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Surgery on Anosov flows using bi-contact geometry

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Abstract. Using bi-contact geometry, we define a new type of Dehn surgery on an Anosov flow with orientable weak invariant foliations. The Anosovity of the new flow is strictly connected to contact geometry and the Reeb dynamics of the defining bi-contact structure. This approach gives new insights into the properties of the flows produced by Goodman surgery and clarifies under which conditions Goodman's construction yields an Anosov flow. Our main application gives a necessary and sufficient condition to generate a contact Anosov flow by Foulon–Hasselblatt Legendrian surgery on a geodesic flow. In particular, we show that this is possible if and only if the surgery is performed along a simple closed geodesic. As a corollary, we have that any positive skewed $\mathbb{R}$-covered Anosov flow obtained by a single surgery on a closed orbit of a geodesic flow is orbit equivalent to a positive contact Anosov flow.

Key words: surgery, contact geometry, Anosov flows, Reeb dynamics
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1. *Introduction*

The use of surgery has tremendously advanced our understanding of Anosov flows on 3-manifolds. In the groundbreaking work of Handel and Thurston [19], the first example of non-algebraic transitive Anosov flow was constructed performing surgery on a geodesic flow. Inspired by Handel and Thurston's work, Goodman [17] defined a Dehn type surgery near a closed orbit and constructed the first examples of an Anosov flow on a hyperbolic 3-manifold.

In the present work, we introduce a new type of Dehn surgery on Anosov flows using a supporting *bi-contact structure*. Mitsumatsu [25] first noticed that the generating vector field of an Anosov flow belongs to the intersection of a pair of transverse contact structures (ξ_-, ξ_+) rotating towards each other along the flow and asymptotic to the stable and unstable foliations. These pairs are called bi-contact structures and the associated flows are called *projectively Anosov flows*. Projectively Anosov flows are more abundant than

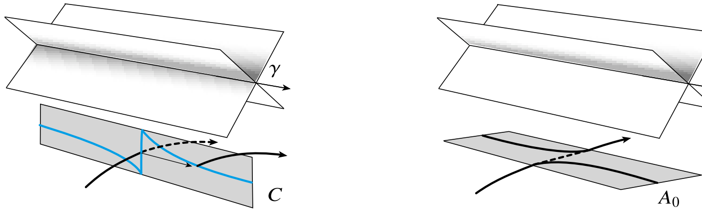

FIGURE 1. On the left, the surgery annulus C in Goodman surgery near a closed orbit γ. The intersecting surfaces are the weak stable and unstable leaves containing γ. The left and the right edge of every strip are identified. The trajectories of the Anosov flow (black curves) are transverse to C. After surgery, the endpoints of the semi-trajectories that hit the annulus are identified with the staring points of the semi-trajectories on the other side of the annulus. This identification is achieved by a shear map that is a Dehn twist (in the picture, this map identifies the vertical segment on one side of C with the curve on the other side of C). On the right, the surgery annulus A_0 used in the bi-contact surgery. The solid black lines are the trajectories of the new Anosov flow, while the dashed lines are those of the original flow. Note that while C is transverse to the Anosov flow, A_0 is tangent to the flow (colour online).

Anosov flows: it can be shown indeed that every closed 3-manifolds admits a projectively Anosov flow, while not every 3-manifold admits an Anosov flow.

A knot K in a bi-contact structure is called *Legendrian-transverse* if its tangents are contained in one of the contact structures and they are transverse to the other one. There are plenty of Legendrian-transverse knots in a bi-contact structure supporting an Anosov flow. Recently, Hozoori [20] has shown that if a flow is Anosov with orientable weak invariant foliations, there are supporting bi-contact structures such that near any periodic orbit γ, there is always a Legendrian-transverse knot isotopic to γ. Additionally, there are many Legendrian-tranverse knots that are not isotopic to any closed orbit. Examples of these knots can be found on the torus fibre of an Anosov suspension flow or considering the circles of the Seifert fibration on the unit tangent bundle of an hyperbolic surface.

Let M be a 3-manifold equipped with a bi-contact structure (ξ_-, ξ_+) defining an Anosov flow and let K be a knot that is Legendrian for ξ_- and is transverse to ξ_+. For a $q \in \mathbb{Z}$, we introduce a *bi-contact surgery*, which is a special type of $(1, q)$-Dehn surgery along an annulus A_0 tangent to the flow and containing K (see Figure 1 for a comparison with Goodman surgery). More precisely, this operation consists in cutting the manifold M along A_0 and gluing back the two sides of the cut adding a $(1, -q)$-Dehn twist along the core of the annulus. (See remark Theorem 5.7 for the minus sign ahead of the coefficient q.) This operation produces a new manifold $\tilde{M}$ and a pair of plane field distributions $(\tilde{\xi}_-, \tilde{\xi}_+)$ on the new manifold. In general, the new plane field distributions does not define a bi-contact structure. However, we have the following theorem.

THEOREM 1.1. (See Theorem 5.11 for the proof) *For $q \in \mathbb{N}$, the bi-contact surgery produces a new pair $(\tilde{\xi}_-, \tilde{\xi}_+)$ of plane field distributions which is a bi-contact structure.*

Since being defined by a bi-contact structure is not a sufficient condition to be Anosov, it is not immediately clear if the bi-contact structures produced in Theorem 1.1 support an Anosov flow. The following result shows that the Anosovity of the new flow is strictly connected to bi-contact geometry.

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THEOREM 1.2. (See Theorem 5.11) *Let ϕ^t be a volume preserving Anosov flow. There is a bi-contact structure (ξ_-, ξ_+) defining ϕ^t such that if a bi-contact surgery along a Legendrian-transverse knot K yields a new bi-contact structure $(\tilde{\xi}_-, \tilde{\xi}_+)$, then $(\tilde{\xi}_-, \tilde{\xi}_+)$ defines an Anosov flow.*

In particular, in the hypothesis of Theorem 1.2, a $(1, q)$-bi-contact surgery yields new Anosov flows for every $q \in \mathbb{N}$.

The construction of Theorem 1.1 naturally fits into the framework set up by Hozoori in [**21**], where the author gives a contact geometric characterization of the bi-contact structures that define Anosov flows. Using this characterization, we give a proof of Theorem 1.2 that relies only on the properties of the Reeb dynamics of an underlying bi-contact structure. (In particular, we do not use the *cone field criterion of hyperbolicity.*)

The construction of Theorem 1.1 gives new insight into the properties of the flows produced by Goodman surgery in virtue of the following equivalence.

THEOREM 1.3. (See Theorem 7.1 for a more precise statement) *Let K be a Legendrian-transverse knot in a bi-contact structure defining an Anosov flow. Suppose also that K is isotopic and close to a closed orbit of the Anosov flow. Any Anosov flow generated by a $(1, q)$-bi-contact surgery along a tangent annulus A_0 containing K is orbit equivalent to an Anosov flow generated by $(1, q)$-Goodman surgery along a transverse annulus C containing K.*

Goodman's operation consists in cutting an Anosov flow along a transverse annulus C in a neighbourhood of a closed orbit and gluing back the two sides of the cut adding a $(1, q)$-Dehn twist (Figure 1). Goodman proved that such an annulus comes with a *preferred direction* in the sense that there is a sign of the twist that always produces hyperbolicity. For example, along an annulus with positive preferred direction, a $(1, q)$-Goodman surgery produces an Anosov flow for every integer $q > 0$.

For a fixed transverse annulus with positive preferred direction, it is not known in general if a $(1, q)$-Goodman surgery with $q < 0$ generates hyperbolicity. Theorems 1.2 and 1.3 allow us to study the Anosovity of the flows generated by surgeries performed in the direction opposite to the preferred one using tools from contact geometry. Let C be an embedded transverse annulus with positive preferred direction centred on a Legendrian-transverse knot K (non-necessarily isotopic to a closed orbit). To a flow-box neighbourhood Λ of C (Λ is constructed by flowing the transverse annulus C using the Anosov flow), we associate a positive real number k depending just on the behaviour of the contact structure ξ_+ on $\partial\Lambda$. This number k can be interpreted as the *slope* of the *characteristic foliation* induced by ξ_+ on $\partial\Lambda$. For a volume preserving Anosov flow, we have the following theorem.

THEOREM 1.4. (Theorem 6.3) *Let $k > 0$ be the slope of the characteristic foliation induced by ξ_+ on $\partial\Lambda$. For every $q > -k$, a $(1, q)$-Goodman surgery along C produces an Anosov flow.*

Let ϕ^t be a volume preserving Anosov flow with weak orientable invariant foliations. Suppose that there is an embedded *quasi-transverse* annulus $C_{-\infty}$ bounded on one side by

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a Legendrian-transverse knot K and on the other side by a closed orbit γ of the flow. We have the following corollary of Theorem 1.4.

COROLLARY 1.5. (See Corollary 7.3.) *There is a sequence of nested annuli* $C_{-1} \subset C_{-2} \subset \cdots \subset C_p \subset \cdots \subset C_{-\infty}$ *with* $p \in \mathbb{Z}^-$. *Each* C_p *is transverse to the Anosov flow and bounded on one side by the Legendrian-transverse knot K. An annulus* C_p *of the sequence has the property that a* $(1, q)$*-Goodman surgery along* C_p *yields an Anosov flow for* $q \geq p$. *The annulus* $C_{-\infty}$ *is quasi-transverse, and it is bounded by K on one side and by the closed orbit* γ *on the other side.*

In particular, given an annulus C_p, $p \in \mathbb{Z}^-$ of the sequence, if we want to construct an annulus C_{p-1} such that a $(1, p-1)$-Goodman surgery produces an Anosov flow, we need to extend C_p towards the closed orbit γ (see Figure 12).

We use Theorem 1.4 and its corollaries to study surgeries in the context of *contact Anosov flows* that are Anosov flows preserving a contact form. The condition of being contact Anosov has a number of remarkable geometric consequences (see [**13**, **18**, **22**]) including exponential decay of correlations. In the context of 3-manifolds, Barbot has shown that every contact Anosov flow is *skewed* $\mathbb{R}$-covered. (An Anosov flow is $\mathbb{R}$-*covered* if the stable (or the unstable) weak foliation lifts to a foliation in the universal cover which has leaf space homeomorphic to $\mathbb{R}$.) The relation between these two classes of flows is expected to be stronger.

Conjecture 1.6. (Barbot–Barthelmé) If ϕ^t is a positively skewed $\mathbb{R}$-covered Anosov flow, then ϕ^t is orbit equivalent to a contact Anosov flows. (Marty [**24**] posted a proof of this statement after the completion of the present work.)

For almost half of a century since the seminal work of Anosov, the only known examples of contact Anosov flows were the geodesic flows of Riemannian or Finsler manifolds. Foulon and Hasselblatt [**14**] showed that it is not only possible to construct new examples of contact Anosov flows performing Goodman surgery on a geodesic flow, but also that it could be done producing hyperbolic manifolds.

The construction of Foulon and Hasselblatt uses a special class of arbitrarily thin transverse annuli with positive preferred direction containing a knot L that is Legendrian for the contact structure preserved by the geodesic flow. Such annuli are located far from a closed orbit and the set of negative surgery coefficients that yield a contact Anosov flow is not known. If the knot L is associated to a simple closed geodesic there is an embedded quasi-transverse annulus containing L and Corollary 1.5 applies. As a consequence, we have the following theorem.

THEOREM 1.7. (See Theorem 8.2) *Choose a closed orbit* γ *in the geodesic flow on the unit tangent bundle of a hyperbolic surface S. If* $q < 0$, *a* $(1, q)$*-Goodman surgery generates a flow that is orbit equivalent to a contact Anosov flow if and only if the orbit* γ *is a lift of a simple closed geodesic on S. (Our proof of Anosovity does not rely on Barbot's version of the cone field criterion* [**5**] *and can be used to give an alternate proof of* [**14**, Theorem 4.3].*)*

Theorem 1.7 represents the counterpart in the category of contact Anosov flows of results recently proven by Bonatti and Iakovoglou [11, Theorem 1] and Marty [23, Theorem J] in the category of $\mathbb{R}$-covered Anosov flows. Using Theorem 1.7 and work of Asaoka, Bonatti and Marty [4], we prove a version of Conjecture 1.6 for a single surgery along a closed orbit of the geodesic flow. Let ϕ^t be the geodesic flow with the orientation that makes it a positively skewed $\mathbb{R}$-covered Anosov flow.

THEOREM 1.8. (See Theorem 9.5) *Any positive skewed $\mathbb{R}$-covered Anosov flow obtained by a single surgery along a closed orbit of a geodesic flow is orbit equivalent to a positive contact Anosov flow.*

Note that Theorem 1.7 has a natural interpretation in the context of contact and symplectic geometry. Foulon and Hasselblatt construction is an example of a *contact* surgery in the sense that given a Legendrian knot L in a contact 3-manifold (M, η), where η is the contact form preserved by an Anosov flow ϕ^t (in particular, η is not part of a supporting bi-contact structure for ϕ^t), it produces a new contact 3-manifold performing a Dehn type surgery along L. Historically, a $(1, -1)$-contact surgery is called *Legendrian* surgery. Legendrian surgeries preserve tightness [31] and have a natural interpretation from a symplectic point of view. Foulon and Hasselblatt show [14] that *positive* Goodman surgeries (therefore, *not* Legendrian) always generate contact Anosov flows. If the geodesic is *filling*, the new manifold is hyperbolic. Since a filling geodesic is non-simple, a consequence of Theorem 1.7 is the impossibility of generating a positive contact Anosov flow on a hyperbolic manifold performing a single Legendrian surgery on a geodesic flow. In contrast, it is possible to generate a positive contact Anosov flow performing Legendrian surgery along any simple closed geodesic. This operation yields a graph manifold and the associated flows are orbit equivalent to those described by Handel and Thurston in [19] for a shear with $a_j < 0$ and short enough geodesic γ_j.

