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SOME EXAMPLES OF NORMAL MOORE SPACES

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1. Introduction. A normal Moore space is non-metrizable only if it fails to be λ-collectionwise normal for some uncountable cardinal λ [**1**].

For each uncountable cardinal λ we present a class $\mathscr{S}_\lambda$ of normal, locally metrizable Moore spaces and a particular space S_λ in $\mathscr{S}_\lambda$. If there is any space of class $\mathscr{S}_\lambda$ which is not λ-collectionwise normal, then S_λ is such a space. The conditions for membership in $\mathscr{S}_\lambda$ make a space in $\mathscr{S}_\lambda$ behave like a subset of a product of a Moore space with a metric space. The class $\mathscr{S}_\lambda$ is sufficiently large to allow us to prove the following. Suppose Y is a locally compact, 0-dimensional Moore space (not necessarily normal) with a basis of cardinality λ and M is a metric space which is 0-dimensional in the covering sense. If there is a normal, not λ-collectionwise normal Moore space X where $X \subset Y \times M$, then S_λ is a normal, not λ-collectionwise normal Moore space.

It is consistent with the usual axioms of set theory that there exist, for each uncountable cardinal λ, a normal, not λ-collectionwise Hausdorff Moore space [**3**]. It follows from the results quoted above that it is consistent that each S_λ be a normal, non-metrizable Moore space.

We also present, for each uncountable cardinal λ, a locally metrizable Moore space T_λ related to S_λ. If there is a first-countable, normal, not λ-collectionwise normal space, then T_λ also fails to be λ-collectionwise normal. P. Nyikos has shown that T_λ is normal if and only if it is metrizable, so T_λ itself cannot be an example of a normal, non-metrizable Moore space.

Both S_λ and T_λ are purely set theoretic in nature, being built from families of subsets of λ.

Either a proof that some S_λ is not collectionwise normal or a proof that it is consistent that all T_λ are collectionwise normal (or, equivalently, normal) would settle the normal Moore space conjecture.

2. The class $\mathscr{S}_\lambda$ and the normality lemma. The class $\mathscr{S}_\lambda$ was defined as a result of investigation of normal Moore spaces which are subsets of a product of a Moore space with a metric space. Theorem 1 below gives a description of some of the spaces of this type which are in $\mathscr{S}_\lambda$. The conditions listed for membership in $\mathscr{S}_\lambda$ are those which allow our proof of normality to work.

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Suppose that λ is an uncountable cardinal. The class $\mathscr{S}_\lambda$ will contain all Moore spaces S for which the following hold.

(1) There is a discrete family $\{C_\alpha\}_{\alpha\in\lambda}$ of closed sets in S.
(2) $D = S - \bigcup_{\alpha\in\lambda} C_\alpha$ is discrete.
(3) If $A \subset \lambda$, there are disjoint open sets U and V in S with $\bigcup_{\alpha\in A} C_\alpha \subset U$ and $\bigcup_{\beta\in\lambda-A} C_\beta \subset V$.
(4) There is a metric space M and, for each $\alpha \in \lambda$, a subspace M_α of M and homeomorphism h_α from M_α onto C_α.
(5) There is a clopen basis $\mathscr{B}$ for M such that $\mathscr{B} = \bigcup_{i\in\omega}\mathscr{B}_i$ where each $\mathscr{B}_i$ is a discrete open cover of M; and, for each $i \in \omega$, there is a (discrete) family $\{O_B | B \in \mathscr{B}_i\}$ of open sets in S such that $h_\alpha(B \cap M_\alpha) \subset O_B$ and $(\bar{O}_B \cap C_\alpha) \subset h_\alpha(\bar{B} \cap M_\alpha)$ for each α in λ. (Note that $\bar{B} = B$ and $\bar{O}_B = O_B$. It is written in the preceding form for future reference.)

If there is a normal, non-metrizable Moore space, there is one which has properties (1)–(3) above. If there is a locally metrizable, normal, non-metrizable Moore space, there is one which has properties (1)–(4) above. Property (5) is a contrived assumption which makes a space in $\mathscr{S}_\lambda$ look enough like a subset of a product so that our proofs of normality and Theorem 3 below work.

A somewhat larger class of spaces could be obtained by weakening (5) by assuming only that $\mathscr{B}$ is an open, rather than clopen, basis for M and that each $\mathscr{B}_i$ is a discrete family of open sets which may not cover M. The proof below suffices to show that any space in this larger class is still normal.

NORMALITY LEMMA. *If S is in class $\mathscr{S}_\lambda$, then S is normal.*

Proof. First note that it is sufficient to show that every pair of disjoint closed sets which lie in $\bigcup_{\alpha\in\lambda} C_\alpha$ can be separated by disjoint open sets.

Let H and K be disjoint closed subsets of $\bigcup_{\alpha\in\lambda} C_\alpha$. We show that H and K can be separated by producing two countable collections $\{U_i\}_{i\in\omega}$ and $\{V_i\}_{i\in\omega}$ of open sets such that $H \subset \bigcup_{i\in\omega} U_i$, $K \subset \bigcup_{i\in\omega} V_i$, and, for each $i \in \omega$, $\bar{U}_i \cap K = \emptyset$ and $\bar{V}_i \cap H = \emptyset$.

