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A FUNCTIONAL ANALYTIC DESCRIPTION OF NORMAL SPACES

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Throughout the paper, X will denote a completely regular (Hausdorff) topological space and $C(X)$ the **R**-algebra of all real-valued continuous functions on X. When this algebra carries the continuous convergence structure [**1**], we write $C_c(X)$. We note that $C_c(X)$ is a complete [**5**] convergence **R**-algebra [**1**].

Our description of normality reads as follows. A completely regular topological space X is normal if and only if $C_c(X)/J$ (endowed with the obvious quotient structure; see § 1) is complete for every closed ideal $J \subset C_c(X)$.

1. Residue class algebras. For a closed non-empty subset $A \subset X$, let $I(A)$ denote the ideal in $C(X)$ consisting of all functions in $C(X)$ vanishing on A. Since the kernel of the restriction map

$$r\colon C(X) \rightarrow C(A),$$

sending each $f \in C(X)$ into its restriction $f|A$, is $I(A)$, we have the following commutative diagram of **R**-algebra homomorphisms:

$$\text{(I)} \qquad \begin{array}{ccc} C(X) & \xrightarrow{\ r\ } & C(A) \\ \Big\downarrow{\scriptstyle \pi} & \nearrow{\scriptstyle \bar{r}} & \\ C(X)/I(A) & & \end{array}$$

where π is the natural projection map and $\bar{r}$ the unique map factoring r. A filter θ converges to zero in $C_c(X)$ if and only if, for each point $p \in X$ and each positive real number ϵ, there is an element F in θ and a neighborhood U of p with

$$|f(x)| \leqq \epsilon,$$

for every $x \in U$ and every $f \in F$. With $C_c(X)/I(A)$ we denote $C(X)/I(A)$ endowed with the natural quotient structure (in the category of convergence spaces) of $C_c(X)$ with respect to π. This means that a filter converges to zero in $C_c(X)/I(A)$ if and only if it is finer than the image (under π) of a filter converging to zero in $C_c(X)$. Endowing $C(X)$ and $C(A)$ with the continuous convergence structure, all the maps in diagram (I) are continuous.

PROPOSITION 1. *The* **R**-*algebra monomorphism* $\bar{r}$ *is a homeomorphism from* $C_c(X)/I(A)$ *onto a subspace of* $C_c(A)$.

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Proof. All we have to show is that a filter $\bar{\theta}$ on $C(X)/I(A)$ for which $\bar{r}(\bar{\theta})$ converges to zero in $C_c(A)$ also converges to zero in $C_c(X)/I(A)$. That is, we must construct a filter θ on $C_c(X)$ converging to zero with the property that $\pi(\theta)$ is coarser than $\bar{\theta}$.

Let $\bar{\theta}$ be a filter on $C(X)/I(A)$ with $\bar{r}(\bar{\theta})$ convergent to zero in $C_c(A)$. Hence, for each $p \in A$ and each positive real number ϵ, there is a neighborhood $U_{p,\epsilon}$ of p in X and an $F'_{p,\epsilon} \in \bar{r}(\bar{\theta})$ contained in $r(C(X))$ with

$$|f'(q)| \leqq \epsilon,$$

for all $f' \in F'_{p,\epsilon}$ and all $q \in U_{p,\epsilon} \cap A$. Without loss of generality, we can assume that each $U_{p,\epsilon}$ is a cozero-set in X. To facilitate the construction of our filter, we choose inside of each $U_{p,\epsilon}$ a zero-set neighborhood $\tilde{U}_{p,\epsilon}$ in X of p. Furthermore, to each y in $X\backslash A$ there exists, disjoint from A, a cozero-set neighborhood V_y of y in X inside of which we fix a zero-set neighborhood $\tilde{V}_y$ of y in X. We intend to show that all the sets of the form

$$(*) \quad F_{p,y,\epsilon} = \{f \in C(X): f|A \in F'_{p,\epsilon}, f(\tilde{U}_{p,\epsilon}) \subset [-2\epsilon, 2\epsilon], \text{ and } f(\tilde{V}_y) = \{0\}\},$$

for $p \in A$, $y \in X\backslash A$, and ϵ a real number greater than 0, generate the desired filter. We first demonstrate that

$$(**) \qquad r\left(\bigcap_{i=1}^{n} F_{p_i,y_i,\epsilon_i}\right) \supset \bigcap_{i=1}^{n} F'_{p_i,\epsilon_i},$$