2. *Anosov flows, projectively Anosov flows and bi-contact structures*

Anosov flows are an important class of dynamical system characterized by structural stability under $\mathcal{C}^1$-small perturbations (see [1, 2, 29]). Beyond their interesting dynamical properties, there is evidence of an intricate and beautiful relationship with the topology of the manifold they inhabit (see the survey [8] for classic and more recent developments [6, 7, 9, 10] by Fenley, Barbot and Barthelmé). Geometrically, they are distinguished by the contracting and expanding behaviour of two invariant directions

Definition 2.1. Let M be a closed manifold and $\phi^t : M \to M$ a $\mathcal{C}^1$ flow on M. The flow ϕ^t is called *Anosov* if there is a splitting of the tangent bundle $TM = E^{uu} \oplus E^{ss} \oplus \langle X \rangle$ preserved by $D\phi^t$, and positive constants A and B such that

$$\|D\phi^t(v^u)\| \geq Ae^{Bt}\|v^u\| \quad \text{for any } v^u \in E^{uu},\ t \geq 0,$$

$$\|D\phi^t(v^s)\| \leq Ae^{-Bt}\|v^s\| \quad \text{for any } v^s \in E^{ss}, t \geq 0.$$

Here, $\|\cdot\|$ is induce by a Riemmanian metric on TM. We call E^{uu} and E^{ss} respectively the *strong unstable bundle* and the *strong stable bundle*.

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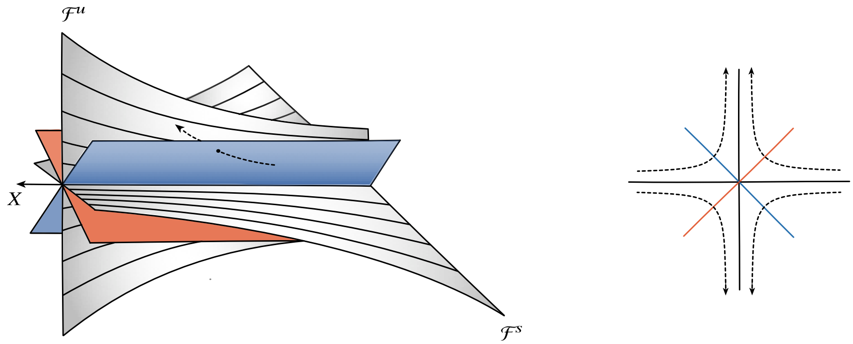

FIGURE 2. On the left, the tangent bundle in a neighbourhood of a flow line of an Anosov flow. The oblique plane fields define the bi-contact structure. The vertical plane is the leaf of the unstable weak foliation $\mathcal{F}^u$. On the right, the normal bundle $TM/\langle X\rangle$ (colour online).

Classic examples of Anosov flows are the geodesic flow on the unit tangent bundle of a hyperbolic surface and the suspension flows of hyperbolic linear automorphisms of the torus.

The definition above has remarkable geometric consequences. Anosov showed that the distributions E^{ss} and E^{uu} are uniquely integrable and the associated foliations are denoted by $\mathcal{F}^{ss}$ and $\mathcal{F}^{uu}$. Moreover, the *weak stable bundle* $E^s = E^{ss} \oplus \langle X\rangle$ and the *weak unstable bundle* $E^u = E^{uu} \oplus \langle X\rangle$ are also uniquely integrable and the codimension one associated foliations are dented with $\mathcal{F}^s$ and $\mathcal{F}^u$ (see Figure 2).

Mitsumatsu [25] first noticed that an Anosov flow with orientable weak invariant foliations is tangent to the intersection of two transverse contact structures (see also Eliashberg and Thurston [12]). We will call such pairs *bi-contact structures*. However, the converse statement is not true and there are bi-contact structures that do not define Anosov flows.

2.1. *Contact structures and Reeb flows.* A co-oriented contact structure is a plane field distribution that is maximally non-integrable in the sense that it can be described as the kernel of a $\mathcal{C}^1$ 1-form satisfying the relation $\alpha \wedge d\alpha \neq 0$. By Frobenious theorem, contact structures can be thought of as polar opposite of foliations: there is not a subsurface S such that $TS = \ker\alpha$, even in a neighbourhood of a point. Contact structures come in two types, positive and negative. A positive contact structure is a plane field distribution ξ_+ described by a $\mathcal{C}^1$ 1-form satisfying the relation $\alpha_+ \wedge d\alpha_+ > 0$. A negative contact structure ξ_- is described instead by a $\mathcal{C}^1$ 1-form such that $\alpha_- \wedge d\alpha_- < 0$.

An important property of contact structures is that they do not have local invariants. Indeed, by Darboux theorem, positive (negative) contact structures are all locally contactomorphic to the positive (negative) *standard contact structure in* $\mathbb{R}^3$ described by $\xi^+_{std} = \ker dz - y\,dx$ and $\xi^-_{std} = \ker dz + y\,dx$. Therefore, we can locally picture a positive contact (negative) structure as a plane field whose plane rotates counterclockwise (clockwise) along the x-axis. Associated to the defining contact forms, there is an important class of flows called *Reeb flows*. Given a contact form α, we define the Reeb

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vector field R_α as the unique vector field satisfying the equations

$$\alpha(R_\alpha) = 1, \quad d\alpha(R_\alpha, \cdot) = 0.$$

These relations imply that R_α is transverse to $\ker \alpha$ and $\mathcal{L}_{R_\alpha}\alpha = 0$ (R_α preserves α).

We conclude this section introducing a very important class of vector fields that is well studied and has remarkable geometric properties [14, 22].

Definition 2.2. A Reeb flow that is also an Anosov flow is called a *contact Anosov* flow.

In other words, a contact Anosov flow is an Anosov flow preserving a contact form.

2.2. *Bi-contact structures and projectively Anosov flows.* We now give an example of a pair of opposite and transverse contact structures that does not define an Anosov flow.

Example 2.3. We construct a bi-contact structure on T^3 using a recipe introduced by Mitsumatsu in [25, 26]. Consider the contact forms defined on $T^2 \times I$,

$$\alpha_n = \cos(2n\pi z)\, dx - \sin(2n\pi z)\, dy,$$

$$\alpha_{-m} = \cos(2m\pi z)\, dx + \sin(2m\pi z)\, dy.$$

They are not transverse to each other on the tori defined by $\{z = 0\}$, $\{z = \frac{1}{4}\}$, $\{z = \frac{1}{2}\}$, $\{z = \frac{3}{4}\}$. If we introduce a perturbation $\epsilon(z)\, dz$ such that $\epsilon(z)$ is a function that does not vanish on the tori, the contact forms $\alpha_+ = \alpha_n + \epsilon(z)\, dz$ and $\alpha_- = \alpha_{-m}$ define transverse contact structures of opposite orientations.

As Plante and Thurston showed in [30], the fundamental group of a manifold that admits an Anosov flow grows exponentially. Since the ambient manifold is T^3, any flow defined by the pair of contact structures $(\ker \alpha_-, \ker \alpha_+)$ is not Anosov.

Definition 2.4. (Mitsumatsu [25]) Let M be a closed manifold and $\phi^t : M \to M$ a C^1 flow on M. The flow ϕ^t is called *projectively Anosov* if there is a splitting of the projectified tangent bundle $TM/\langle X\rangle = \mathcal{E}^u \oplus \mathcal{E}^s$ preserved by $D\phi^t$ and positive constants A and B such that

$$\frac{\|D\phi^t(v^u)\|}{\|D\phi^t(v^s)\|} \geq Ae^{Bt}\frac{\|v^u\|}{\|v^s\|} \quad \text{for any } v^u \in \mathcal{E}^u \text{ and } v^s \in \mathcal{E}^s,\ t \geq 0.$$

Here, $\|\cdot\|$ is induce by a Riemmanian metric on TM. We call $\mathcal{E}^u$ and $\mathcal{E}^s$ respectively the *unstable bundle* and the *stable bundle*.

The invariant bundles $\mathcal{E}^u$ and $\mathcal{E}^s$ induce invariant plane fields E^u and E^s on M. These plane fields are continuous and integrable, but unlike the Anosov case, the integral manifolds may not be unique (see [12]). However, when they are smooth, they also are uniquely integrable. We call these flows *regular projectively Anosov* (see [3, 27, 28] for a complete classification).

Definition 2.5. We say that a flow ϕ^t is *defined* (or *supported*) by a bi-contact structure (ξ_-, ξ_+) if the generating vector field X belongs to the intersection $\xi_- \cap \xi_+$.

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The following result gives a geometric characterization of the class of projectively Anosov flows.

THEOREM 2.6. (Mitsumatsu [25]) *Let X be a* $\mathcal{C}^1$ *vector field on M. Then, X is projectively Anosov if and only if it is defined by a bi-contact structure.*

Remark 2.7. We refer to [20, 21] for a more complete overview of the connection between bi-contact geometry, symplectic geometry and projectively Anosov flows, and for a precise discussion on the regularity of the plane fields, bundles and foliations involved.

2.3. *Reeb dynamics of a bi-contact structure defining an Anosov flows.* We now present a characterization of Anosov flows due to Hoozori [21] that uses the Reeb dynamics of the underling bi-contact structure.

Definition 2.8. (Hozoori [21]) Consider a bi-contact structure (ξ_-, ξ_+) on M and a supported vector field *X*. We say that *X* is *dynamically positive (negative)* if at every point $p \in M$, its image into $TM/\langle X\rangle$ lies in the interior of the region defined by the stable and unstable bundle. More precisely, we say that X is dynamically positive if its image into $TM/\langle X\rangle$ lies in the interior of the region defined by the stable and unstable bundle in clock-wise order and considering the flow pointing into the page (see Figure 2).

THEOREM 2.9. (Hozoori [21]) *Let* ϕ^t *be a projectively Anosov flow on M. The following are equivalent.*

(1) *The flow* ϕ^t *is Anosov.*
(2) *There is a pair* (α_-, α_+) *of positive and negative contact forms defining* ϕ^t *such that the Reeb vector field of* α_+ *is dynamically negative.*
(3) *There is a pair* (α_-, α_+) *of positive and negative contact forms defining* ϕ^t *such that the Reeb vector field of* α_- *is dynamically positive.*

In particular, we have the following sufficient condition of Anosovity.

COROLLARY 2.10. *Let* $(\ker \alpha_- = \xi_-, \xi_+)$ *be a bi-contact structure such that* $R_{\alpha_-} \in \xi_-$. *The flow* ϕ^t *defined by* (ξ_-, ξ_+) *is Anosov.*

It is important to remark that under these circumstances, the knowledge of the position of stable and unstable invariant direction in $TM/\langle X\rangle$ is *not* needed since under this condition, R_{α_-} is automatically dynamically positive.

A *volume preserving* Anosov flow is an Anosov flow preserving a continuous volume form. It is known that if the flow is $\mathcal{C}^k$, such a volume form is automatically $\mathcal{C}^k$ (see [20] for more references).

Hozoori shows [20] the following characterization of bi-contact structures defining volume preserving Anosov flows.

THEOREM 2.11. (Hozoori [20]) *Let* ϕ^t *be a projectively Anosov flow on M. Here,* ϕ^t *is a volume preserving Anosov flow if and only if there are pairs of positive and negative contact forms defining* ϕ^t *such that* $R_{\alpha_-} \in \ker \alpha_+$ *and* $R_{\alpha_+} \in \ker \alpha_-$.

At the time of writing this work, it is not known if a general Anosov flow is always supported by a bi-contact structure (ξ_-, ξ_+) such that $R_{\alpha_-} \in \ker \alpha_+$.

2.4. *Flexibility of bi-contact structures and structural stability.* The interplay between structural stability of an Anosov flow and the flexibility of its underlying bi-contact structure can be used to construct $\mathcal{C}^1$-paths of Anosov flows.

LEMMA 2.12. *Suppose that M is equipped with a pair of contact structures* $(\ker \alpha_- = \xi_-, \xi_+)$ *defining an Anosov flow* ϕ^t *such that the Reeb vector field* R_{α_-} *of* α_- *belongs to* ξ_+. *An isotopy of* ξ_+ *along the flow-lines of* R_{α_-} *defines a path of orbit equivalent Anosov flows.*

Proof. Assume that $R_{\alpha_-} \in \xi_+$. An isotopy of $\xi_+ = \ker \alpha_+$ as above defines a $\mathcal{C}^1$-path of bi-contact structures $(\xi_-, \ker(\alpha_+)_t)_{t\in[0,1]}$ such that $R_{\alpha_-} \in (\xi_+)_t = \ker(\alpha_+)_t$ in N. The statement follows from Theorem 2.9 and the $\mathcal{C}^1$-structural stability of Anosov flows. □

In particular, all the flows along the path are orbit equivalent with an orbit equivalence isotopic to the identity. The above result will be used in the proof of Theorems 5.6 and 6.4.

3. *The geodesic flow on the unit tangent bundle of a hyperbolic surface*

In this section, we introduce the prototype of an Anosov flow: the geodesic flow on the unit tangent bundle of a hyperbolic surface (see [**1**]). This object is particularly important in our context because it is naturally supported by a bi-contact structure: it can be considered the motivating example for constructing a bi-contact theory of Anosov flow. Moreover, for a long time, it was the only known example of a contact Anosov flow.