Using $\mathscr{B}$ as described in (5), for each $B \in \mathscr{B}$ let

$$H_B = \{\alpha \in \lambda | h_\alpha(B \cap M_\alpha) \cap H \neq \emptyset \quad \text{but} \quad h_\alpha(\bar{B} \cap M_\alpha) \cap K = \emptyset\}.$$

Similarly let $K_B = \{\alpha \in \lambda | h_\alpha(B \cap M_\alpha) \cap K \neq \emptyset \text{ but } h_\alpha(\bar{B} \cap M_\alpha) \cap H = \emptyset\}$. By (3), there are disjoint open sets X_B and Y_B in S with $\bigcup_{\alpha\in H_B} C_\alpha \subset X_B$ and $\bigcup_{\alpha\in\lambda-H_B} C_\alpha \subset Y_B$. Also there are disjoint open sets Z_B and W_B in S with $\bigcup_{\alpha\in K_B} C_\alpha \subset Z_B$ and $\bigcup_{\alpha\in\lambda-K_B} C_\alpha \subset W_B$.

Let $U_i = \bigcup_{B\in\mathscr{B}_i}(O_B \cap X_B)$ and $V_i = \bigcup_{B\in\mathscr{B}_i}(O_B \cap Z_B)$ where O_B is defined as in (5).

Observe that $\{U_i\}_{i\in\omega}$ covers H. For suppose $p \in C_\alpha \cap H$. There is an $i \in \omega$ and $B \in \mathscr{B}_i$ such that $p \in h_\alpha(B \cap M_\alpha)$ but $h_\alpha(\bar{B} \cap M_\alpha) \cap K = \emptyset$. Thus $p \in O_B \cap X_B \subset U_i$. Similarly $\{V_i\}_{i\in\omega}$ covers K.

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To show that $\bar{V}_i \cap H = \emptyset$, again assume $p \in C_\alpha \cap H$. Since $\{O_B | B \in \mathscr{B}_i\}$ is discrete, there is at most one $B \in \mathscr{B}_i$ with $p \in \bar{O}_B$. If $p \in h_\alpha(\bar{B} \cap M_\alpha)$, then $p \in W_B$ and $p \notin \bar{Z}_B$. Thus, since $(\bar{O}_B \cap C_\alpha) \subset h_\alpha(\bar{B} \cap M_\alpha)$, $p \notin \bar{V}_i$. Similarly $K \cap \bar{U}_i = \emptyset$.

Thus U_i and V_i have the desired properties.

3. Some spaces in $\mathscr{S}_\lambda$. In the theorem below we describe a natural construction which yields spaces in $\mathscr{S}_\lambda$ and which, in fact, helped motivate the definition of $\mathscr{S}_\lambda$.

THEOREM 1. *Let Y be a Moore space with a discrete family $\{C_\alpha'\}_{\alpha \in \lambda}$ of closed sets such that each C_α' is compact and 0-dimensional. Let M' be a metric space which is 0-dimensional in the covering sense. Let X' be a subspace of $Y \times M'$. Let X be the space obtained from X' by making $((Y - \bigcup_{\alpha \in \lambda} C_\alpha) \times M') \cap X'$ discrete.*

Then if X is normal, X belongs to $\mathscr{S}_\lambda$.

Proof. Since X is a Moore space, we proceed to check that X satisfies properties (1)–(5). Each C_α in (1) is $(C_\alpha' \times M') \cap X$. The D in (2) equals $((Y - \bigcup_{\alpha \in \lambda} C_\alpha') \times M') \cap X$. Property (3) is guaranteed by the normality of X.

Let E be the Cantor set. Then the M of (4) is $E \times M'$. Note that each C_α' is a compact, 0-dimensional metric space; hence there is a subset E_α of E and a homeomorphism h_α' from E_α onto C_α'. Then $h_\alpha'' = h_\alpha' \times \mathrm{id}_{M'} : E_\alpha \times M' \rightarrow C_\alpha' \times M'$ is a homeomorphism. Now M_α in (4) is $(h_\alpha'')^{-1}(X \cap (C_\alpha' \times M_\alpha'))$ and h_α in (4) is h_α'' restricted to M_α.

Let $\mathscr{B}' = \bigcup_{i \in \omega} \mathscr{B}_i'$ be a nested basis for E so that each $\mathscr{B}_i'$ is a discrete open cover of E. Let $\mathscr{B}'' = \bigcup_{i \in \omega} \mathscr{B}_i''$ be such a basis for M'. Each $\mathscr{B}_i$ in (5) equals $\{B' \times B'' | B' \in \mathscr{B}_i'$ and $B'' \in \mathscr{B}_i''\}$. And $\mathscr{B} = \bigcup_{i \in \omega} \mathscr{B}_i$ will be the basis of $M = E \times M'$ required in (5).

For each $i \in \omega$, let $\mathscr{B}_i' = \{B_{ij}'\}_{j=1}^{n_i}$. By the normality of X, there are disjoint open sets $\{O_{ij}'\}_{j=1}^{n_i}$ in X so that for each j, $\bigcup_{\alpha \in \lambda} h_\alpha((B_{ij}' \times M') \cap M_\alpha) \subset O_{ij}'$. Let $O_{B_{ij}' \times B''} = O_{ij}' \cap (Y \times B'')$. This collection of O's satisfies (5).