where p_i, y_i, and ϵ_i are as above. To this end, let

$$f' \in \bigcap_{i=1}^{n} F'_{p_i,\epsilon_i}$$

and j be a fixed integer between 1 and n. We now choose an element $f \in C(X)$ for which $r(f) = f'$ and associate to this function the sets

$$P_j = \{q \in \tilde{U}_{p_j,\epsilon_j}: |f(q)| \geqq 2\epsilon_j\}, \text{ and}$$
$$Q_j = \{q \in X: |f(q)| \leqq \epsilon_j\} \cup (X\backslash U_{p_j,\epsilon_j}).$$

It is clear that $Q_j \supset A$ and, furthermore, that P_j and Q_j are disjoint zero-sets in X. Hence, there is a function $h_j \in C(X)$ separating P_j and Q_j; that is,

$$h_j(q) = 0 \quad \text{for all } q \in P_j, \text{ and}$$
$$h_j(q) = 1 \quad \text{for all } q \in Q_j.$$

Without loss of generality, we may assume that $h_j(X) \subset [-1, 1]$. Similarly, we pick a function $k_j \in C(X)$ with the property that

$$k_j(q) = 0 \quad \text{for all } q \in \tilde{V}_{y_j}, \text{ and}$$
$$k_j(q) = 1 \quad \text{for all } q \in X\backslash V_{y_j},$$

and $k_j(X) \subset [-1, 1)$. The function $g = f \cdot h_1 \cdot h_2 \cdot \ldots \cdot h_n \cdot k_1 \cdot \ldots \cdot k_n$ is an element of $\bigcap_{i=1}^{n} F_{p_i, y_i, \epsilon_i}$ and extends f'. Now the filter θ generated on $C(X)$ by all the sets of the form (*) obviously converges to zero in $C_c(X)$. Because (**) is satisfied, $\pi(\theta)$ is coarser than $\bar{\theta}$, and thus the proof is complete.

Next, we will investigate the universal representation [**2**] of $C_c(X)/I(A)$, i.e., the **R**-algebra $C_c(Hom_c\, C_c(X)/I(A))$ and the **R**-algebra homomorphism

$$d\colon C_c(X)/I(A) \to C_c(Hom_c\, C_c(X)/I(A)),$$

where $Hom_c\, C_c(X)/I(A)$ denotes the space of all continuous **R**-algebra homomorphisms from $C_c(X)/I(A)$ onto **R** together with the continuous convergence structure. The map d sends each element $\bar{f} \in C_c(X)/I(A)$ to the function defined by $d(\bar{f})(h) = h(\bar{f})$, for each $h \in Hom_c\, C_c(X)/I(A)$.

We intend to establish a relationship between $Hom_c\, C_c(X)/I(A)$ and A. The homomorphism π induces a continuous map

$$\pi^*\colon Hom_c\, C_c(X)/I(A) \to Hom_c\, C_c(X),$$

sending each $h \in Hom_c\, C_c(X)/I(A)$ to $h \circ \pi$. By $Hom_c\, C_c(X)$ we mean the collection of all continuous **R**-algebra homomorphisms from $C_c(X)$ onto **R** together with the continuous convergence structure. As pointed out in [**3**], the map

$$i_X\colon X \to Hom_c\, C_c(X),$$

defined by the relation $i_X(p)(f) = f(p)$, for all $f \in C(X)$ and all $p \in X$, is a homeomorphism. Hence, the map $i_X^{-1} \circ \pi^*$ maps $Hom_c\, C_c(X)/I(A)$ continuously into X. In fact, the range of this map is in A, since $(i_X^{-1} \circ \pi^*)(h)$ for any $h \in Hom_c\, C_c(X)/I(A)$, is sent to zero by all the functions in $I(A)$, and A is a closed subset of a completely regular space. Next, we show that $i_X^{-1} \circ \pi^*$ is actually a bijection onto A. Because π is surjective, the map $i_X^{-1} \circ \pi^*$ is clearly injective. For the surjectivity, choose a point $p \in A$. The homomorphism $i_X(p)\colon C_c(X) \to \mathbf{R}$ annihilates all the functions in $I(A)$, and therefore can be factored to a continuous homomorphism h on $C_c(X)/I(A)$. It is clear that $(i_X^{-1} \circ \pi^*)(h) = p$.