3.1. *Geometric structures on UTS.* The geodesic flow on the unit tangent bundle of a hyperbolic surface S carries some remarkable geometric structure that can be interpreted in the context of bi-contact geometry. We follow the discussion of [**15**]. Using the identification of $\mathrm{UT}\mathbb{H}^2$ with $\mathrm{PSL}(2, \mathbb{R})$, it is possible to show that there is a canonical framing consisting of the vector field X that generates the flow, the periodic vector field V pointing in the fibre direction and the vector field defined by $H := [V, X]$. This frame satisfies the following relations:

$$[V, X] = H, \quad [H, X] = V, \quad [H, V] = X. \tag{3.1}$$

A consequence of the structure equations is that the strong stable and unstable bundles $E^{\pm}$ are spanned by the vectors $e^{\pm} = V \pm H$. It is not difficult to show that equation (3.1) implies the existence of three 1-forms α_-, α_+ and β_+ defining mutually transverse contact structures (see Figure 3 and [**15**] for more details). They are defined by

$$\begin{aligned} &\beta_+(V) = 0 = \beta_+(H), && \alpha_+(X) = 0 = \alpha_+(V), && \alpha_-(X) = 0 = \alpha_-(H),\\ &\beta_+(X) = 1, && \alpha_+(H) = 1, && \alpha_-(V) = 1,\\ &d\beta_+(X, \cdot) = 0, && d\alpha_+(H, \cdot) = 0, && d\alpha_-(V, \cdot) = 0. \end{aligned}$$

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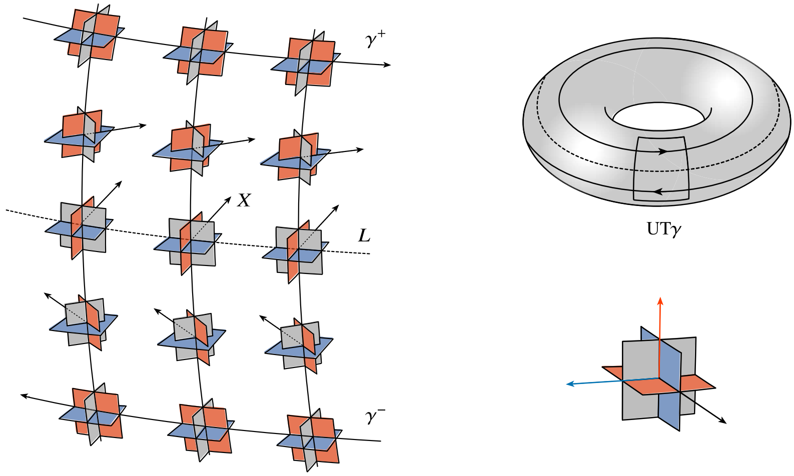

FIGURE 3. On the top right, a Birkhoff torus associated to simple closed geodesic γ. On the right side, the contact structures described in §3.2. The plane field transverse to the arrows is the contact structure $\eta_+ = \ker \beta_+$ preserved by the geodesic flow. The closed orbits γ^+ and γ^- are Legendrian knots for the bi-contact structure (ξ_-, ξ_+) that generates the geodesic flow. The special knot L is represented by the dashed line (colour online).

The relations above show that the vector fields X, H and V are respectively the Reeb vector fields of the contact forms β_+, α_+ and α_-. In particular, the geodesic flow is the Reeb flow of β_+. The pair of contact forms (α_-, α_+) define a bi-contact structure that supports X.

3.2. *Structures on a Birkhoff torus and Legendrian knots.* Let $\mathrm{S} = \mathbb{H}^2/\Gamma$ be a hyperbolic surface and let γ be a closed geodesic. The lift of γ to the unit tangent bundle of γ is an immersed *Birkhoff* torus UTγ with two closed orbits $\{\gamma_1, \gamma_2\}$, in the sense that UTγ is an immersed torus transverse to the geodesic flow in UT$\gamma \setminus \{\gamma_1, \gamma_2\}$ and tangent to the geodesic flow along $\{\gamma_1, \gamma_2\}$. If the geodesic γ is also simple, UTγ is an embedded torus.

We now describe a special knot L on the Birkhoff torus that plays a very important role in numerous applications. This knot is defined by the angle $\theta = \pi/2$ on each fibre along γ and has the following remarkable properties (see Figures 2 and 4).

(1) L is a Legendrian knot for the contact structure η_+ preserved by the flow.
(2) L is transverse to the weak stable and unstable foliations.
(3) L is a Legendrian-transverse knot with respect to the bi-contact structure $(\ker \alpha_-, \ker \alpha_+)$.

LEMMA 3.1. *Let ϕ^t be the geodesic flow on the unit tangent bundle UTS of an oriented hyperbolic surface S. In a neighbourhood $N = A_0 \times (-\epsilon, \epsilon)$ of a Birkhoff annulus associated to a simple closed geodesic, there is a coordinate system (w, s, v) such that the natural contact forms $(\alpha_-, \alpha_+, \beta_+)$ have the following expressions:*

$$\alpha_- = dw + v\, ds;$$

$$\alpha_+ = e^{(1/2)v^2}(\cos w\, ds - \sin w\, dv);$$

$$\beta_+ = e^{(1/2)v^2}(-\sin w\, ds - \cos w\, dv).$$

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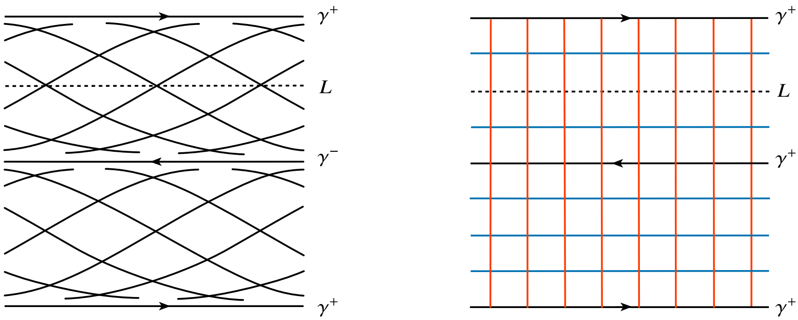

FIGURE 4. On the left, the foliation on an embedded Birkhoff torus induced by the stable and unstable weak foliations. On the right, the characteristic foliations induced by the bi-contact structure (ker α_+, ker α_-) (colour online).

Here, A_0 is constructed as follows. Consider the Legendrian-transverse knot L constructed in §3.2 and let A_0 be an annulus tangent to the flow ϕ^t of X constructed by flowing L (using the flow ϕ^t). Let (s, v) be a coordinate system on A_0 such that the s-curves are parallel to ∂A_0. We extend (s, v) to a coordinate system (s, v, w) on N using the flow of R_{α_-}. (This can always be done since, by construction, R_{α_-} is transverse to A_0.)

Proof. Let $(\alpha_-, \alpha_+, \beta_+)$ be the contact forms defined above. It is enough to show that if $V = R_{\alpha_-}$, $H = R_{\alpha_+}$ and $X = R_{\beta_+}$, we have

$$\beta_+(V) = \beta_+(H) = \alpha_-(X) = \alpha_-(H) = \alpha_+(X) = \alpha_+(V) = 0.$$

This shows that the contact forms $(\alpha_-, \alpha_+, \beta_+)$ define a local model of the geodesic flow near a quasi-transverse torus, as described in §3.2. An elementary calculation gives the following expressions for the Reeb vector fields of $(\alpha_-, \alpha_+, \beta_+)$ in N:

$$V = R_{\alpha_-} = \frac{\partial}{\partial w};$$
$$H = R_{\alpha_+} = e^{-(1/2)v^2}\bigg(\cos w\,\frac{\partial}{\partial s} - \sin w\,\frac{\partial}{\partial v} - v\cos w\,\frac{\partial}{\partial w}\bigg);$$
$$X = R_{\beta_+} = e^{-(12)v^2}\bigg(-\sin w\,\frac{\partial}{\partial s} - \cos w\,\frac{\partial}{\partial v} + v\sin w\,\frac{\partial}{\partial w}\bigg).$$

We check that the vector field $V = R_{\alpha_-} = \partial/\partial w$ satisfies the requirements of the (unique) Reeb vector field of α_-,

$$\alpha_-(V) = (dw + v\,ds)\bigg(\frac{\partial}{\partial w}\bigg) = 1,$$

and since $d\alpha_- = dv \wedge ds$, we have $d\alpha_-(V, \cdot) = 0$. Moreover, since α_+ and β_+ do not have a dw-term, $\alpha_+(V) = 0$ and $\beta_+(V) = 0$. We leave to the reader to check the other identities. □

4. *Foulon–Hasselblatt contact surgery*

In this section, we introduce a construction of Foulon and Hasselblatt that will play a fundamental role in the definition of the bi-contact surgery. Remarkably, the bi-contact surgery

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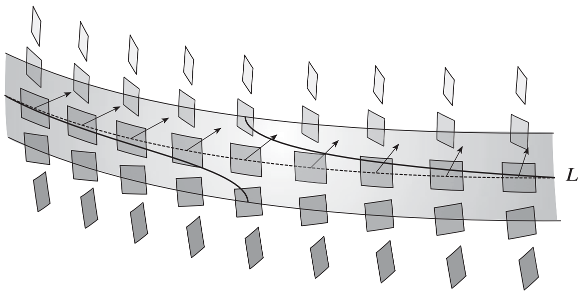

FIGURE 5. The surgery annulus in Foulon–Hasselblatt construction.

will allow us to extend the classic application of Foulon and Hasselblatt construction to a broader class of (contact) Anosov flows.

Let α be a $\mathcal{C}^k$ contact form with Reeb vector field R_α, and consider a Legendrian knot L for $\xi = \ker \alpha$. Foulon and Hasselblatt describe in [14] a family of contact surgeries of the *Dehn* type that generates a new $\mathcal{C}^k$ contact form $\tilde{\alpha}$ in the new manifold $\tilde{M}$ and a $\mathcal{C}^{k-1}$ vector field $R_{\tilde{\alpha}}$ such that:

(1) $\tilde{\xi} = \ker \tilde{\alpha}$ is isotopic to a contact structure obtained from $\xi = \ker \alpha$ by classic contact surgery and such that $\tilde{\alpha} = \alpha$ outside a neighbourhood of an annulus C transverse to the flow of R_α;

(2) $R_{\tilde{\alpha}}$ is the Reeb vector field of $\tilde{\alpha}$. Moreover, $R_{\tilde{\alpha}}$ is collinear to R_α in $M \setminus C$.

In other words, the Foulon and Hasselblatt construction is a contact surgery that allows us to have some control on the direction of the resulting Reeb vector field.

4.1. *Definition of the contact surgery.* In the following, we recall the main features of the Foulon–Hasselblatt construction (see [14, 15] for more details). Given a 3-manifold with a contact structure $\xi = \ker \alpha$ and a Legendrian knot L, there is a coordinate system

$$(t, s, w) \in N = (-\delta, \delta) \times S^1 \times (-\epsilon, \epsilon),$$

where the parameters (s, w) are defined on the *surgery annulus* $C = \{0\} \times S^1 \times (-\epsilon, \epsilon)$ (see Figure 5). More precisely, $s \in S^1$ is the parameter of L and w belongs to some interval $(-\epsilon, \epsilon)$. The transverse parameter t is such that the Reeb vector field of γ satisfies $R_\alpha = \partial/\partial t$; therefore, N is a flow-box chart for R_α. In this coordinate system, a contact form defining a contact structure takes the particularly simple expression $\gamma = dt + w\, ds$.

The surgery can be thought of as first cutting the manifold M along the annulus C and then gluing back the two sides of the cut in a different way. In particular, we glue the point that, on one side of the cut, is described by coordinates (s, w) to that described by coordinates $(s + f(w), w)$ on the other side. Here, $f : [-\epsilon, \epsilon] \to S^1$ is a non-decreasing function such that $f(-\epsilon) = 0$ and $f(\epsilon) = 2\pi q$, and satisfying a number of additional requirements. The transition map $F : C \to C,\ (s, w) \to (s + f(w), w)$ is often called *shear*. Since

$$F_* \frac{\partial}{\partial t} = \frac{\partial}{\partial t},$$

the shear defines a smooth vector field from R_α. Note that if we restrict our attention to the vector field R_α, Foulon–Hasselblatt construction can be interpreted as a Goodman surgery on R_α.

Let us denote with $(\alpha)^-$ the contact form defined on one side of the flow-box chart and with $(\alpha)^+$ the contact form on the other side, we easily see that

$$F^*(\alpha)^+ = (\alpha)^- + wf'(w)\,dw;$$

therefore, the shear defines a smooth vector field from R_α, but does not define a smooth contact structure on the new manifold. This issue is addressed by Foulon and Hasselblatt introducing a deformation that yields to a 1-form $(\tilde{\alpha})^+ = (\alpha)^+ - dh$ of class $\mathcal{C}^1$ and dh is the differential of the following function:

$$h(t, w) = \lambda(t) \int_{-\epsilon}^{w} xf'(x)\,dx.$$

Here, $\lambda : \mathbb{R} \to [0, 1]$ is a $\mathcal{C}^1$ bump function with support in some interval $(0, \delta)$ with $\delta > 0$ that takes value 1 in a neighbourhood of 0 and takes value 0 in a neighbourhood of δ. With these choices, we have $F^*(\alpha - dh)^+ = (\alpha)^-$. Therefore, $\tilde{\alpha}$ is a well-defined 1-form of class $\mathcal{C}^1$ in $\tilde{M}$. Note that the deformation depends on the shear and it is not immediately clear if the plane field distribution $\ker\tilde{\alpha}$ still defines a contact structure. The authors show that if we choose $0 < \epsilon < \delta/2\pi q$, this is in fact the case. Moreover, the Reeb vector field of $\tilde{\alpha}$ takes the form

$$R_{\tilde{\alpha}} = \frac{R_\alpha}{1 - dh(R_\alpha)}.$$

4.2. *Application to contact Anosov flows.* The Foulon–Hasselblatt construction is purely contact geometric, it does not require the Reeb vector field of α to be Anosov and it can be performed in a neighbourhood of any Legendrian knot.

If applied to a geodesic flow on the unit tangent bundle of an hyperbolic surface, we have the following theorem.

