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Introduction

In everyday language, an object is said to be chiral whenever it lacks mirror symmetry. Human hands and shells of snails are examples of objects regarded as chiral. Formally, it is understood that a subset of an orientable geometric space is chiral if no orientation-reversing isometry of the space preserves it. This definition is more restrictive than just discarding symmetries acting locally as mirror reflections.

The notion of 'polytope' arises as a generalisation of the polygon and polyhedron. In the classical theory, polygons are two-dimensional objects bounded by edges, whereas polyhedra are three-dimensional objects bounded by polygons. One can continue by considering four-dimensional objects bounded by polyhedra (these are the 4-polytopes). The intuitive idea is that an n-polytope is an n-dimensional object bounded by $(n-1)$-polytopes. If we know some $(n-1)$-polytope $\mathcal{P}$ then we can easily understand some n-polytopes like the prism or the pyramid over $\mathcal{P}$.

The present book studies combinatorial objects known as *abstract polytopes*, with particular attention to those possessing a high degree of symmetry. Abstract polytopes are an abstraction of the initial geometric notion of polytopes. They consist of elements (called 'faces') of different ranks, and an incidence relation satisfying some of the main properties of the faces of convex polytopes. In particular, the face-lattice of every convex polytope constitutes an abstract polytope. Naturally, abstract polytopes of ranks 2 and 3 are called *abstract polygons* and *abstract polyhedra*, respectively. In this context, the term *chiral* denotes not just all orientable abstract polytopes lacking orientation-reversing automorphisms, but we also require the polytopes to be symmetric by all possible *abstract rotations*. The term 'chiral' for such highly symmetric abstract polytopes was first used by Schulte and Weiss in [247]; it had a fast acceptance by the community, which was reflected by the number of subsequent related papers. It is important to keep in mind, however, that in other areas of mathematics and in other sciences chirality does not imply any particular symmetry.

The study of chiral abstract polytopes is a natural follow-up to the theory of the most symmetric class of abstract polytopes, called *regular polytopes*, which are extensively treated in the monograph [188]. It has become clear since the definitions and first results that it is impossible to talk in detail about chiral polytopes without talking about regular polytopes as well.

Abstract regular polytopes admit all possible symmetry by *abstract reflections*. In the same way as their geometric counterparts, composition of two abstract reflections fixing a point yields an abstract rotation; this ensures that regular polytopes are also symmetric by all abstract rotations. Moreover, precisely half of the symmetries of a regular orientable polytope are products of abstract rotations. In this sense, the degree of symmetry of regular polytopes doubles the degree of symmetry of chiral polytopes. The only constraint for an abstract chiral polytope to become regular is precisely the lack of orientation-reversing symmetry.

Regular polytopes in their geometric version have been studied since antiquity, including convex regular polygons and the five Platonic solids. This is best illustrated by Euclid's *Elements* [85, Book XIII], although some features of regular polytopes were known even before Euclid's time. Since then, several mathematicians developed ideas leading to the study of the combinatorial structure of regular polytopes, leaving aside convexity and other geometric properties. The first big step in this direction was taken by Johannes Kepler (1571–1630), who allowed polyhedra to have star-shaped faces [152]. He observed that the symmetry of the small stellated dodecahedron and the great stellated dodecahedron have much in common with that of the Platonic solids, despite the fact that Kepler's polyhedra are not convex. This idea was taken further by Louis Poinsot (1777–1859), who considered faces meeting in a star-shape manner around vertices [232]. Ludwig Schläfli (1814–1895) and Edmund Hess (1843–1903) explored polytopes outside the plane and space, and classified the higher-dimensional analogues of regular convex and star polyhedra [126, 237].

In the twentieth century, J. F. Petrie and H. S. M. Coxeter admitted the possibility of polyhedra having infinitely many vertices, edges and faces [61]. Branko Grünbaum considered non-planar faces as building blocks, also allowing faces to meet around a vertex in a non-planar arrangement [104]. From this viewpoint, the faces of polyhedra are no longer associated to particular surfaces embedded in 3-space; instead, they become the embedding of combinatorial structures consisting of vertices and edges. The final list of 48 regular polyhedra satisfying the definition dictated by Grünbaum was given by Andreas W. M. Dress [83, 84]. In recent years Peter McMullen has made a lot of progress in the description of regular polytopes in higher-dimensional Euclidean spaces, extending the key ideas of Grünbaum's concept of polyhedron [172–174, 176, 177]. The final jump to a purely combinatorial definition was done in [76], where L. Danzer and E. Schulte established the axioms of abstract polytopes. The monograph [188] comprises the necessary theory to understand the most symmetric among these objects, that is, abstract regular polytopes. Many others have made contributions to the construction of the current theory of abstract regular polytopes. A more detailed account of historical remarks can be found in [188, Section 1A].

