log L(\Phi y) = \log \mathcal{T}^u(y) + \log \frac{g(\Phi y)}{g(y)}. \tag{4.6}$$

Now $I(\Phi^{-1}\eta|\eta)\geq 0$ and by proposition 2.5 $\log J^u \in L^1(M,\mu)$. Consequently (4.6) implies

$$\log^- \frac{g\circ\Phi}{g} \in L^1(M,\mu).$$

But the entropy $H(\Phi^{-1}\eta|\eta)$ is given by

$$H(\Phi^{-1}\eta|\eta)=\int_M I(\Phi^{-1}\eta|\eta)(y)\,d\mu(y)$$

and (2.6) follows immediately from (4.6) and from proposition 2.2.

(4.3). We give now the proof of proposition 4.2.

Let $Z(y)$ denote the $\mathcal{M}_{\Phi^{-1}\eta}$ measurable function defined by

$$Z(y)=\mu_{C_\eta(y)}(C_{\Phi^{-1}\eta}(y)).$$

We shall write the family of conditional measures with respect to $\Phi^{-1}\eta$ in two ways.

Firstly as $\Phi^{-1}\eta\geq\eta$, we can write for any Borel subset K of M

$$\mu_{C_{\Phi^{-1}\eta}(y)}(K)=\frac{\mu_{C_\eta(y)}(K\cap C_{\Phi^{-1}\eta}(y))}{\mu_{C_\eta(y)}(C_{\Phi^{-1}\eta}(y))}=\frac{1}{Z(y)}\int_{C_{\Phi^{-1}\eta}(y)\cap K} g(z)\,d\mu_y(z). \tag{4.7}$$

Secondly by invariance of μ we also have for any Borel subset K of M

$$\mu_{C_{\Phi^{-1}\eta}(y)}(K)=\mu_{C_\eta(\Phi y)}(\Phi(K)). \tag{4.8}$$

Therefore we get from (4.7) and (4.8) for any Borel subset K of M that

$$\frac{1}{Z(y)}\int_{C_{\Phi^{-1}\eta}(y)\cap K} g(z)\,d\mu_y(z)=\int_{C_\eta(\Phi y)\cap\Phi(K)} g(z)\,d\mu_{\Phi(y)}(z)$$
$$=\int_{\Phi(C_{\Phi^{-1}\eta}(y)\cap K)} g(z)\,d\mu_{\Phi(y)}(z)=\int_{C_{\Phi^{-1}\eta}(y)\cap K} g(\Phi z)\mathcal{T}^u(z)\,d\mu_y(z).$$

The last equality follows from proposition 2.1.

Consequently we get for μ almost every $y\in M$

$$\frac{1}{Z(y)}g(z)=g(\Phi z)\mathcal{T}^u(z)$$

for μ_y almost every z in $C_{\Phi^{-1}\eta}(y)$. Thus the function

$$L\circ\Phi=\frac{1}{Z}$$

is $\Phi^{-1}\eta$ measurable and proves proposition 4.2. This achieves the proof of theorem 1.2. □

(4.4). Let us finally point out that this last proposition 4.2 can be written as follows:

PROPOSITION 4.2 (bis). *Let η be a measurable partition with the properties from proposition* 3.1. *Then there exists a strictly positive, measurable function h such that the function*

$$\mathcal{T}^u\frac{h\circ\Phi}{h}$$

is η measurable.

This result was proved in a completely different way for Anosov systems by A. N. Livsic and Ya. G. Sinaï (see [**10**] and [**3**]). It turns out that it is true in a much larger setting.

(4.5). After this paper was finished, F.L. proved that the conjecture formulated after formula (1.4) is true in the case when no characteristic exponent is zero. A proof will appear elsewhere (see [**24**]).

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