THEOREM 4.1. (Foulon–Hasselblatt–Vaugon [15]) *Select a closed geodesic γ on a hyperbolic surface S and consider the geodesic flow on UTS. Consider the knot L defined by the angle $\theta = \pi/2$ on each fibre along γ. Let 2ϵ be the width of the surgery annulus.*

(1) *The $(1, q)$-Dehn surgery along L defined in §4.1 does produce an Anosov flow if $q > 0$ regardless of the width of the surgery annulus.*
(2) *It does not produce an Anosov if $-q/\epsilon$ is large enough, that is, if either $q < 0$ is fixed and ϵ is small enough or if $\epsilon > 0$ is fixed and $q < 0$ with $|q|$ big enough.*

The following result shows that the geodesic γ can be chosen in such a way that the result of the surgery is hyperbolic.

THEOREM 4.2. (Calegari/Folklore [14]) *Let γ a closed, filling geodesic in an hyperbolic surface S. Then, the complement of its image in the unit tangent bundle of S is hyperbolic.*

Note that the Birkhoff torus corresponding to a closed, filling geodesic is self-intersecting. However, for $q > 0$, we can choose $\epsilon > 0$ small enough to ensure that the surgery annulus C_ϵ is embedded in M.

5. *Bi-contact surgery on Anosov flows*

We introduce a new type of Dehn surgery on a projectively Anosov flow with weak orientable invariant foliations. Our construction is defined in a neighbourhood N of a knot K that is simultaneously Legendrian for ξ_- and transverse for ξ_+. Furthermore, we require that in N, the Reeb vector field of a contact form α_- such that $\ker \alpha_- = \xi_-$ is contained in ξ_+. As Hozoori shows in [20], if the flow is Anosov and volume preserving, there is a supporting bi-contact structure $(\ker \alpha_- = \xi_-, \ker \alpha_+ = \xi_+)$ that satisfies the above property everywhere. However, if the flow is just Anosov, Hozoori [20] shows that the condition $R_{\alpha_-} \in \ker \alpha_+ = \xi_+$ can be always achieved in a neighbourhood of a closed orbit.

THEOREM 5.1. (Hozoori [20]) *Let ϕ^t be a $\mathcal{C}^1$-Hölder Anosov flow with orientable weak invariant foliations and let γ be a periodic orbit. There is a pair of contact forms (α_-, α_+), such that $(\ker \alpha_-, \ker \alpha_+)$ is a supporting bi-contact structure for ϕ^t and, in a neighbourhood N of γ, the Reeb vector field R_{α_-} of α_- satisfies the condition $\alpha_+(R_{\alpha_-}) = 0$.*

We use the following corollary to construct a Legendrian-transverse knot from a closed orbit γ.

COROLLARY 5.2. *Let γ be a closed orbit of a flow defined by a bi-contact structure $(\ker \alpha_- = \xi_-, \ker \alpha_+ = \xi_+)$ such that in a neighbourhood N of γ, the Reeb vector field R_{α_-} belongs to $\ker \alpha_+ = \xi_+$. Let ψ^w be the flow of R_{α_-}. The knot $K = \psi^w(\gamma)$ is a Legendrian-transverse knot for $0 < w$ sufficiently small.*

Definition 5.3. Consider a closed orbit γ and a pair of contact forms (α_-, α_+) such that in a neighbourhood N of γ, the Reeb vector field R_{α_-} belongs to $\ker \alpha_+ = \xi_+$. We call a *Legendrian-transverse push-off of γ* a knot K that is simultaneously Legendrian for $\xi_- = \ker \alpha_-$ and transverse for $\xi_+ = \ker \alpha_+$, and it is obtained by translating γ using the flow of R_{α_-} (see Figure 6).

5.0.1. *Sketch of the construction of a bi-contact surgery.* Let K be a Legendrian-transverse knot in a bi-contact structure (ξ_-, ξ_+) defining a vector field X with flow ϕ^t. Suppose that in a neighbourhood N of K, we have $R_{\alpha_-} \in \xi_+$ for some contact form α_- such that $\ker \alpha_- = \xi_-$. Since K is Legendrian for ξ_-, we can apply Foulon–Hasselblatt contact surgery to the pair (α_-, R_{α_-}) using a shear F on a surgery annulus A_0 constructed by translating the knot K using the flow ϕ^t. Note that with the above assumptions, A_0 is transverse to R_{α_-} and tangent to X. After surgery, we have a new contact form $\tilde{\alpha}_-$ with Reeb vector field $R_{\tilde{\alpha}_-}$ that is collinear to R_{α_-} in $M \setminus A_0$. Consider now the contact structure $\ker \alpha_+ = \xi_+$ transverse to the knot K. If the surgery coefficient q is positive, we will show that there is a new contact form $\tilde{\alpha}_+$ such that the Reeb vector field $R_{\tilde{\alpha}_-}$ belongs

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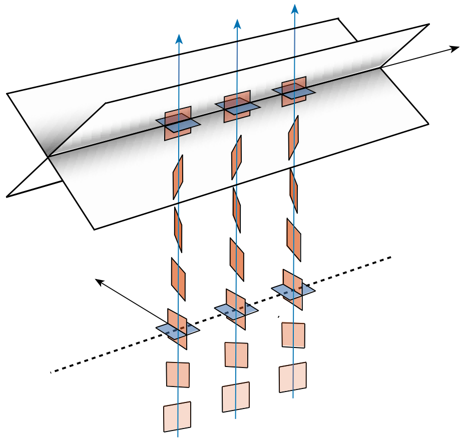

FIGURE 6. Legendrian-transverse push-off of a closed orbit. The vertical arrows represent R_{α_-}. In black, the Anosov flow (colour online).

to $\ker \tilde{\alpha}_+ = \tilde{\xi}_+$. Suppose now that the initial defining bi-contact structure is such that $R_{\alpha_-} \in \ker \alpha_+$ everywhere (therefore, the supported flow is Anosov by Corollary 2.10. The new bi-contact structure $(\ker \tilde{\alpha}_-, \ker \tilde{\alpha}_+)$ has the same property; therefore, the supported vector field $\tilde{X}$ is Anosov.

Remark 5.4. The condition $R_{\alpha_-} \in \ker \alpha_+$ *everywhere* can be always achieved in a volume preserving Anosov flow (see Theorem 2.11). At the time of writing this work, it is not known if the condition can be achieved on any Anosov flow with orientable weak foliations.

5.1. *Definition of the bi-contact surgery.* We first construct a special coordinate system in a neighbourhood N of the Legendrian-transverse knot K, where two supporting contact forms (α_-, α_+) take a very simple expression.

LEMMA 5.5. *Suppose M is a 3-manifold endowed with a projectively Anosov flow ϕ^t defined by a bi-contact structure (ξ_-, ξ_+) and consider a knot K that is simultaneously Legendrian for ξ_- and transverse for ξ_+. Suppose also that there is a contact form α_- defining ξ_- such that the Reeb vector field of α_- is contained in ξ_+ in a neighbourhood $N = A_0 \times (-\epsilon, \epsilon)$ of K, where A_0 is an annulus tangent to the Anosov flow and $\epsilon > 0$. (A_0 is also tangent to the contact structure along its core K.) We can choose N equipped with coordinates (s, v, w) and a contact form α_+ supporting ξ_+ such that in N, we have*

$$\alpha_- = dw + v\, ds,$$

$$\alpha_+ = ds - b(s, v, w)\, dv \quad \text{with} \quad \alpha_+ = ds \quad \text{on} \quad A_0.$$

Here, s and v are coordinates on A_0 such that $s \in S^1$ is the parameter describing K and $v \in (-\delta, \delta)$ with $\delta \in \mathbb{R}^+$, while $w \in (-\epsilon, \epsilon)$ is the transverse parameter to A_0 defined by the flow of R_-.

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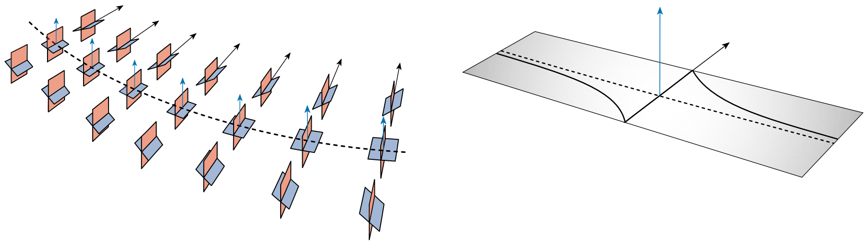

FIGURE 7. The surgery annulus A_0 spanned by the flow. The dashed line is the Legendrian-transverse knot K. The vertical arrows represent R_{α_-}. In black, the Anosov flow (colour online).

Proof. The construction of the surgery annulus A_0 is straightforward. We start with the Legendrian-transverse knot $K : S^1 \to M$ and we define the tangent annulus

$$A_0 = \bigcup_{t\in[-\tau,\tau]} \phi^t(K)$$

spanned by the flow ϕ^t for $t \in [-\tau, \tau]$ with $\tau > 0$. Let α_- be the 1-form specified in the statement of the lemma. Since K is Legendrian, we can choose a small enough τ such that the Reeb vector field R_{α_-} is transverse to A_0 (see Figure 7).

Let ψ^w be the flow of the Reeb vector field R_{α_-} of α_-. There is an auxiliary coordinate system (s, u, w) in a neighbourhood of A such that

$$\alpha_- = dw + a(s, u)\, ds.$$

Since K is a Legendrian knot, we have $a(s, 0) = 0$ and since $\partial/\partial u a(s, u) \neq 0$, the transformation of coordinates

$$(s, u) \to (s, a(s, u)) =: (s, v)$$

is non-singular and therefore we have (see [14])

$$\alpha_- = dw + v\, ds.$$

Since the flow lines of R_{α_-} are Legendrian for ξ_+ by hypothesis and the v-curves are transverse to ξ_+ in a neighbourhood N of K, there is a contact form α_+ supporting ξ_+ that in N can be written

$$\alpha_+ = ds - b(s, v, w)\, dv.$$

In general, the v-curves do not describe the same codimension-two foliation described by the flowlines of X. However, these codimension-two foliations coincide on the surgery annulus. Therefore, $b(s, v, 0) = 0$ and $\alpha_+ = ds$. □

LEMMA 5.6. *In the hypothesis of Lemma 5.5, there is a neighbourhood $\Lambda' \subset N$ of K such that the contact form $\alpha_+^0 = ds - b(s, v, w)\, dv$ can be isotoped along the flow lines of R_{α_-} to a contact form independent on the s- and v-coordinate. In particular, we can write*

$$\alpha_+^1 = ds - h(w)\, dv.$$

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Proof. Since ξ_+ is a positive contact structure, $b(s, v, w)$ is strictly increasing with w. Therefore, $b(s, v, 0) = 0$ implies that $b(s, v, w) > 0$ for $w > 0$. Choose a smooth function $h : \mathbb{R} \to \mathbb{R}$ such that $h(w)$ is strictly increasing with w and $h(0) = 0$. For every point $(s, v) \in A_0$ and k small enough, there is function $g_k : A_0 \to \mathbb{R}$ such that $g_k(s, v) > 0$ and $b(s, v, g_k(s, v)) = h(k)$. The graph of g_k, namely $A_{g_k} = \{(s, v, w) | (s, v) \in A_0, w = g_k(s, v)\}$, is an annulus above A_0. More generally, for $w \in [0, k]$, we have a family of annuli $A_{g_w} = \{(s, v, w) | (s, v) \in A_0, w = g_w(s, v)\}$, where $g_w(s, v)$ is such that $b(s, v, g_w(s, v)) = h(w)$. Now, let $\mathcal{F}_{A_{g_w}}$ be the characteristic foliation induced by ξ_+ on A_{g_w} for $w \in [0, k]$. Since $h(w)$ is independent on the *s*- and *v*-coordinate, the projection of $\mathcal{F}_{A_{g_w}}$ along the *w*-curves onto A_0 is a foliation of A_0 that is invariant in the *s*- and *v*-direction. Consider the union $\Lambda = \bigcup_{w \in [0,k]} A_{g_w}$ of all the annuli A_{g_w} for $w \in [0, k]$ and the subset $\Lambda' \subset \Lambda$ with $\Lambda' = \bigcup_{w \in [0,k]} A'_{g_w}$, where $A'_{g_w} = \{(s, v, w) | (s, v) \in A'_0, w = g_w(s, v)\}$ and A'_0 is defined by $v \in [-\delta_1, \delta_1] \subset (-\delta, \delta)$. Let ψ^t be a smooth isotopy with support in Λ along the flow-lines of R_{α_-} such that $\psi^1(A'_{g_w})$ is the set $A'_w = \{(s, v, w) | (s, v) \in A'_0, w = \text{cost}\}$. Consider on $A_w = \psi^1(A_{g_w})$ the foliation $\mathcal{F}_{A_w} = \psi^1(\mathcal{F}_{A_{g_w}})$ and let $\mathcal{F}_{A'_w}$ be its restriction on A'_w. Since the isotopy has been realized deforming each annulus A_{g_w} along the *w*-curves, the projection of $\mathcal{F}_{A_w}$ on A_0 is independent on the *s*- and *v*-coordinate. Since A'_w is parallel to A'_0, the foliation $\mathcal{F}_{A'_w}$ (not just its projection) is invariant in the *s*- and *v*-coordinate. Note that by construction, $\mathcal{F}_{A_w}$ is the characteristic foliation induced on A_w by the contact structure $\psi^1_* \xi_+$. Since $\mathcal{F}_{A'_w}$ on each A'_w, $w \in [0, k]$ is independent on the *s*- and *v*-coordinates, there is a contact form defining $\psi^1_* \xi_+$ on Λ' that is independent on the *s*- and *v*-coordinates. In particular, we can write $\psi^1_* \xi_+ = \ker \alpha^1_+$ with $\alpha^1_+ = ds - h(w)\, ds$. □

Remark 5.7. Consider a path of projectively Anosov flows defined by an isotopy of the contact structure ξ_+ along the flow-lines of R_{α_-} as in Lemma 5.6. Every flow along the path is Anosov. In particular, any two flows ϕ^t_0 and ϕ^t_1 in the path are orbit equivalent by a diffeomorphism isotopic to the identity (see Theorem 2.12).