4. The description of S_λ. In this section we describe the space S_λ in $\mathscr{S}_\lambda$ which is a normal, non-metrizable Moore space if any space in $\mathscr{S}_\lambda$ is such a space. (See Theorem 3 below.)

Assume that λ is an uncountable cardinal. We think of 2^{2^λ} as the set of all collections of subsets of λ. We think of $(2^{2^\lambda})^\lambda$ as the set of all functions from λ into 2^{2^λ}. The metric space M associated with the space S_λ is obtained by taking the product (with the product topology) of ω copies of $(2^{2^\lambda})^\lambda$ with the discrete topology. As a countable product of discrete spaces, $M = ((2^{2^\lambda})^\lambda)^\omega$ is metrizable. A function $f : \omega \rightarrow (2^{2^\lambda})^\lambda$ is a point of M and the nth basic open set for f is $B_n(f) = \{g \in M | g \upharpoonright n = f \upharpoonright n$ where $n = \{0, 1, 2, \ldots, n-1\}\}$. The set $\mathscr{B}_n = \{B_n(f) | f \in M\}$ is an open cover of M by disjoint clopen sets: a discrete family in the strongest sense.

We now turn to the definition of S_λ.

For $\alpha \in \lambda$, $M_\alpha = \{f: \omega \to (2^{2^\lambda})^\lambda |$ if $A \subset \lambda$, then there is an $n \in \omega$ with $A \in f(n)(\alpha)\}$. Let $C_\alpha = \{\langle \alpha, f\rangle | f \in M_\alpha\}$ and $C = \bigcup_{\alpha\in\lambda} C_\alpha$. Then C will be the set of all nondiscrete points of S_λ.

The set of discrete points D is divided into ω pieces. So $D = \bigcup_{n\in\omega} D_n$. Each D_n is divided into pieces indexed by unordered pairs $\{\alpha, \beta\}$ of elements of λ. The part of D_n associated with $\{\alpha, \beta\}$ contains the points of potential intersection of a basic open set of a point in C_α with a basic open set of a point in C_β. The precise definition below of D_n is technical and the reason for its being defined as it is will not become apparent until we check that S_λ has property (3).

For $n \in \omega$, let $D_n = \{\langle\{\alpha, \beta\}, f, n\rangle | \alpha \in \lambda, \beta \in \lambda, f \in M$, and, if $A \in f(i)(\alpha) \cap f(j)(\beta)$ for some $i < n$ and $j < n$, then $\alpha \in A$ if and only if $\beta \in A\}$. Recall that $D = \bigcup_{n\in\omega} D_n$ and that D is the set of discrete points of S_λ.

For each $n \in \omega$ and $\langle \alpha, f\rangle \in C$, we define the nth basic open set for $\langle \alpha, f\rangle$ to be $U_n(\langle \alpha, f\rangle) = \{\langle \alpha, g\rangle \in C_\alpha | g \upharpoonright n = f \upharpoonright n\} \cup \{\langle\{\alpha, \beta\}, h, m\rangle \in D | \beta \in \lambda, h \upharpoonright n = f \upharpoonright n$ and $m \geqq n\}$.

The space S_λ equals $C \cup D$ topologized by using $D \cup \{U_n(\langle \alpha, f\rangle) | \langle \alpha, f\rangle \in C$ and $n \in \omega\}$ as a basis.

We want to prove that S_λ is a Moore space of class $\mathscr{S}_\lambda$; we begin by showing that S_λ satisfies (1)–(5).

Certainly D is open and discrete and $\{C_\alpha\}_{\alpha\in\lambda}$ is a discrete family of closed sets of S_λ.

Define M and $\mathscr{B}_n$ as in the first paragraph of this section. If we topologize M_α as a subspace of M, then $h_\alpha : M_\alpha \to C_\alpha$ defined by $h_\alpha(f) = \langle \alpha, f\rangle$ is a homeomorphism. Let $\mathscr{B} = \bigcup_{n\in\omega} \mathscr{B}_n$ and for $B = B_n(f) \in \mathscr{B}$, let $O_B = \bigcup_{\alpha\in\lambda} U_n(\langle \alpha, f\rangle)$. Then $\{O_B | B \in \mathscr{B}_n\}$ is a cover of C by disjoint open (hence clopen) sets. Thus, since $h_\alpha(B \cap M_\alpha) = U_n(\langle \alpha, f\rangle) \cap C$ and B and O_B are both clopen, (5) holds.

