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On a relation between the Fitting length of a soluble group and the number of conjugacy classes of its maximal nilpotent subgroups

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In a finite soluble group G, the Fitting (or nilpotency) length $h(G)$ can be considered as a measure for how strongly G deviates from being nilpotent. As another measure for this, the number $\nu(G)$ of conjugacy classes of the maximal nilpotent subgroups of G may be taken. It is shown that there exists an integer-valued function f on the set of positive integers such that $h(G) \leqq f(\nu(G))$ for all finite (soluble) groups of odd order. Moreover, if all prime divisors of the order of G are greater than $\nu(G)(\nu(G) - 1)/2$, then $h(G) \leqq 3$. The bound $f(\nu(G))$ is just of qualitative nature and by far not best possible. For $\nu(G) = 2$, $h(G) = 3$, some statements are made about the structure of G.

1. In various papers by Gross [3], Hoffman [5], and Thompson [9], bounds were given for the nilpotency length of a finite soluble group in terms of the group exponent (in the case of certain pq-groups) the order of a fixed-point-free p-automorphism, or of the number of primes (not necessarily different) dividing the order of a soluble, π'-automorphism group of a π-group. The main result of this paper is

THEOREM. *Let $h(G)$ be the Fitting length of a finite group G of odd order, $\nu(G)$ the number of different conjugacy classes of the maximal*

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nilpotent subgroups of G *, then there exists a function* f *, defined on the set of natural numbers and taking natural numbers as values, such that* $h(G) \leq f(\nu(G))$.

Rose [8] has, however, shown that there is no lower bound for $h(G)$ in terms of $\nu(G)$. The statement of the theorem is merely of qualitative nature, for $\nu(G) = 2,3$ sharp bounds are given, and for $\nu(G) = 2$, $h(G) = 3$ some statements are made about the structure of G . All the groups are assumed to be finite and soluble in this paper.

2. In order to prove the theorem, we start with some lemmas.

LEMMA 1. *If* $N \triangleleft G$ *, then* $\nu(G/N) \leq \nu(G)$ *. Every maximal nilpotent subgroup of* G/N *can be written as* VN/N *where* V *is maximal nilpotent in* G *. If* $\nu(G/N) = \nu(G)$ *, and* V *is maximal nilpotent in* G *, then* VN/N *is maximal nilpotent in* G/N .

Proof. Let W/N be maximal nilpotent in G/N . Then there exists a nilpotent subgroup W_1 in G such that $W = W_1N$. Let V be a maximal nilpotent subgroup of G such that $W_1 \subset V$. Then $W/N = W_1N/N \subset VN/N$ whence $W = VN$ by maximality of W/N . The other statements of the lemma follow.

LEMMA 2. *Let* $N \triangleleft G$ *,* N *nilpotent,* V *maximal nilpotent in* G *,* $V \not\supset N$ *,* $W \subset V$ *,* $(|W|, |N|) = 1$ *, and* $W \not\subset \overline{V}$ *for every nilpotent subgroup* $\overline{V}$ *of* G *containing* N *. Then either* $C_N(W) = V \cap N$ *or there exists a maximal nilpotent subgroup* T *of* G *,* $T \not\supset N$ *such that* $W \subset T$ *,* $V \cap N < T \cap N = C_N(W)$ *. If, in particular,* $V \cap N$ *is maximal among* $\{T \cap N \mid T$ *maximal nilpotent in* G *,* $T \not\supset N\}$ *, then* $C_N(W) = V \cap N$.

Proof. Since $(|W|, |N|) = 1$ we have $V \cap N \subset C_N(W)$, and $C_N(W) \times W$ is a nilpotent subgroup of G . Let T be a maximal nilpotent subgroup of G such that $T \supset C_N(W) \times W$. Then $T \not\supset N$ as $W \subset T$, by assumption. Also $(|W|, |N|) = 1$ implies $T \cap N \subset C_N(W)$ and so $T \cap N = C_N(W)$. The second statement of Lemma 2 is then obvious.