5.1.1. *Deformations and gluings.* Our procedure can be thought of as first cutting the manifold M along the annulus A_0 and then gluing back the two sides of the cut in a different way. In particular, we glue the points described by the coordinates (s, v) on the side of the cut where $w \leq 0$ to points $(s + f(v), v)$ on the side of the cut where $w \geq 0$. More precisely, this can be stated as the following remark.

Remark 5.8. (Notation) Let $w \in [-\epsilon, \epsilon]$ be the parameter of the flow generated by R_{α_-}. In the following, by A^-_0, we mean the surgery annulus as a subset of the flow-box chart $(-\epsilon, 0] \times A_0$, while by A^+_0, we mean the surgery annulus as a subset of the flow-box chart $[0, \epsilon) \times A_0$. Moreover, $(\alpha_-)|_{A^-_0}$ is the restriction of α_- to A^-_0, and $(\alpha_-)^-$ is the restriction of α_- to $(-\epsilon, 0] \times A_0$. Similarly, $(\alpha_-)^+$ is the restriction of α_- to $(0, \epsilon] \times A_0$.

We define

$$f : \mathbb{R} \to S^1, \quad v \to f(v),$$

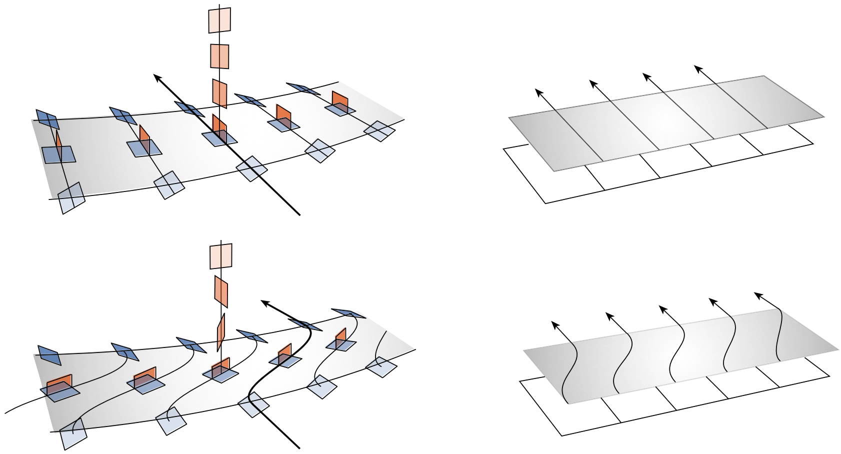

FIGURE 8. On the top left, the bi-contact structure of the original flow. On the bottom left, the bi-contact structure after deformation. On the right, the corresponding flow-lines on the two sides of the surgery annulus A_0 (colour online).

where $f : \mathbb{R} \to [0, 2\pi]$ monotone in $(-\delta, \delta)$, smooth and such that $f((-\infty, -\delta)) = 0$, $f([\delta, \infty)) = -2q\pi$, where $q \in \mathbb{Z}$, and we define the *shear map*

$$F : A_0^- \to A_0^+, \ (s, v) \to (s + f(v), v).$$

Remark 5.9. Note that for a $(1, q)$-bi-contact surgery with $q > 0$, the function f used to define the shear is *non-increasing*, while the function used in §4 to define a $(1, q)$-Foulon–Hasselblatt surgery is non-decreasing. This is because Foulon–Hasselblatt construction was originally performed along a transverse annulus C, while the Legendrian-transverse surgery is performed on an annulus A_0 tangent to the flow. With this convention, if $K \subset C \cap A_0$, the two operations produce homeomorphic manifolds for the same q.

As noticed in §4.1, the shear $F : A_0^- \to A_0^+$ defines a smooth new 3-manifold $\tilde{M}$, but it does not preserve the contact form α_- since on the surgery annulus A_0,

$$F^*(\alpha_-)|_{A_0^+} = dw + v\, d(s + f(v)) = (\alpha_-)^- + vf'(v)\, ds.$$

Here, by $F^*(\alpha_-)|_{A_0^+}$, we mean the image of $(\alpha_-)|_{A_0^+}$ under the pull-back map F^*.

An analogous calculation shows that α_+ is not preserved either since

$$F^*(\alpha_+)|_{A_0^+} = F^*\, ds = d(s + f(v)) = ds + f'(v)\, dv = (\alpha_+)|_{A_0^-} + f'(v)\, dv.$$

The strategy is to deform each of the contact structures independently on $M \setminus A_0$ in such a way that they remain transverse to each other and after the application of the shear, they define two new (transverse) contact structures (see Figure 8).

The deformation that we apply to α_- is the same one used by Foulon and Hasselblatt to define their contact surgery.

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We define $(\tilde{\alpha}_-)^- = (\alpha_-)^-$ in the flow-box charts $(-\epsilon, 0] \times A_0$ and $(\tilde{\alpha}_-)^+ = (\alpha_-)^+ - dh$ in $[0, \epsilon) \times A_0$. Here, $h : (-\delta, \delta) \times [0, \epsilon) \to \mathbb{R}$ is such that

$$h(v, w) = \lambda_1(w) \int_{-\delta}^{v} x f'(x)\, dx.$$

Here, $\lambda_1 : \mathbb{R} :\to [0, 1]$ is a smooth bump function supported in $[0, \epsilon)$ such that $\lambda_1(0) = 1$ and $|\lambda_1'| < 1/\epsilon + \iota$ with ι arbitrarily small. Note that in a neighbourhood of A_0^+, we have [**15**]

$$\tilde{\alpha}_- = dw + v\, ds - f'(v) v\, dv;$$

therefore,

$$F^*((\alpha_-)^+ - dh)|_{A_0^+} = (\alpha_-)|_{A_0^-}$$

and we have a well-defined 1-form on the new manifold $\tilde{M}$.

The deformation that we apply to α_+ is defined as follows. We set $(\tilde{\alpha}_+)^- = (\alpha_+)^-$ and $(\tilde{\alpha}_+)^+ = (\alpha_+)^+ - \sigma$, and consider the 1-form

$$\sigma = \lambda_2(w)\, f'(v)\, dv, \tag{5.1}$$

where $\lambda_2 : \mathbb{R} \to [0, 1]$ is a smooth bump function, with support in $[0, \epsilon)$ that takes the value 1 in a neighbourhood of 0 and value 0 in a neighbourhood of ϵ. More explicitly,

$$(\tilde{\alpha}_+)^+ = ds - (b(w) + \lambda_2(w)\, f'(v))\, dv.$$

Note that since $f'(v)$ has support in $(-\delta, \delta)$, the 1-form σ is smooth and vanishing in the complement of $A_0 \times [0, \epsilon)$.

We check that on the surgery annulus A_0, we have $F^*((\alpha_+)^+ - \sigma)|_{A_0^+} = (\alpha_+)|_{A_0^+}$. A direct computation shows that

$$\begin{aligned} F^*(\tilde{\alpha}_+)|_{A_0^+} &= F^*(ds - b(w)\, dv - \lambda_2(w)\, f'(v)\, dv)|_{A_0^+} \\ &= ds + f'(v)\, dv - b(w)\, dv - f'(v)\, dv = (\alpha_+)|_{A_0^-}; \end{aligned}$$

therefore, $\tilde{\alpha}_+$ is a well-defined 1-form on $\tilde{M}$.

Remark 5.10. The plane field distribution $\tilde{\xi}_+ = \ker(\tilde{\alpha}_+)$ and the plane field distribution $\tilde{\xi}_- = \ker(\tilde{\alpha}_-)$ are transverse since $\tilde{\xi}_+$ contains $R_{\tilde{\alpha}_-}$ that is a vector field always transverse to $\tilde{\xi}_- = \ker(\tilde{\alpha}_-)$.

The following statement encompasses Theorems 1.1 and 1.2 in the introduction.

THEOREM 5.11. *Let* $(\xi_- = \ker \alpha_-, \xi_+ = \ker \alpha_+)$ *be a bi-contact structure such that in a neighbourhood of a Legendrian-transverse knot* K*, we have* $R_{\alpha_-} \subset \ker \alpha_+$*. If* $q > 0$*, a bi-contact* $(1, q)$*-surgery along* K *produces a new bi-contact structure* $(\tilde{\xi}_- = \ker \tilde{\alpha}_-, \tilde{\xi}_+ = \ker \tilde{\alpha}_+)$*. Moreover,* $R_{\tilde{\alpha}_-} \subset \ker \tilde{\alpha}_+$.

Proof. We first show that the $(1, q)$-surgery produces a 1-form $\tilde{\alpha}_-$ that is contact for every $q \in \mathbb{Z}$. This is done using the contact surgery introduced by Foulon and Hasselblatt in [**14**], and interpreting the surgery annulus A_0 as a transverse annulus to the Reeb vector

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field of α_-. We use a slightly different argument to that used by Foulon and Hasselblatt. In particular, we show that we do not need any bound on the derivative of f in the definition of the shear in §5.1.1. As shown in [14], we have

$$\tilde{\alpha} \wedge d\tilde{\alpha} = \left(-1 + \frac{\partial h}{\partial w} \right) dV,$$

with $dV = ds \wedge dv \wedge dw$. Therefore, the condition $|\partial h/\partial w| < 1$ ensures that $\tilde{\alpha}$ is contact.

$$\left| \frac{\partial h}{\partial w} \right| = \left| \lambda_1' \int_{-\delta}^{v} x f'(x)\, dx \right|,$$

integrating by parts and since $f(-\delta) = 0$, we get

$$\left| \frac{\partial h}{\partial w} \right| = \left| \lambda_1' \left(v f(v) + \delta f(-\delta) - \int_{-\delta}^{v} f(x)\, dx \right) \right| = \left| \lambda_1' \left(v f(v) - \int_{-\delta}^{v} f(x)\, dx \right) \right|$$

since $|\lambda_1'| < 1/\epsilon + \iota$ with ι arbitrarily small, $-\delta < v < \delta$ (by definition of v), $|f(v)| < 2\pi |q|$ (by definition of f) and $|\int_{-\delta}^{v} f(x)\, dx| \le |\int_{-\delta}^{\delta} f(x)\, dx)| < 4\pi\delta |q|$, we have

$$\left| \frac{\partial h}{\partial w} \right| \le \left(\frac{1}{\epsilon} + \iota \right) (\delta 2\pi |q| + 4\delta\pi |q|) = \left(\frac{1}{\epsilon} + \iota \right) (6\delta\pi |q|),$$

and since ι is arbitrarily small, we have

$$\left| \frac{\partial h}{\partial w} \right| \le \frac{\delta}{\epsilon} (6\pi |q| + 1).$$

This shows that the 1-form $\tilde{\alpha}_-$ is contact if

$$0 < \delta < \frac{\epsilon}{6\pi |q| + 1}. \tag{5.2}$$

Remark 5.12. Note that condition (5.2) holds independently of the sign of q and the positivity of α_-. Moreover, δ can be chosen arbitrarily small, in particular, the tangent surgery annulus A_0 can be chosen arbitrarily thin.

We now show that for $q > 0$, the 1-form $\tilde{\alpha}_+$ defines a contact form on $\tilde{M}$. The 1-form $\tilde{\alpha}_+$ defines a positive contact form if and only if [12]

$$b'(w) + \lambda_2'(w)\, f'(v) > 0. \tag{5.3}$$

Since $b'(w) > 0$ by the contact condition and $\lambda_2'(w) \le 0$ by the definition of the bump function λ_2, the inequality is always satisfied if $f'(v) \le 0$. Therefore, the family of $(1, q)$-Dehn surgeries produces bi-contact structures for $q > 0$.

Finally, since $\tilde{\alpha}_+ = ds - (b(w) + \lambda_2(w)\, f'(v))\, dv$ and $R_{\tilde{\alpha}_+}$ is parallel to R_{α_+} on $\tilde{M} \setminus A_0$, we have $R_{\tilde{\alpha}_+} \in \ker \tilde{\alpha}_+$. □

The proof of Theorem 5.11 shows that there is a choice of the direction of the twist that strengthens the contact condition of the contact structure transverse to K. Performing the surgery in the opposite direction may result in a violation of the contact condition. An analogous phenomenon emerges in Goodman surgery (see [5, 15, 19]), where there is a choice of the direction of the shear that weakens the hyperbolicity of the flow. The following result shows that there is indeed a strong connection between the Anosovity of the new flows produced by the bi-contact surgery and contact geometry.

THEOREM 5.13. *Let M be a 3-manifold endowed with a bi-contact structure* $(\ker\alpha_-, \ker\alpha_+)$ *such that $R_{\alpha_-} \in \ker\alpha_+$ everywhere. If the Legendrian-transverse surgery yields a new bi-contact structure, it yields an Anosov flow.*

Proof. The proof is a straightforward application of Hoozori criterion of Anosovity. Since by construction, we have $R_{\tilde{\alpha}_-} \subset \tilde{\xi}_+$, the Reeb vector field $R_{\tilde{\alpha}}$ is dynamically positive everywhere (Corollary 2.10). □

6. *New Anosov flows for negative surgery coefficients*

We now give a sufficient condition for a bi-contact $(1, q)$-surgery to yield an Anosov flow when the operation is performed in the direction that weakens the hyperbolicity. Let K be a Legendrian-transverse knot in a bi-contact structure $(\xi_- = \ker\alpha_-, \xi_+ = \ker\alpha_+)$ such that $R_{\alpha_-} \subset \xi_+$ everywhere. The condition we give for a bi-contact $(1, q)$-surgery to yield an Anosov flow is purely contact geometric and involves the behaviour of ξ_+ along the boundary of a neighbourhood Λ of K. To this end, we introduce the following definition.