It remains to check (3), so assume that $A \subset \lambda$. If $p = \langle \alpha, f\rangle \in C_\alpha$, by the definition of M_α there is an $n_p \in \omega$ such that $A \in f(n_p - 1)(\alpha)$. Let $U = \bigcup\{U_{n_p}(p) | p \in C_\alpha$ and $\alpha \in A\}$ and $V = \bigcup\{U_{n_q}(q) | q \in C_\beta$ and $\beta \in \lambda - A\}$. Certainly $\bigcup_{\alpha\in A} C_\alpha \subset U$ and $\bigcup_{\beta\in\lambda-A} C_\beta \subset V$. To check that $U \cap V = \emptyset$ assume that $\alpha \in A$, $\beta \in \lambda - A$, $p = \langle \alpha, f\rangle$, and $q = \langle \beta, g\rangle$. Suppose a point $x \in U_{n_p}(p) \cap U_{n_q}(q)$. Then $x \notin C$ so $x = \langle\{\alpha, \beta\}, h, m\rangle$ where $m \geqq n_p$, $m \geqq n_q$, $h \upharpoonright n_p = f \upharpoonright n_p$, and $h \upharpoonright n_q = g \upharpoonright n_q$. By definition of n_p and n_q, $A \in f(n_p - 1)(\alpha) \cap g(n_q - 1)(\beta) = h(n_p - 1)(\alpha) \cap h(n_q - 1)(\beta)$. But since $\alpha \in A$ and $\beta \in \lambda - A$, the supposed point $\langle\{\alpha, \beta\}, h, m\rangle$ does not belong to D_m. This proves that $U \cap V = \emptyset$ and also shows why D_m was defined as it was.

To see that S_λ is a Moore space, for each $p = \langle \alpha, f\rangle \in C_\alpha$ choose a $k_p \in \omega$ with $\{\alpha\} \in f(k_p - 1)(\alpha)$. If $q = \langle \beta, g\rangle \in C_\beta$ for some $\beta \neq \alpha$, there is an $i \in \omega$ with $\{\alpha\} \in g(i - 1)(\beta)$; thus $U_i(q) \cap U_{k_p}(p) = \emptyset$. Hence $\{U_n(p) | n > k_p\}$ is a

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clopen basis for p in S_λ contained in the metric space $C_\alpha \cup D$. From this it is easy to check that S_λ is T_1 and regular and that $\mathscr{G} = \bigcup_{n\in\omega}\mathscr{G}_n$ where $\mathscr{G}_n = D \cup \{U_n(p)|p \in C\}$ is a development for S_λ; therefore, S_λ is a Moore space of class $\mathscr{S}_\lambda$.

5. The universality of S_λ in $\mathscr{S}_\lambda$. In this section we prove that if S_λ is collectionwise normal, then every space in $\mathscr{S}_\lambda$ is collectionwise normal. The following lemma is used in the proof.

LEMMA 2. *Let $\{C_\alpha\}_{\alpha\in\lambda}$ be a family of disjoint sets in a space X. Let $C = \bigcup_{\alpha\in\lambda}C_\alpha$. For each n in ω let $\{U_\alpha{}^n\}_{\alpha\in\lambda}$ be a discrete collection of open sets in X so that for each α, $(\bar{U}_\alpha{}^n \cap C) \subset C_\alpha$ and $C_\alpha \subset \bigcup_{n\in\omega}U_\alpha{}^n$.*

Then the C_α's can be mutually separated by disjoint open sets.

Proof. For $n \in \omega$ and $\alpha \in \lambda$, let $Z_\alpha{}^n = U_\alpha{}^n - \bigcup\{\bar{U}_\beta{}^i|i \leqq n \text{ and } \beta \neq \alpha\}$. Let $Z_\alpha = \bigcup_{n\in\omega}Z_\alpha{}^n$. Then for each α in λ, $C_\alpha \subset Z_\alpha$ and the Z_α's are disjoint open sets, so the lemma is proved.

THEOREM 3. *If S_λ is collectionwise normal, then every space in class $\mathscr{S}_\lambda$ is collectionwise normal.*

Proof. Let $X \in \mathscr{S}_\lambda$. Let $C_\alpha{}^*$, $h_\alpha{}^*$, $M_\alpha{}^*$, $\mathscr{B}_n{}^*$, and $O_B{}^*$ be as described in conditions (1)–(5) for X. Assume further that for each $n \in \omega$, $\mathscr{B}_{n+1}{}^*$ refines $\mathscr{B}_n{}^*$. Let C_α, h_α, M_α, $\mathscr{B}_n$, and O_B be the related objects for S_λ.

Note that it is sufficient to prove that the $C_\alpha{}^*$'s can be mutually separated by disjoint open sets.

Since X is a Moore space and each $h_\alpha{}^*$ is a homeomorphism, we assume that for each $\alpha \in \lambda$, and $B \in \mathscr{B}^*$, we have a neighborhood $N(B, \alpha)$ of $h_\alpha{}^*(B \cap M_\alpha{}^*)$ contained in $O_B{}^*$ such that $\{N(B, \alpha)|B \in \mathscr{B}^*\}$ is a basis in X for the points of $C_\alpha{}^*$ and $N(B, \alpha) \cap C_\beta{}^* = \emptyset$ for $B \in \mathscr{B}^*$ and $\beta \neq \alpha$. We also assume that if $B_1 \in \mathscr{B}^*$, $B_2 \in \mathscr{B}^*$, and $B_1 \subset B_2$, then $N(B_1, \alpha) \subset N(B_2, \alpha)$.

By condition (3), for each $A \subset \lambda$, there are disjoint open sets U_A and V_A in X such that $\bigcup_{\alpha\in A}C_\alpha{}^* \subset U_A$ and $\bigcup_{\beta\in\lambda-A}C_\beta{}^* \subset V_A$.