The history of chiral abstract polytopes is considerably younger. There are classical polyhedra with a large group of rotations allowing no orientation-reversing symmetries. The snub cube and the snub dodecahedron have as full symmetry groups the rotation group of the cube and the rotation group of the dodecahedron, respectively (see Figure 1.1). However, these polyhedra fail our definition of chirality, since no non-trivial rotation around the axis through a vertex and the centre of the polyhedron acts like a symmetry. Also, in both cases there are triangles whose centres do not belong to axes of 3-fold rotations preserving the polyhedron. Informally, we could say that the snub cube and the snub dodecahedron are not chiral because it is not possible to rotate around all vertices and centres of faces.

Under certain sets of conditions, imposing all possible symmetry by rotations to polytopes forces symmetry by reflections. The objects listed below are witnesses of this phenomenon.

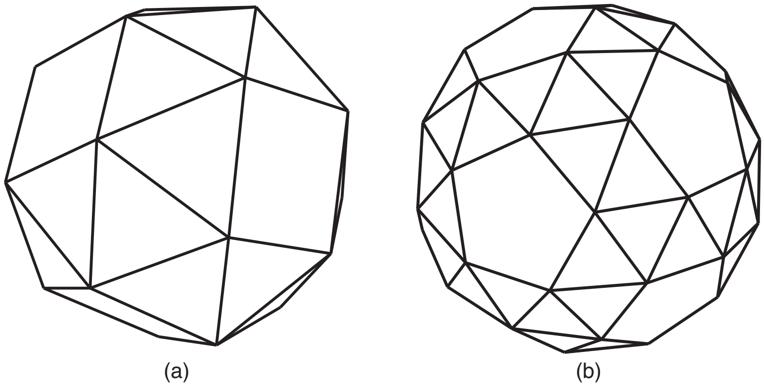

Figure 1.1 (a) Snub cube and (b) snub dodecahedron.

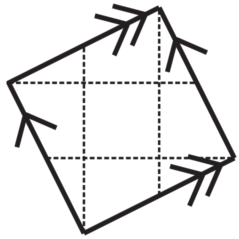

Figure 1.2 The toroidal polyhedron $\{4,4\}_{(2,1)}$.

- Convex polytopes [168].
- Tessellations of Euclidean or hyperbolic spaces [48].
- Finite polyhedra in the Euclidean space $\mathbb{E}^3$ [245].
- Planar polyhedra [245].
- Apeirohedra (infinite polyhedra) in $\mathbb{E}^3$ with planar faces [246].
- Tessellations of the d-dimensional torus for $d \geq 3$ [122].

Thus, the existence of chiral polytopes in geometric spaces is restricted to several constraints. As a result, chiral polytopes did not appear in the classical theory of polytopes. For instance, the discovery and full classification of chiral polyhedra in $\mathbb{E}^3$ took place only in the current century [245, 246].

Figure 1.2 shows the simplest chiral polytope; it is the polyhedron called $\{4,4\}_{(2,1)}$, and it is represented as a map on the torus. It contains 5 vertices, 10 edges and 5 faces. Later in this book it will become clear that it allows no reflection, but it is symmetric under all 4-fold rotations around vertices and centres of faces.

The formal study of abstract chiral polytopes started in the first half of the twentieth century with the works of Brahana and Coxeter on chiral maps on the torus (see [13, 49]). Most attention was given to polytopes of rank 3, in part because of the difficulty of working combinatorially with these objects in higher ranks, and in part because rank-3 chiral abstract polytopes are a particular case of chiral maps on surfaces, which are interesting objects on their own. In Section 2.3 we present a detailed summary of relevant discoveries on chiral maps and polytopes.

It is a striking contrast that numerous mathematicians since the early Greeks knew about regular polytopes, whereas the study of chiral polytopes was initiated only in the past century. When trying to construct examples of chiral polytopes we usually find that among all polytopes that are invariant under all possible combinatorial rotations, those with a simple description tend to be also invariant under reflections, and so they are regular instead of chiral. As a result, few chiral polytopes were known before the computer era, in comparison to the wealth of regular polytopes that had been studied by that time. Even with the help of computers, only finitely many chiral polytopes of ranks 6 and higher have been described; while on the other hand, cubes and tessellations by cubes are well-understood examples of regular polytopes that exist in arbitrarily large ranks.

Despite the initial lack of examples of chiral polytopes, they are the most-studied abstract polytopes among those that are not regular. The fact that they are invariant under all possible abstract rotations makes them similar in many aspects to their regular counterparts, and provides tools to study them. This is a good reason to consider chiral polytopes as the natural class to investigate after the class of the regular ones.

Thus, the study of abstract chiral polytopes constitutes an area of recent development that has a wide interaction with the topic of maps on surfaces. These objects can be defined in multiple equivalent ways, and motivate pleasant interactions of distinct branches of mathematics, like combinatorics of partially ordered sets, finitely presented groups, algebraic topology of simplicial complexes, as well as Euclidean, projective and hyperbolic geometry. Mathematicians looking for fields of research may be encouraged by several interesting open questions about chiral polytopes, listed in Appendix C (some of them taken from [220, 250]). In Chapters 2 and 3 the reader will find the basic theory of chiral polytopes, whereas the main directions of research are suggested in Chapters 4, 5 and 6. Throughout, the concepts and results are illustrated by plenty of examples.