Definition 6.1. Let Λ be a neighbourhood of a Legendrian-transverse knot K such that the leaves of the characteristic foliation $\mathcal{F}_{\partial\Lambda}$ induced by ξ_+ on $\partial\Lambda$ are simple closed curves in the homology class $r[\mu] + s[K]$, where μ bounds a disk in Λ. We call $k = s/r > 0$ the *slope* of the characteristic foliation $\mathcal{F}_{\partial\Lambda}$.

Remark 6.2. Let N be a neighbourhood of a Legendrian-transverse knot K as in Lemma 5.5. After an isotopy of the ξ_+ along the flow-lines of α_- that induces an orbit equivalence isotopic to the identity, we assume that $\alpha_+ = ds - h(w)\, dv$ by Lemma 5.6.

THEOREM 6.3. *Let K be a Legendrian-tansverse knot in a bi-contact structure* $(\xi_- = \ker\alpha_-, \xi_+ = \ker\alpha_+)$ *such that $R_{\alpha_-} \subset \xi_+$ everywhere. There is a neighbourhood $\Lambda \subset N$ of K such that if $k > 0$ is the slope of the characteristic foliation $\mathcal{F}_{\partial N}$ induced by ξ_+ on ∂N, a bi-contact $(1, q)$-surgery with $q > -k$ yields an Anosov flow.*

Proof. Let ϕ^t be an Anosov flow supported by $(\ker\alpha_- = \xi_-, \xi_+)$ such that $R_{\alpha_-} \in \xi_+$. Call ψ^w the flow of R_{α_-}. Consider a flow-box neighbourhood $\Lambda \subset N$ of $K = K_0$ bounded at the bottom by the surgery annulus $A_{K_0} = A_0$, bounded on the sides by the annuli (transverse to the flow ϕ^t) C_{in} and C_{out} constructed by flowing the boundary components of A_0 using the flow of R_{α_-}, and bounded on the top by an annulus (see Figure 9)

$$A_{K_\epsilon} = \bigcup_{t\in[-\tau,\tau]} \phi^t(\psi^\epsilon(K)).$$

In particular, A_{K_ϵ} is constructed as follows. Let $K_\epsilon = \psi^\epsilon(K)$ be the knot obtained by translating K using the flow ψ^w for a (small) time ϵ. The annulus A_{K_ϵ} is constructed translating the knot $\psi^\epsilon(K)$ using the Anosov flow ϕ^t for $t \in [-\tau, \tau]$. Moreover, we can assume that $\mathcal{F}_{\partial\Lambda}$ is composed by closed curves. This is possible since $\mathcal{F}_{\partial\Lambda}$ is a linear foliation since it is independent on the s-coordinate. If the slope is irrational, after a $\mathcal{C}^1$-small perturbation along the flow-lines of R_{α_-}, we can assume that $\mathcal{F}_{\partial\Lambda}$ has closed leaves. Let $k = s/r > 0$ be the slope of $\mathcal{F}_{\partial\Lambda}$. Let p be the integral part of $-k$ if $-k$ is not an integer and set $p = -k + 1$ if $-k$ is an integer. Since $p > -k$, we can choose a function $f_p : [-\delta, \delta] \to S^1$ with $f_p'(-\delta) = f_p'(\delta) = 0$ such that the curve $\mathfrak{f}_p = \{(v, s) \in A_0 | v \in [-\delta, \delta],\ s = f_p(v)\}$, defined by the graph of f_p, has the following properties:

(1) the projection $\pi_w(\mathfrak{f}_p)$ of $\mathfrak{f}_p$ into A_{K_ϵ} along the w-curves intersects transversely the curves of the characteristic foliation $\mathcal{F}_{A_{K_\epsilon}}$ obtained by restricting $\mathcal{F}_{\partial\Lambda}$ to A_{K_ϵ};

(2) the foliation on $\partial\Lambda$ constructed by replacing $\mathcal{F}_{A_{K_\epsilon}}$ with a family of curves parallel to $\pi_w(\mathfrak{f}_p)$ has slope p.

The shear map

$$F : A_0 \to A_0, \quad (s, v) \to (s + f_p(v), v)$$

induces a $(1, p)$-Dehn surgery. Similarly to §5.1.1, we set

$$(\tilde{\alpha}_+)^+ = ds - (b(w) + \lambda_p(v, w)\, f_p'(v))\, dv.$$

On A_{K_ϵ}, the characteristic foliation induced by $\xi_+ = \ker(ds - b(w)\, dv)$ is directed by $v = (\partial/\partial s) b(g_{K_\epsilon}(v)) + \partial/\partial v + C_1(v)\partial/\partial w$, where $g_{K_\epsilon}(v) : [-\delta, \delta] \to [0, \epsilon]$ is such that $A_{K_\epsilon} = \{v \in [-\delta, \delta], w = g_{K_\epsilon}(v)\}$ and some $C_1 : [-\delta, \delta] \to \mathbb{R}$. Since the curve $\pi_w(\mathfrak{f}_p)$ defined above positively intersects $\mathcal{F}_{A_{K_\epsilon}}$, we have $b(g_{K_\epsilon}(v)) > f_p'(v)$. Moreover, $b(0) = 0$ on A_0 and $b'(w) > 0$ in Λ. Therefore, there is an annulus $A_{\mathfrak{f}_p}$ between A_{K_ϵ} and A_0 such that the characteristic foliation $\mathcal{F}_{A_{\mathfrak{f}_p}}$ induced by ξ_+ on $A_{\mathfrak{f}_p}$ is directed by $u = (\partial/\partial s) f_p'(v) + \partial/\partial v + C_2(v)\partial/\partial w$ with $C_2 : [-\delta, \delta] \to \mathbb{R}$. We have $A_{\mathfrak{f}_q} = \{v \in [-\delta, \delta], w = h_{\mathfrak{f}_p}(v)\}$ with $h_{\mathfrak{f}_p} : [-\delta, \delta] \to [0, \epsilon]$ such that $b(h_{\mathfrak{f}_p}(v)) = f_p'(v)$. We now define the smooth function $\lambda_p : \Lambda \to [0, 1]$. (Since α_+ is invariant in the s-direction, λ_p can be chosen to be constant along the coordinate s, so we will write for simplicity $\lambda_p(v, w)$ instead of $\lambda_p(s, v, w)$.) We set $\lambda_p(v, w) = 0$ near A_{K_ϵ}. Let $\Lambda' \subset \Lambda$ be the portion of Λ between A_0 and $A_{\mathfrak{f}_p}$. We set $\lambda_p(v, w) = 1$ on a subset $\Lambda'' \subset \Lambda$ that contains Λ'. Therefore, we have $\lambda_p(v, w) = 1$ near $A_{\mathfrak{f}_q}$ and $\lambda_p(v, w) = 0$ near A_{K_ϵ}. Since in the region $\Lambda \setminus \Lambda''$ we have $b(w) > f_p'(v)$, we can define $\lambda_p : \Lambda \to [0, 1]$ such that

$$b(w) + \lambda_p(v, w)\, f_p'(v) > 0, \quad h_{\mathfrak{f}_p}(v) < w < g_{K_\epsilon}(v).$$

In particular, we can choose $\lambda_p(v, w) < -b(w)/f_p'(v)$ with $\partial\lambda_p/\partial w < -b'(w)/f_p'(v)$. With these choices, we have

$$b'(w) + \frac{\partial\lambda_p}{\partial w}\, f_p'(v) > 0, \tag{6.1}$$

which is equivalent to say that $\tilde{\alpha}_+$ is contact. Since the w-curves are flow lines of R_{α_-}, Anosovity follows by Hozoori's criterion. □

6.1. *Bi-contact surgery near a closed orbit.* We now show that if the knot K is a Legendrian-transverse push off of a closed orbit γ of the flow, it is possible to perform the bi-contact surgery with *any* integral surgery coefficient. First, we give a version of Lemma 5.5 in a neighbourhood of a closed orbit γ.

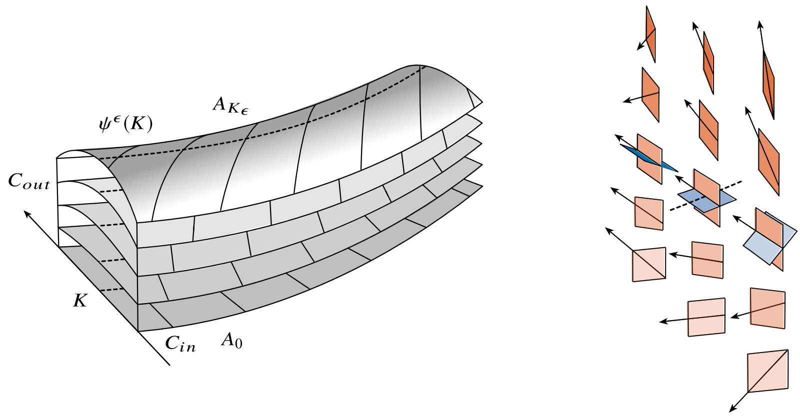

FIGURE 9. On the right, the positive contact structure rotates along the w axis. As usual, the Legendrian-transverse knot is denoted by a dashed line. On the left, the annuli A_{K_w} spanned by the flow (colour online).

LEMMA 6.4. *Suppose M is a 3-manifold endowed with a projectively Anosov flow ϕ^t defined by a bi-contact structure* $(\ker\alpha_- = \xi_-, \xi_+)$ *and consider a knot K that is a Legendrian-transverse push-off of a closed orbit γ. Suppose also that the Reeb vector field of α_- is contained in ξ_+ in a neighbourhood $N = A_0 \times (-\epsilon, \epsilon)$ of K, where A_0 is an annulus tangent to the Anosov flow and $\epsilon > 0$. We can choose N equipped with coordinates (s, v, w) and a contact form α_+ supporting ξ_+ such that in N, we have*

$$\alpha_- = dw + v\,ds,$$

$$\alpha_+ = (\overline{\epsilon} - w)\,ds - w\,dv.$$

Here, s and v are coordinates on A_0 such that $s \in S^1$ is the parameter describing K and $v \in (-\delta, \delta)$ with $\delta \in \mathbb{R}^+$, while $w \in (-\epsilon, \epsilon)$ is the transverse parameter to A_0 defined by the flow of R_-. Finally, $\overline{\epsilon} \in (0, \epsilon)$.

Proof. The proof is similar to that of Lemma 5.5. Let ψ^w be the flow of R_{α_-}. Since K is a Legendrian-transverse push-off of γ, we can write $K = \psi^{-\overline{\epsilon}}(\gamma)$ for some $\overline{\epsilon} > 0$. Since γ is a closed orbit of the flow, γ is a knot that is Legendrian for both ξ_- and ξ_+. Similarly to the proof of Lemma 5.5, we can write

$$\alpha_+ = b_1(s, v, w)\,ds - b_2(s, v, w)\,dv,$$

with $b_2(s, v, 0) = 0$ and $b_1(s, v, \overline{\epsilon}) = 0$ since γ is Legendrian for ξ_+. Using the same argument of Lemma 5.6, we can isotope ξ_+ along the w-curves obtaining the desired expression for α_+.

□

COROLLARY 6.5. *Let γ be a closed orbit of a flow defined by a bi-contact structure* $(\xi_- = \ker\alpha_-, \xi_+ = \ker\alpha_+)$ *such that $R_{\alpha_-} \subset \xi_+$ and let K be a Legendrian-transverse push-off of γ. For every $p \in \mathbb{Z}^-$, there is a flow-box neighbourhood Λ_p of K such that a bi-contact $(1, q)$-surgery along K yields an Anosov flow for every $q > p$.*

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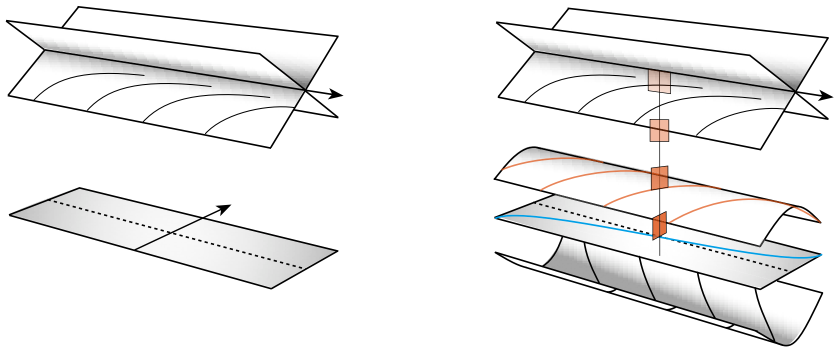

FIGURE 10. The positive contact structure ξ_+ in the proximity of a closed orbit. The curves are the leaves of the characteristic foliation induced by ξ_+ on A_ϵ (colour online).