For $x \in C_\alpha{}^*$ we choose a point $p_x = \langle\alpha, f\rangle \in C_\alpha$ as follows. For $n \in \omega$ and $\beta \in \lambda$, let $f(n)(\beta) = \{A \subset \lambda|N(B, \beta) \subset U_A \text{ or } N(B, \beta) \subset V_A$ where $B \in \mathscr{B}_n{}^*$ and $x \in N(B, \alpha)\}$. Since $\{N(B, \alpha)|B \in \mathscr{B}^*\}$ contains a basis for x, for each $A \subset \lambda$ there is an n so that $A \in f(n)(\alpha)$. Thus $f \in M_\alpha$ and $p_x = \langle\alpha. f\rangle \in C_\alpha$.

If S_λ is collectionwise normal, there is a family $\{W_\alpha\}_{\alpha\in\lambda}$ of disjoint open sets in S_λ with $C_\alpha \subset W_\alpha$ for each α. For each point p in C_α choose $i(p) \in \omega$ so that $U_{i(p)}(p) \subset W_\alpha$.

Now we can choose an integer $j(x)$ for each point x in $C_\alpha{}^*$ as follows. Let $j(x) = i(p_x)$. This integer will tell us the size of the neighborhood of x which we need. Let $B(x)$ be the open set in $\mathscr{B}_{j(x)}{}^*$ which contains $h_\alpha{}^{*-1}(x)$. Let $W(x) = N(B(x), \alpha)$.

For each $n \in \omega$ and $\alpha \in \lambda$, let $W_\alpha{}^n = \bigcup\{W(x)|x \in C_\alpha{}^* \text{ and } j(x) = n\}$. Note that for each α, $C_\alpha{}^* \subset \bigcup_{n\in\omega}W_\alpha{}^n$. To finish the proof we will modify each

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collection $\{W_\alpha^n\}_{\alpha\in\lambda}$ so that the new collections will satisfy the hypotheses of Lemma 2.

First we show that $\{W_\alpha^n\}_{\alpha\in\lambda}$ is a disjoint collection of open sets. To this end suppose that $x \in C_\alpha^*$, $y \in C_\beta^*$, $p_x = \langle \alpha, f\rangle$, $p_y = \langle \beta, g\rangle$, and $j(x) = j(y) = n$. We will show that $W(x) \cap W(y) = \emptyset$. If $B(x) \neq B(y)$, then $W(x) = N(B(x), \alpha) \subset O_{B(x)}^*$ and $W(y) = N(B(y), \beta) \subset O_{B(y)}^*$, but $O_{B(x)}^* \cap O_{B(y)}^* = \emptyset$ so $W(x) \cap W(y) = \emptyset$.

Thus we assume that $B(x) = B(y)$. Therefore, $f \restriction n = g \restriction n$. Recall that $n = i(p_x) = i(p_y)$. Therefore, back in S_λ now, $U_n(p_x) \cap U_n(p_y) = \emptyset$. In particular, $\langle\{\alpha, \beta\}, f, n\rangle \notin U_n(p_x) \cap U_n(p_y)$. There must therefore be $r < n$, $s < n$, and $A \subset \lambda$ such that $A \in f(r)(\alpha) \cap f(s)(\beta)$ but exactly one of α and β belongs to A. Suppose $\alpha \in A$ and $\beta \in \lambda - A$. By the way we associated $\langle \alpha, f\rangle$ with x, we know that $N(B, \alpha) \subset U_A$ where $x \in N(B, \alpha)$ and $B \in \mathscr{B}_r^*$. By the nesting properties of the $N(B, \alpha)$'s, we know that $N(B(x), \alpha) \subset U_A$. Similarly, $N(B(y), \beta) \subset V_A$. Since $U_A \cap V_A = \emptyset$, $W(x) \cap W(y) = \emptyset$. This shows that $\{W_\alpha^n\}_{\alpha\in\lambda}$ is a disjoint set.

The final step is to modify each W_α^n slightly in order to get a discrete set as required in Lemma 2.

For each $B \in \mathscr{B}_n^*$, let $A(B) = \{\alpha \in \lambda | N(B, \alpha) = W(x) \text{ for some } x \in C_\alpha^*\}$. Let $U_{A(B)}$ and $V_{A(B)}$ be disjoint open sets so that $\bigcup_{\alpha\in A(B)} C_\alpha^* \subset U_{A(B)}$ and $\bigcup_{\beta\in\lambda - A(B)} C_\beta^* \subset V_{A(B)}$. Let $Y(B) = (\bigcup_{\alpha\in A(B)} N(B, \alpha)) \cap U_{A(B)}$. Let $Z_\alpha^n = (\bigcup_{B\in\mathscr{B}_n^*} Y(B)) \cap W_\alpha^n$. The collections $\{Z_\alpha^n\}_{\alpha\in\lambda}$ meet the requirements of of Lemma 2, proving the theorem.

A consequence of Theorem 3 is, of course, that if any normal, non-metrizable Moore space can be constructed as described in Theorem 1, then an S_λ is a normal, non-metrizable Moore space.