PROPOSITION 2. *The map*

$$i_X^{-1} \circ \pi^*\colon Hom_c\, C_c(X)/I(A) \to A$$

is a homeomorphism.

Proof. Since $i_X^{-1} \circ \pi^*$ is a continuous bijection, it remains to verify that $(i_X^{-1} \circ \pi^*)^{-1}$ is also continuous. We have the commutative diagram

$$\begin{array}{ccc} A & \xrightarrow{i_A} & Hom_c\, C_c(A) \\ & {\scriptstyle (i_X^{-1} \circ \pi^*)^{-1}} \searrow \quad \swarrow {\scriptstyle \bar{r}^*} & \\ & Hom_c\, C_c(X)/I(A), & \end{array}$$

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where $\bar{r}^*$ sends each $h \in Hom_c\, C_c(A)$ to $h \circ \bar{r}$. Since both i_A and $\bar{r}^*$ are continuous, the proposition is established.

2. Closed C-embedded subsets. A closed non-empty subset A of a space X is said to be *C-embedded* if every continuous real-valued function defined on A has a continuous extension to X, that is to say

$$r: C(X) \to C(A)$$

is surjective. For example, every compact subset of X is C-embedded.

THEOREM 1. *A closed non-empty subset A of a completely regular topological space X is C-embedded if and only if $C_c(X)/I(A)$ is complete.*

Proof. If A is a C-embedded subset of X, then the map $\bar{r}$ is surjective. Since $C_c(A)$ is complete and $\bar{r}$ is a homeomorphism (see Proposition 1), $C_c(X)/I(A)$ is complete. Conversely, assume that $C_c(X)/I(A)$ is complete. Proposition 1 implies that $\bar{r}(C_c(X)/I(A))$ is a closed subalgebra of $C_c(A)$. By a type of Stone-Weierstrass theorem proved in [**5**], which states that a closed subalgebra of $C_c(Y)$ that contains the constant functions and determines the topology (see [**6**, p. 39]) of the completely regular topological space Y is all of $C(Y)$, we conclude that the map $\bar{r}$ is surjective. Thus, A is C-embedded.

PROPOSITION 3. *A closed non-empty subset A of a completely regular topological space X is compact if and only if $C_c(X)/I(A)$ is normable.*

Proof. For A compact, $C_c(A)$ is a normed algebra under the supremum norm (this can be verified directly from the definition of the continuous convergence structure). It follows from Proposition 1 that $C_c(X)/I(A)$ is normable. On the other hand, if $C_c(X)/I(A)$ is normable, then $Hom_c\, C_c(X)/I(A)$ is a compact topological space (see [**7**]) and, hence, A is compact by Proposition 2.

COROLLARY. *Let A be a closed non-empty subset of a completely regular topological space X. If $C_c(X)/I(A)$ is normable, then it is complete.*

3. Normal spaces. A completely regular topological space is normal if and only if every non-empty closed subset is C-embedded (see [**6**, p. 48]). In view of Theorem 1, we know that the space X is normal if and only if $C_c(X)/I(A)$ is complete for every non-empty closed subset $A \subset X$. Since every closed ideal in $C_c(X)$ is of the form $I(A)$ for a non-empty closed subset A of X (see [**4**]), we state

THEOREM 2. *A completely regular topological space X is normal if and only if $C_c(X)/J$ is complete for every closed ideal $J \subset C_c(X)$.*

REFERENCES

1. E. Binz and H. H. Keller, *Funktionenräume in der Kategorie der Limesräume*, Ann. Acad. Sci. Fenn. Ser. AI *383* (1966), 1–21.

2. E. Binz, *Bemerkungen zu limitierten Funktionenalgebren*, Math. Ann. *175* (1968), 169–184.
3. ——— *Zu den Beziehungen zwischen* c-*einbettbaren Limesräumen und ihren limitierten Funktionenalgebren*, Math. Ann. *181* (1969), 45–52.
4. ——— *On closed ideals in convergence function algebras*, Math. Ann. 182 (1969), 145–153.
5. ——— *Notes on a characterization of function algebras*, Math. Ann. *186* (1970), 314–326.
6. L. Gillman and M. Jerison, *Rings of continuous functions* (Van Nostrand, Princeton, N.J. 1960).
7. M. Schroder, *Continuous convergence in a Gelfand theory for topological algebras*, Ph.D. thesis, Queen's University, Kingston, Ontario, 1971.

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