Proof. Consider the flow-box neighbourhood Λ as in the proof of Theorem 6.3. Since $\psi^{\overline{\epsilon}}(K) = \gamma$, we have that if w approaches $\overline{\epsilon}$, the contact form α_+ approaches the plane field $\ker(-\overline{\epsilon}\, dv)$. In particular, the slope of the characteristic foliation induced by α_+ on Λ can be made arbitrarily large taking w close enough to $\overline{\epsilon}$. We finally use Theorem 6.3 and the fact that either p is the integral part of $-k$ or $p = -k + 1$ (with k the slope of the characteristic foliation induced by ξ_+ on Λ defined in the proof of Theorem 6.3). □

Remark 6.6. In a neighbourhood N of a closed orbit γ of a flow defined by a bi-contact structure $(\xi_- = \ker \alpha_-, \xi_+ = \ker \alpha_+)$ such that $R_{\alpha_-} \subset \xi_+$, there is a sequence of nested flow-box neighbourhoods of the Legendrian-transverse knot K namely $\Lambda_{-1} \subset \Lambda_{-2} \subset \cdots \subset \Lambda_p \subset \cdots \subset \Lambda_{-\infty}$, $p \in \mathbb{Z}^-$ such that a bi-contact $(1, q)$-surgery yields an Anosov flow for every $q \geq p$. Here,

$$\Lambda_p = \bigcup_{w \in [0, \epsilon_p]} A_{K_w},$$

and

$$A_{K_w} = \bigcup_{t \in [-\tau, \tau]} \phi^t(\psi^w(K))$$

with ϵ_p sufficiently close to $\overline{\epsilon}$. Moreover,

$$\Lambda_{-\infty} = \overline{\bigcup_{w \in [0, \overline{\epsilon})} A_{K_w}}.$$

All the neighbourhoods are bounded on one side by A_0. For $p \in \mathbb{Z}^-$, the neighbourhood Λ_p is a flow-box (see Figure 10), while $\Lambda_{-\infty}$ is not a flow-box, since it contains the closed orbit γ.

7. *Relations with Goodman surgery*

In this section, we study the relationship between the bi-contact surgery and the surgery introduced by Goodman [17]. At a first glance, the two operations look very different. First of all, the surgery annulus used in the bi-contract surgery is tangent to the generating vector

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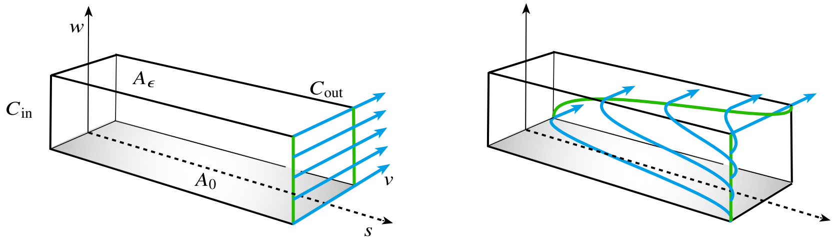

FIGURE 11. The flow-box N. For simplicity, we depicted the top layer A_ϵ as flat. The two vertical segments on the right side of the left picture represent two leaves of $\mathcal{F}$, while on the right, two leaves of the new foliation $\tilde{\mathcal{F}}$ are represented. A negative Dehn twist on A_0 (represented by the curve on the bottom of the picture on the right) corresponds to a positive Dehn twist on C_{out}. Therefore, a flow constructed by a $(1, q)$-bi-contact surgery along A_0 is orbit equivalent to a flow generated by $(1, q)$-Goodman surgery along C_{out} (colour online).

field X, while in Goodman's construction, we use an annulus transverse to the flow. Second, in the bi-contact surgery construction, the deformation of the bi-contact structure induces a three-dimensional perturbation of the flow in a neighbourhood of the surgery annulus. However, in Goodman's surgery, the flow lines of the new flow coincide with the old ones outside the (transverse) surgery annulus. Despite these differences, the flows generated by bi-contact surgery and those generated by Goodman's construction are orbit equivalent.

THEOREM 7.1. *Suppose that K is a Legendrian-transverse push-off of a closed orbit in a bi-contact structure (ξ_-, ξ_+) defined by two contact forms (α_-, α_+) such that R_{α_-} is contained in ξ_+. An Anosov flow generated by a bi-contact $(1, q)$-surgery along the tangent annulus A_0 is orbit equivalent to an Anosov flow generated by a $(1, q)$-Goodman surgery along a transverse annulus C.*

Proof. As in the proof of Theorem 6.3, let ϕ^t be an Anosov flow supported by $(\ker \alpha_- = \xi_-, \xi_+)$ such that $R_{\alpha_-} \in \xi_+$. Call ψ^w the flow of R_{α_-}. Let N be a neighbourhood of a Legendrian knot K with a coordinate system (s, v, w) defined as in Lemma 5.5, where the contact form α_+ can be assumed (by Lemma 5.6) to be independent on the s-coordinate. Consider a flow-box neighbourhood $\Lambda \subset N$ of $K = K_0$ bounded at the bottom by the tangent surgery annulus $A_{K_0} = A_0$, bounded on the sides by the transverse annuli C_{in} and C_{out} constructed by flowing the boundary components of A_0 using the flow of R_{α_-}, and bounded on the top, by a tangent annulus (see Figure 9)

$$A_{K_\epsilon} = \bigcup_{t \in [-\tau, \tau]} \phi^t(\psi^\epsilon(K)).$$

Let $\mathcal{F}_{\text{out}}$ be the foliation obtained by projecting on C_{out} (using the flow of X) the w-curves defined on C_{in}. Consider now a bi-contact $(1, q)$-surgery along the tangent annulus A_0 with shear

$$F : A_0 \to A_0, \quad (v, s) \to (v, s + f(v))$$

with $f(-\delta) = 0$ and $f(\delta) = -2\pi q$. Let $\tilde{X}$ be the new vector field supported by the new bi-contact structure $(\tilde{\xi}_-, \tilde{\xi}_+)$. Note that after the surgery procedure the flow-box Λ is transformed in another flow-box $\tilde{\Lambda}$, bounded by the same annuli as Λ. Let $\tilde{\mathcal{F}}_{\text{out}}$ be the foliation on C_{out} obtained by projecting (using the flow of $\tilde{X}$) the w-curves defined on C_{in}.

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We want to show that a leaf $\tilde{l}_{\text{out}}$ of $\tilde{\mathcal{F}}_{\text{out}}$ has the same endpoints of a leaf l_{out} of $\mathcal{F}_{\text{out}}$, but they differ by a Dehn twist along the core of C_{out}. Consider a segment l_{in} of a segment of a w-curve with endpoints $p \in \partial A_0$ and $q \in \partial A_{K_\epsilon}$, and let l_{out} and $\tilde{l}_{\text{out}}$ be its projection into C_{out} using respectively the flow of X and $\tilde{X}$. Since on A_{K_ϵ} the vector field X and $\tilde{X}$ coincide, l_{out} and $\tilde{l}_{\text{out}}$ share the same endpoint on the intersection of ∂C_{out} and ∂A_{K_ϵ}. Since the foliation induced by $\tilde{X}$ on A_0 is directed by $\tilde{W}_0 = f'(v)\, \partial/\partial s + \partial/\partial v$, while the one induced by X on A_0 is directed by $W_0 = \partial/\partial v$, a segment of a flow-line of $\tilde{X}$ on A_0 starting from the point $p \in \partial A_0$ differs from that of X (staring from p) by a full Dehn twist along the core of A_0; therefore, the flow-line segments directed by W_0 and $\tilde{W}_0$ share also the endpoint on the intersection of ∂A_0 and ∂C_{out}. Thus, $\tilde{l}_{\text{out}}$ and l_{out} have the same endpoints. Since $\tilde{\Lambda}$ and Λ are both toroidal flow-boxes with the same transverse and tangent faces, and since the flow-line of $\tilde{X}$ on A_0 differs from that of X by a full Dehn twist along the core of A_0, $\tilde{l}_{\text{out}}$ must differ from l_{out} by a full Dehn twist along the core of C_{out}. Consider the flow obtained by collapsing to a point each of the flow-line segments of $\tilde{X}$ in $\tilde{\Lambda}$. The resulting flow is clearly orbit equivalent to $\tilde{X}$ and also it is orbit equivalent to a flow obtained by removing $\tilde{\Lambda}$ and gluing C_{in} to C_{out} using a map that identifies each segment of w-curves l_{in} on C_{in} to the corresponding $\tilde{l}_{\text{out}}$ on C_{out}. Call $\tilde{X}_\delta$ the family of flows obtained by bi-contact surgery for different choices of $\delta \in [\tilde{\delta}, 0)$ (2δ represents the width of the surgery annulus A_0). Since this family defines a smooth path of Anosov flows, they are all orbit equivalent by an orbit equivalence isotopic to the identity. Taking δ small enough, we can make the annuli C_{in} and C_{out} to be $\mathcal{C}^1$-close to the annulus C (this is the annulus transverse to the flow and bounded by $K = K_0$ on the bottom and by K_ϵ on the top). However, l_{in} is $\mathcal{C}^1$-close to l_{out}. Since l_{in} is a w-segment, l_{out} is $\mathcal{C}^1$-close to a w-segment and since $\tilde{l}_{\text{out}}$ is obtained adding a Dehn twist to l_{out} and the knot $K_w,\ w \in [0, \epsilon]$ is Legendrian (the fact that K_w is Legendrian is relevant since if we take δ small, we can assume that the foliation in annuli of Λ and $\tilde{\Lambda}$ (defined in Remark 6.6) are $\mathcal{C}^1$-close), the identification of l_{in} with $\tilde{l}_{\text{out}}$ (for δ small enough) gives a flow that is $\mathcal{C}^1$-close to one constructed by cutting X along C and gluing back the two sides C^- and C^+ of C using the shear map

$$G : C^- \to C^+, \quad (w, s) \to (w, s + g(w)),$$

with $g : \mathbb{R} \to [0, 2\pi]$, smooth, and such that $g(0) = 0$ and $g(\epsilon) = 2\pi q$. The procedure just described is Goodman surgery (see Figure 11). □

COROLLARY 7.2. *Suppose that K is a Legendrian-transverse knot in a bi-contact structure* $(\xi_- = \ker \alpha_-, \xi_+)$ *such that* $R_{\alpha_-} \in \xi_+$ *everywhere. Choose* $p \in \mathbb{Z}^-$. *There is a transverse annulus* C_p *such that a* $(1, q)$*-Goodman surgery on* C_p *yields an Anosov flow for every* $q \geq p$.

Proof. For every flow-box Λ_p as in Remark 6.6, consider the core annulus C_p of Λ_p,

$$C_p = \bigcup_{w \in [0, \epsilon_p]} K_w,$$

where $K_w = \psi^w(K)$ and use the proof of Theorem 7.1. □

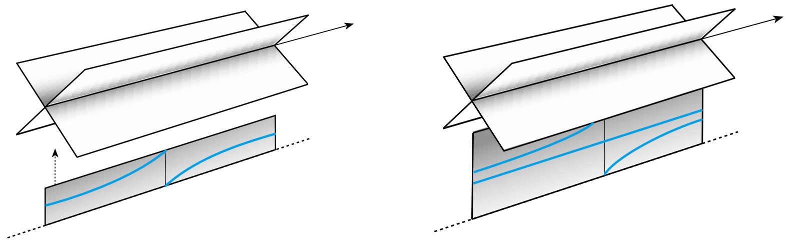

FIGURE 12. If a $(1, p)$-Goodman surgery along an annulus C_p produces an Anosov flow for a fixed $p < 0$, a $(1, p - 1)$-Goodman produces an Anosov flow on an annulus C_{p-1} obtained by expanding C_p towards the closed orbit. The curve on the left represents a $(1, -1)$ Goodman surgery, while the curve on the right is associated to a $(1, -2)$ Goodman surgery (colour online).

COROLLARY 7.3. *In the hypothesis of Theorem 7.2, there is a sequence of nested annuli $C_{-1} \subset C_{-2} \subset \cdots \subset C_p \subset \cdots \subset C_{-\infty}$ bounded on one side by the Legendrian-transverse knot K. Every C_p is the core of Λ_p and it is transverse to the flow if $p \in \mathbb{Z}$. The annulus $C_{-\infty}$ is quasi-transverse and it is bounded by K on one side and by the closed orbit γ on the other side (see Figure 12). A $(1, q)$-Goodman surgery along C_p produces an Anosov flow for every $q \geq p$.*

8. *Surgery along simple closed geodesics in a geodesic flow*

Let X be the generating vector field of the geodesic flow on the unit tangent bundle of an oriented hyperbolic surface S and let β_+ be the contact form preserved by $X = R_{\beta_+}$. Select a closed geodesic γ (eventually self intersecting) on S and consider the knot L defined by the angle $\theta = \pi/2$ on each fibre along γ. Here, L is a Legendrian knot for $\ker \beta_+$. Let $C_\epsilon = S^1 \times [-\epsilon, \epsilon] \subset \mathrm{UT}\gamma$ be a transverse annulus with ϵ small enough to ensure that C_ϵ is not self intersecting. As shown by Foulon and Hasselblatt for any $q > 0$, their construction ensures that a $(1, q)$-Goodman surgery produces a contact Anosov flow.

If L is associated to a simple closed geodesic γ, a recent result of Marty [23] shows that Goodman surgery produces a positively skewed $\mathbb{R}$-covered Anosov flow regardless of the sign of the surgery if and only if the closed orbit is the lift of a simple closed geodesic. We now prove a counterpart of Marty's result in the contact category.

We prove the following lemma showing that while the Foulon and Hasselblatt construction requires an arbitrarily thin annulus to produce a contact flow for positive surgery coefficients, in the case of negative surgery coefficients, this requirement is not necessary.