THEOREM 4. *Let Y be a locally compact*, 0-*dimensional Moore space* (*not necessarily normal*) *with a basis of cardinality* λ. *Let M be a metric space which is* 0-*dimensional in the covering sense. Let X be a normal Moore space such that $X \subset Y \times M$.*

Then if S_λ is collectionwise normal, X is collectionwise normal. (*Note that X is not necessarily in* $\mathscr{S}_\lambda$.)

Proof. Let $\mathscr{G}$ be an open cover of Y so that for each $B \in \mathscr{G}$, $\bar{B}$ is compact. Let $\{\mathscr{D}_n\}_{n\in\omega}$ be a σ-discrete closed refinement of $\mathscr{G}$ [**1**].

For each $n \in \omega$, let X_n be the space obtained from X by making the points $X \cap ((Y - \bigcup\{D | D \in \mathscr{D}_n\}) \times M)$ discrete.

By Theorem 1, X_n belongs to $\mathscr{S}_\lambda$. By Theorem 3, each X_n is collectionwise normal. We are now ready to prove that X is collectionwise normal using Lemma 2.

Let $\{H_\alpha\}_{\alpha\in\mu}$ be a discrete collection of closed sets in X. For each $n \in \omega$, let $\{U_{\alpha n}\}_{\alpha\in\mu}$ be a disjoint collection of open sets in X_n such that $H_\alpha \subset U_{\alpha n}$. There are open sets $\{V_{\alpha n}\}_{\alpha\in\mu}$ in X so that for each α, $(H_\alpha \cap (\bigcup_{D\in\mathscr{D}_n} D \times M)) \subset V_{\alpha n}$.

By the normality of X, there is a discrete family $\{W_{\alpha n}\}_{\alpha \in \mu}$ of open sets in X so that $(H_\alpha \cap (\bigcup_{D \in \mathscr{D}_n} D \times M)) \subset W_{\alpha n}$ for each $\alpha \in \mu$ and $\bar{W}_{\alpha n} \cap H_\beta = \emptyset$ for $\alpha \neq \beta$. By Lemma 2, X is collectionwise normal.

6. The consistency of S_λ not being collectionwise Hausdorff. It is known to be consistent with the usual axioms for set theory that any normal Moore space be collectionwise Hausdorff [**2**]. It is also known to be consistent that there be a normal Moore space which is not ω_1-collectionwise Hausdorff [**3**]. Thus the following theorem shows that it is consistent that S_{ω_1} fail to be collectionwise Hausdorff.

THEOREM 5. *If there is a first-countable space X which is normal but not λ-collectionwise Hausdorff, then S_λ is not collectionwise Hausdorff.*

Proof. Let $\{x_\alpha\}_{\alpha \in \lambda}$ be a closed discrete set of points in X which cannot be separated by disjoint open sets. Let $\{N_i(x_\alpha)\}_{i \in \omega}$ be a nested countable basis for x_α.

Since X is normal, for each $A \subset \lambda$ there are disjoint open sets U_A and V_A such that $\{x_\alpha | \alpha \in A\} \subset U_A$ and $\{X_\beta | \beta \in \lambda - A\} \subset V_A$.

We will use the same notation here in referring to the parts of S_λ as was used in its original description in Section 4.

We choose an $f \in M$ as follows. If $n \in \omega$ and $\beta \in \lambda$, let

$$f(n)(\beta) = \{A \subset \lambda | N_n(x_\beta) \subset U_A \text{ or } N_n(x_\beta) \subset V_A\}.$$

Suppose that $A \subset \lambda$. If $\beta \in A$, there is an $i \in \omega$ with $N_i(x_\beta) \subset U_A$. If $\beta \notin A$, there is an $i \in \omega$ with $N_i(x_\beta) \subset V_A$. Thus for each $\beta \in \lambda$ there is an $i \in \omega$ with $A \in f(i)(\beta)$. Therefore, $f \in M_\beta$ for all $\beta \in \lambda$.

The subset $\{\langle \alpha, f \rangle\}_{\alpha \in \lambda}$ of S_λ is discrete. So, if S_λ is collectionwise Hausdorff, for each $\alpha \in \lambda$ there is an $i_\alpha \in \omega$ such that $\{U_{i\alpha}(\langle \alpha, f \rangle)\}_{\alpha \in \lambda}$ are disjoint. We claim that in this case the x_α's could be separated by disjoint open sets. To see this note first that for each $n \in \omega$, $\{N_{i\alpha}(x_\alpha) | i_\alpha = n\}$ are disjoint. This is true since if $i_\alpha = i_\beta = n$, $\langle \{\alpha, \beta\}, f, n \rangle \notin U_{i\alpha}(\langle \alpha, f \rangle) \cap U_{i\beta}(\langle \beta, f \rangle)$. So there is an $A \subset \lambda$ such that $A \in f(n)(\alpha) \cap f(n)(\beta)$ and exactly one of α and β belongs to A. But then, if $\alpha \in A$ and $\beta \in \lambda - A$, $N_n(x_\alpha) \subset U_A$ and $N_n(x_\beta) \subset V_A$ so $N_{i\alpha}(x_\alpha) \cap N_{i\beta}(x_\beta) = \emptyset$.

Using the normality of X, we can find for each $n \in \omega$ a discrete collection $\{W_\alpha | i_\alpha = n\}$ of open sets in X so that for each α with $i_\alpha = n$, $x_\alpha \in W_\alpha \subset N_{i\alpha}(x_\alpha)$ and for $\beta \neq \alpha$, $x_\beta \notin \bar{W}_\alpha$. By Lemma 2 then, the x_α's can be separated by disjoint open sets.

7. A special class of neighborhoods in S_λ. Let $\bar{\mathscr{A}} = \{\mathscr{A} | \mathscr{A}$ is a finite family of subsets of $\lambda\}$. For each $\mathscr{A} \in \bar{\mathscr{A}}$ and $p = \langle \alpha, f \rangle \in C_\alpha$ let $n(\mathscr{A}, p)$ be the smallest integer n such that $\mathscr{A} \subset f(n-1)(\alpha)$. Let $V_{\mathscr{A},\alpha} = \bigcup_{p \in C\alpha} U_{n(\mathscr{A},p)}(p)$.

One hope that S_λ is not λ-collectionwise normal is based on the following fact.

THEOREM 5. *If $\{\mathscr{A}_\alpha\}_{\alpha \in \lambda} \subset \bar{\mathscr{A}}$ then $\{V_{\mathscr{A}_\alpha,\alpha}\}_{\alpha \in \lambda}$ are not disjoint.*

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Proof. By a Δ-system argument [**1**], there are $i \leqq n < \omega$ and an infinite subset L of λ such that, for all $\alpha \neq \beta$ in L:

(a) $\mathscr{A}_\alpha = A_{0\alpha}, A_{1\alpha}, \ldots, A_{n\alpha}$;

(b) for $j < i$, $A_{j\alpha} = A_{j\beta}$;

(c) for $i \leqq j \leqq n$, $A_{j\alpha} \notin \mathscr{A}_\beta$;

(d) for $1 \leqq j \leqq n$, $\alpha \in A_{j\alpha}$ if and only if $\beta \in A_{j\beta}$.

Choose $\alpha \neq \beta$ in L arbitrarily. Choose $f \in M_\alpha \cap M_\beta$ with $\mathscr{A}_\alpha = f(0)(\alpha)$ and $\mathscr{A}_\beta = f(0)(\beta)$. Then $\langle\{\alpha, \beta\}, f, 0\rangle \in U_1(\langle f, \alpha\rangle) \cap U_1(\langle f, \beta\rangle)$. Since $n(\mathscr{A}_\alpha, \langle f, \alpha\rangle) = 1 = n(\mathscr{A}_\beta, \langle f, \beta\rangle)$, $V_{\mathscr{A}_\alpha, \alpha} \cap V_{\mathscr{A}_\beta, \beta} \neq \emptyset$.

8. The description of T_λ**.** There is a metric space M' associated with T_λ obtained by taking the product (with the product topology) of ω copies of 2^{2^λ} with the discrete topology. That is, $M' = (2^{2^\lambda})^\omega$ and, if $f \in M'$, then $B_n(f) = \{g \in M' | f \upharpoonright n = g \upharpoonright n\}$ is the nth basic open set for f. Thus $\mathscr{B}_n' = \{B_n(f) | f \in M'\}$ is an open cover of M' by disjoint clopen sets.

For each $\alpha \in \lambda$, let $M_\alpha' = \{f\colon \omega \rightarrow 2^{2^\lambda} |$if $A \subset \lambda$, there is an $n \in \omega$ with $A \in f(n)\}$. Note that $M_\alpha' = M_\beta'$ for each $\alpha, \beta \in \lambda$. Let $C_\alpha' = \{\langle\alpha, f\rangle | f \in M_\alpha'\}$ and $C' = \bigcup_{\alpha \in \lambda} C_\alpha'$. The points in C' will be the non-discrete points of T_λ.

For $n \in \omega$, let $D_n' = \{\langle\{\langle\alpha, f\rangle, \langle\beta, g\rangle\}, n\rangle | \langle\alpha, f\rangle$ and $\langle\beta, g\rangle$ belong to C' and if $A \in f(i) \cap g(j)$ for some $i < n$ and $j < n$, then $\alpha \in A$ if and only if $\beta \in A\}$. Let $D' = \bigcup_{n \in \omega} D_n'$. The points in D' will be the discrete points in T_λ.

For each $n \in \omega$ and $\langle\alpha, f\rangle \in C'$ let $U_n(\langle\alpha, f\rangle) = \{\langle\alpha, g\rangle \in C' | g \upharpoonright n = f \upharpoonright n\} \cup \{\langle\{\langle\alpha, g\rangle, \langle\beta, h\rangle\}, m\rangle \in D' | m \geqq n$ and $g \upharpoonright n = f \upharpoonright n\}$.

Let T_λ equal $C' \cup D'$ topologized by using $D' \cup \{U_n(\langle\alpha, f\rangle) | n \in \omega$ and $\langle\alpha, f\rangle \in C'\}$ as a basis.

The space T_λ is a Moore space which satisfies conditions (1)–(4) required for a space of class $\mathscr{S}_\lambda$. The same proof given for S_λ proves this fact. However, T_λ does not satisfy (5) and P. Nyikos has shown that normality of T_λ is equivalent to its metrizability.

However we do know the following fact.