LEMMA 8.1. *Suppose that R_{β_+} is the Reeb vector of a contact form β_+ defining a positive contact structure and let L be a Legendrian knot for* $\ker \beta_+$. *For $q < 0$, the Foulon and Hasselblatt construction along a transverse embedded annulus C produces a new contact form $\tilde{\beta}_+$ with Reeb vector field $R_{\tilde{\beta}_+}$ regardless of the thickness of the surgery annulus.*

Proof. Foulon and Hasselblatt show that the vector field $\tilde{X}$ obtained by Goodman surgery with shear

$$G : C \to C, \quad (s, w) \to (w, s + g(w)),$$

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along C preserves a positive contact form

$$\tilde{\beta}_+ = dt + w\, ds - dh$$

with

$$h(t, w) = \lambda(t) \int_{-\epsilon}^{w} x g'(x)\, dx;$$

hence,

$$\tilde{\beta}_+ = dt + w\, ds - \lambda'(t) \int_{-\epsilon}^{w} x g'(x)\, dx\, dt - \lambda(t) w g'(w)\, dw.$$

They indeed show that the contact condition

$$\tilde{\beta}_+ \wedge d\tilde{\beta}_+ = \left(1 - \frac{\partial h}{\partial t}\right) dV > 0 \tag{8.1}$$

is satisfied regardless of the sign of the Dehn twist if the surgery annulus is sufficiently thin. Note that if

$$\frac{\partial h}{\partial t} = \lambda'(t) \int_{-\epsilon}^{w} x g'(x)\, dx\, dt \leq 0, \tag{8.2}$$

the contact condition (8.1) is satisfied independently of the thickness of the surgery annulus. If $g'(w) < 0$ and $g : [-\epsilon, \epsilon] \to S^1$ is odd, we have $\int_{-\epsilon}^{w} x g'(x)\, dx \geq 0$ for $w \in (-\epsilon, \epsilon)$ and since $\lambda'(t) \leq 0$ for $t > 0$, inequality (8.2) is satisfied. This corresponds to $(1, q)$-Foulon–Hasselblatt surgery with $q < 0$. $\square$

THEOREM 8.2. *Let ϕ^t be the geodesic flow on the unit tangent bundle UTS of an oriented hyperbolic surface S. Let $C \subset UT\gamma$ be a quasi-transverse annulus associated to a simple closed geodesic. For every $p < 0$, there is an embedded transverse annulus $C_p \subset C$ centred on L (as defined in §3.2) such that a $(1, p)$-Goodman surgery along C_p produces a contact Anosov flow.*

Proof. We first show that for every $p \in \mathbb{Z}^-$, there are choices such that the bi-contact surgery along a tangent annulus A_0 containing L produces an Anosov flow. Then, we show that these flows are orbit equivalent to flows constructed by $(1, p)$-Foulon–Hasselblatt surgery along an annulus C transverse to the geodesic flow. The first part of the proof is essentially the same as Theorem 7.1. The difference is that in this case, we are not allowed to perform any isotopy to transform the natural form α_+ in a contact form independent on the s and v coordinates since such an isotopy would modify the vector field X defined by (α_-, α_+) in a vector field X' that *a priori* does not preserve β_+. Nevertheless, Theorem 3.1 shows that α_+ and α_- are independent on the s coordinates and this is enough for our purpose. More precisely, consider the (s, v, w)-coordinate system in a neighbourhood N of UTγ as in Theorem 3.1. Let ϕ^t be the geodesic flow supported by (α_-, α_+), the natural contact forms. Call ψ^w the flow of R_{α_-}. Consider a flow-box neighbourhood $\Lambda \subset N$ of $L = L_0$ bounded at the bottom by the annulus

$$A_{L_{-\epsilon}} = \bigcup_{t \in [-\tau, \tau]} \phi^t(\psi^{-\epsilon}(L)),$$

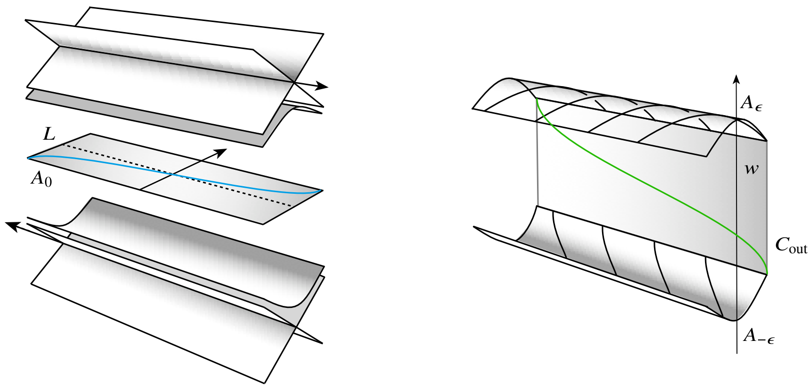

FIGURE 13. On the left, a neighbourhood of a Legendrian knot L associated to a simple closed geodesic on a geodesic flow. The curve on the left is the Dehn twist corresponding to a (negative) Legendrian-transverse surgery (note that the curve has positive slope). On the left, the characteristic foliations induced by ξ_+ on A_{L_ϵ} and $A_{L_{-\epsilon}}$. The curve on the right is the Dehn twist corresponding to the induced Goodman surgery (colour online).

bounded on the sides by the annuli (transverse to the flow ϕ^t) C_{in} and C_{out} constructed by flowing the boundary components of A_0 using the flow of R_{α_-}, and bounded on the top by an annulus (see Figure 9)

$$A_{L_\epsilon} = \bigcup_{t\in[-\tau,\tau]} \phi^t(\psi^\epsilon(L)).$$

Note that in the proof of Theorem 7.1, the bottom of Λ was bounded by the surgery annulus A_0, while here, A_0 is in between $A_{L_{-\epsilon}}$ and A_{L_ϵ}. This is because we want to construct a transverse annulus C centred on L. We can achieve this performing a bi-contact surgery along A_0 *splitting the difference* as in [**14**]. This method consists of introducing half of the deformation of the bi-contact structure on the side of the annulus where $w < 0$ and half on the side of the annulus where $w > 0$. Since by Theorem 3.1 ξ_+ does not depend on the s-coordinate, the characteristic foliation induced by ξ_+ on $\partial\Lambda$ does not depend on the s-coordinate. By Theorem 6.6, for every $p \in \mathbb{Z}$, there is a neighbourhood Λ_p and a new pair of contact structures $(\tilde{\xi}_-, \tilde{\xi}_+)_q$ on the manifold $\tilde{M}_p$ obtained by performing bi-contact $(1, p)$-surgery along A_0 in the manifold M. Note that Λ_p and the pair $(\tilde{\alpha}_-, \tilde{\alpha}_+)_p$ can be chosen to be independent of the s-coordinate. By Theorem 7.1, these flows are orbit equivalent to flows obtained by $(1, p)$-Goodman surgery with shear

$$G : C \to C, \quad (s, w) \to (w, s + g(w)),$$

where we chose $g'(w) < 0$. Since L is Legendrian for $\ker \beta_+$ and β_+ is also not dependent on s and $\ker \beta_-$ contains the w-curves, we can apply the Foulon–Hasselblatt construction with surgery annulus

$$C = \bigcup_{w\in[-\epsilon,\epsilon]} K_{L_\epsilon}$$

and coordinates (s, w, t), where (s, w) are defined on C and t is the parameter given by the geodesic flow (the Reeb flow of β_+). By Theorem 8.1, the resulting flow is contact

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regardless of the thickness of C. Therefore, for every $q < 0$, the resulting flow is Anosov and contact (see Figure 13). □

9. *Skewed $\mathbb{R}$-covered Anosov flows generated by surgery are contact*

In this section, we prove a version of Conjecture 1.6 for surgeries on a closed orbit of a geodesic flow on the unit tangent bundle of an hyperbolic surface S. This result is a consequence of Theorem 8.1 and the work of Asaoka, Bonatti and Marty [4] that we recall in the following subsection.

9.1. *Partial sections, multiplicity and linking numbers.*

Let M be an oriented 3-manifold equipped with a smooth flow ϕ^t. We call a subset P of M a *partial section* if it is the image of a smooth immersion $\iota : \hat{P} \to M$, where $\hat{P}$ is a compact surface and the restriction of ι to the interior of $\hat{P}$ is an embedding transverse to the flow and $\iota(\partial \hat{P})$ is a finite union of closed orbits of ϕ^t. The immersion ι lifts to the manifold $M_{\partial P}$ obtained by blowing up along $\iota(\partial \hat{P}) = \partial P$. A closed orbit in ∂P is the image $\iota(\hat{\gamma})$ of a boundary component $\hat{\gamma}$ of $\partial \hat{P}$. We denote the lift of $\iota(\hat{\gamma})$ to the blow-up with γ^* and we call it the *boundary component immersed in* γ. Let M be an oriented 3-manifold equipped with an Anosov flow with orientable invariant foliations, an immersed boundary component γ^* has two invariants: the multiplicity $\operatorname{mult}(\gamma^*)$ and the linking number $\operatorname{link}(\gamma^*)$ defined as follows:

$$\gamma^* = \operatorname{mult}(\gamma^*)\, \lambda_\gamma + \operatorname{link}(\gamma^*)\, \mu_\gamma.$$

Here, λ_γ is a curve homotopic to a lift of γ in the blow-up manifold (a parallel), while μ_γ is a curve homotopic to a fibre of the projection $\pi_{\partial P} : M_{\partial P} \to M$ (a meridian). If a closed orbit γ is the image of just one boundary components of $\hat{P}$, we can write $\operatorname{mult}(\gamma^*) = \operatorname{mult}(\gamma)$ and $\operatorname{link}(\gamma^*) = \operatorname{link}(\gamma)$. We say that a boundary component γ^* is *positive (negative)* if $\operatorname{mult}(\gamma^*) > 0$ ($\operatorname{mult}(\gamma^*) < 0$). A partial section P is said to be positive (negative) when *all* its boundary components are positive (negative).

THEOREM 9.1. (Asaoka–Bonatti–Marty [4]) *If ϕ^t is positively skewed $\mathbb{R}$-covered, it does not admit a negative partial section.*

In [4], the authors also study how the multiplicity of a boundary component of a partial section varies after a $(1, q)$-Goodman surgery.

THEOREM 9.2. (Asaoka–Bonatti–Marty [4]) *A $(1, q)$-Goodman surgery along a closed orbit in ∂P increases the multiplicity of a boundary component γ^* by $+q \operatorname{link}(\gamma^*)$.*

Example 9.3. Given a non-simple closed geodesic, we have an associated Birkhoff annulus C with self intersection. There is a process called *Fried desingularization* (see [16]) that allows us to obtain from C an immersed partial section P as defined above. The boundary components γ_1 and γ_2 of a partial section C associated to a (simple or non-simple) closed geodesic have both multiplicity 1 and $\operatorname{link}(\gamma^*) = p$, where p is the number of self intersections of the geodesic. By Theorem 9.2, a $(1, q)$-Goodman surgery with $q < 0$ along γ_1 changes the positivity of only one boundary component. The resulting partial section has two boundary components with opposite sign.

Example 9.4. In [23, §1.5.20], Marty describes a method to produce partial sections associated to a closed orbit γ that is the lift of non-simple closed geodesics on an orientable surface of negative curvature. These partial sections have multiple negative boundary components and the only positive boundary component is that associated to γ with $\mathrm{mult}(\gamma) = 1$ and $\mathrm{link}(\gamma) > 0$.

We are now ready to prove the main result of this section. Let ϕ^t be a geodesic flow with the orientation that makes it a positive skewed $\mathbb{R}$-covered Anosov flow.

THEOREM 9.5. *Let ϕ^t be the geodesic flow on the unit tangent bundle of an oriented hyperbolic surface. Any positive skewed $\mathbb{R}$-covered Anosov flow obtained by a single Goodman surgery along a closed orbit γ that is the lift of a simple closed geodesic is orbit equivalent to a positive contact Anosov flow.*

Proof. As a consequence of Theorem 8.1, it is enough to show that a $(1, q)$-Goodman surgery along a non-simple closed geodesic does not produce a positive skewed $\mathbb{R}$-covered Anosov flow if $q < 0$. Given a closed orbit γ associated to a non-simple geodesic, we chose a partial section as described in Example 9.4. Since $\mathrm{mult}(\gamma) = 1$ and $\mathrm{link}(\gamma) > 0$, by Theorem 9.2, a $(1, q)$-Goodman surgery along γ with $q < 0$ yields a negative partial section. By Theorem 9.1, the new flow cannot be positive skewed $\mathbb{R}$-covered. □

Remark 9.6. One important ingredient in proving Theorem 9.5 is the fact that a $(1, q)$-Goodman surgery along a closed non-simple geodesic γ in a positive skewed $\mathbb{R}$-cover Anosov flow does not produce a positive skewed $\mathbb{R}$-covered Anosov flow if $q < 0$. We now state a counterpart of this statement in the contact category.

PROPOSITION 9.7. *It is not possible to construct a contact Anosov flow performing Foulon–Hasselblatt construction if the closed geodesic is non-simple and $q < 0$.*

Sketch of the proof. The idea is the following. To a neighbourhood N of L, Theorem 1.4 associates a positive number k (the slope of the characteristic foliation induced by ξ_+ on the neighbourhood Λ) with the property that a $(1, q)$-Goodman surgery produces a contact Anosov flow if $k + q > 0$. A different neighbourhood N' is associated to a different k'. For instance, a sequence of nested neighbourhoods $N' \subset N'' \subset N'''$ is associate to a sequence of increasing slopes $k''' > k'' > k'$. For a fixed K, we denote with $\overline{k}$ the largest of all the possible values of k. If the geodesic is closed and simple, there is an infinite sequence of nested neighbourhoods as in Remark 6.3 and $\overline{k} = \infty$. If the geodesic is non-simple, we claim that $\overline{k} \leq 1$. A neighbourhood N of L with largest slope $\overline{k}$ contains the closed orbit γ in its boundary ∂N. As a curve in ∂N, it has the form $r[\mu] + s[\lambda]$, where λ is homotopic to L. Since also γ is homotopic to L, we have $s = 1$. Since the geodesic is non-simple, the associated closed orbit intersects C and the slope of the characteristic foliation is finite at least in a neighbourhood of the point of self intersection. Therefore, $\overline{k} \neq \infty$, implying that $\overline{k} = 1/r$. Finally, since $r \in \mathbb{Z}$, we have $\overline{k} \leq 1$. This shows that there does not exist a neighbourhood N of L such that $k + q > 0$ if $q < 0$.

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