THEOREM 7. *If there is any normal, first-countable space X which is not λ-collectionwise normal, then T_λ is not λ-collectionwise normal.*

Proof. Let $\{C_\alpha^*\}_{\alpha \in \lambda}$ be a discrete family of closed sets in X which cannot be separated by disjoint open sets. For each $x \in \bigcup_{\alpha \in \lambda} C_\alpha^*$ let $\{N_i(x)\}_{i \in \omega}$ be a nested neighborhood basis for x.

Since X is normal, for each $A \subset \lambda$ there are disjoint open sets U_A and V_A in X such that $\bigcup_{\alpha \in A} C_\alpha^* \subset U_A$ and $\bigcup_{\beta \in \lambda - A} C_\beta^* \subset V_A$.

Recall in the description of T_λ the definitions of C_α' and $U_n(p)$ for p a point of C_α'. For each $x \in C_\alpha^*$ we choose $p_x = \langle\alpha, f\rangle \in C_\alpha'$ as follows. Let $f(n) = \{A \subset \lambda | N_n(x) \subset U_A$ or $N_n(x) \subset V_A\}$. Since $\{N_i(x)\}_{i \in \omega}$ is a basis for x, for each $A \subset \lambda$ there is an $n \in \omega$ so that $N_n(x) \subset U_A$ if $\alpha \in A$ or $N_n(x) \subset V_A$ if $\alpha \notin A$. Thus f belongs to M_α' so $\langle\alpha, f\rangle$ is a point of C_α'.

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Suppose T_λ is collectionwise normal. Then there is a collection of disjoint open sets $\{W_\alpha\}_{\alpha\in\lambda}$ in T_λ so that $C_\alpha' \subset W_\alpha$ for each α.

For each $p \in C_\alpha'$ there is an $i(p) \in \omega$ such that $U_{i(p)}(p) \subset W_\alpha$. Let $W_\alpha^* = \bigcup_{x\in C_\alpha^*} N_{i(p_x)}(x)$. We will show that $\{W_\alpha^*\}_{\alpha\in\lambda}$ are disjoint and thus that $\{C_\alpha^*\}_{\alpha\in\lambda}$ can be separated by disjoint open sets contrary to assumption.

To this end, assume that $\alpha \neq \beta$, $x \in C_\alpha^*$ and $y \in C_\beta^*$. Let $p = p_x = \langle\alpha, f\rangle$ and $q = p_y = \langle\beta, g\rangle$. We want to show that $N_{i(p)}(x) \cap N_{i(q)}(y) = \emptyset$.

Assume that $i(p) \leqq i(q)$. Let $f' \in M_\alpha'$ so that $f' \restriction i(p) = f \restriction i(p)$ and $f'(j) = \emptyset$ for $i(p) \leqq j < i(q)$. Recall that $U_{i(p)}(p) \cap U_{i(q)}(q) = \emptyset$. Therefore $\langle\{\langle\alpha, f'\rangle, \langle\beta, g\rangle\}, i(q)\rangle \notin U_{i(p)}(p) \cap U_{i(q)}(q)$. Thus there must be an $A \subset \lambda$, $i < i(q)$ and $j < i(q)$ so that $A \in f'(i) \cap g(j)$ and exactly one of α and β belongs to A. But then $i < i(p)$. So $A \in f(i) \cap g(j)$. Thus, if say $\alpha \in A$ and $\beta \in \lambda - A$, then $N_{i(p)}(x) \subset N_i(x) \subset U_A$ and $N_{i(q)}(y) \subset N_j(y) \subset V_A$. But $U_A \cap V_A = \emptyset$; hence $N_{i(p)}(x) \cap N_{i(q)}(y) = \emptyset$.

Peter Nyikos has observed that T_λ is normal if and only if it is metrizable. To see this fact, let $H_i = \{\langle\alpha, f\rangle \in C' | f(j) = \emptyset \text{ for } j < i \text{ and } f_i \neq \emptyset\}$. Then $\{H_i\}_{i\in\omega}$ is a countable, discrete collection of closed sets whose union is C'. Suppose T_λ is normal, then there are disjoint open sets $\{V_i\}_{i\in\omega}$ with $H_i \subset V_i$ for every $i \in \omega$. For each $p \in C'$, let $n(p)$ be an integer such that $U_{n(p)}(p) \subset V_i$ for some i. For each $f \in M'$ define f^+ by $f^+(k+1) = f(k)$ and $f^+(0) = \emptyset$. For each point $\langle\alpha, f\rangle \in C'$, let $m(\langle\alpha, f\rangle) = \max\{n(\langle\alpha, f\rangle), n(\langle\alpha, f^+\rangle)\}$. For each $\alpha \in \lambda$ let $U_\alpha = \bigcup_{p\in C_\alpha'} U_{m(p)}(p)$. Then $\{U_\alpha\}_{\alpha\in\lambda}$ is a disjoint collection of open sets which separate the C_α''s making T_λ the discrete union of metrizable subsets, hence making T_λ metrizable.

Thus one cannot hope that T_λ itself is an example of a normal, non-metrizable Moore space; however, a proof that T_λ is not collectionwise normal would still be of interest since it would provide an example of a Moore space with a normalized collection of closed sets which cannot be mutually separated.

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