LEMMA 3. *Let* F *be the Fitting subgroup of* G *,* $|G|$ *odd, and suppose* F *is an elementary-abelian* p*-group. Let* V *be a maximal nilpotent subgroup of* G *,* $V \not\supset F$ *, and* l *be the largest integer for*

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which there exists a chain of subgroups $V \cap F < V_{l-1} \cap F < V_{l-2} \cap F < \ldots < V_1 \cap F < F$ *where* V_i *is a maximal nilpotent subgroup of* G *for* $i = 1,2,\ldots,l-1$. *Then, for every prime* $q \nmid p$, *every abelian* q*-subgroup of* V *can be generated by at most* l *elements.*

Proof. By induction on l . For $l = 1$, let U be an abelian q-subgroup of V , $z \in U$. By Lemma 2, $V \cap F = C_F(z)$ implying U is cyclic, otherwise $F = \prod_{z\in U\setminus\{1\}} C_F(z) = V \cap F$ (see [2], Th. 6.2.4). Suppose Lemma 3 is true for all integers less than l , and let U be an abelian q-subgroup of V . Let $\underline{\underline{K}} = \{C_F(T) \mid \{1\} < T \subset U\}$. $\underline{\underline{K}}$ is ordered by inclusion. Either $\underline{\underline{K}}$ consists just of one element, then, again by [2], Th. 6.2.4, U is cyclic, or we can choose a second minimal element $C_F(R) = X \cap F$ in $\underline{\underline{K}}$, where $\{1\} < R < U$ and X is maximal nilpotent in G , by Lemma 2. $C_U(X \cap F) \times (X \cap F)$ is nilpotent, hence $C_U(X \cap F) \subset Y$, $X \cap F \subset Y \cap F$ for some maximal nilpotent subgroup Y of G . Also $V \cap F \subset C_F(U) < C_F(R) = X \cap F$ whence $V \cap F < Y \cap F$ so that, by induction, $C_U(X \cap F)$ can be generated by at most $l-1$ elements. It remains to show that $U/C_U(X \cap F)$ is cyclic. If $u \in U$, then $(X \cap F)^u = C_F(R^u) = C_F(R) = X \cap F$, and so $U \subset N_G(X \cap F)$. We claim $U/C_U(X \cap F)$ acts faithfully on $X \cap F/C_F(U)$. Let $u \in U$, and $[u , X \cap F] \subset C_F(U)$. By Maschke's theorem, $X \cap F = C_F(U) \times L$ where L is a U-module. Hence $[u , L] \subset C_F(U) \cap L = 1$, and so $u \in C_U(L) = C_U(X \cap F)$. We claim that $U/C_U(X \cap F)$ acts in a fixed-point-free manner on $X \cap F/C_F(U)$. For, let $u \in U$, $x \in X \cap F$, $[u , x] \in C_F(U)$. We may write $x = yz$, $y \in C_F(U)$, $z \in L$. Then $[u , z] \in C_F(U) \cap L = 1$ and $z \in C_F(u)$. If $C_F(u) \not\supset X \cap F$, then $X \cap F > C_F(u) \cap (X \cap F) = C_F(u) \cap C_F(R) = C_F(\langle u , R\rangle) = C_F(U)$, by definition of $X \cap F$. Hence $z \in C_F(U)$ and so $x \in C_F(U)$. If $C_F(u) \supset X \cap F$, then $u \in C_U(X \cap F)$. Therefore $U/C_U(X \cap F)$ is cyclic.

LEMMA 4 (Thompson). *If* G *is a* p*-group,* $p > 2$ *and every abelian*

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normal subgroup of G *can be generated by* k *elements, then every subgroup of* G *can be generated by* $\frac{k(k+1)}{2}$ *elements.*

Proof. See [6], III. Satz 12.3.

LEMMA 5 (Huppert). *Let* G *be a* p*-soluble group,* V *a vector space of dimension* n *over* $GF(p)$ *, and let* G *be faithfully and irreducibly represented on* V *,* $(n, |G|) = 1$ *. Then* G *is cyclic and* $|G| \mid p^n - 1$ *.*

Proof. See [7].

LEMMA 6. *Let* G *possess a Fitting subgroup* F *such that* F *is the unique minimal normal subgroup of* G *and suppose* F *is a* p*-group. Then there exists a normal subgroup* S *of* G *such that* $h(G/S) = h(G) - 1$ *, for* $h(G) > 1$ *. Moreover, the Fitting subgroup of* G/S *is the unique minimal normal subgroup of* G/S *and is a* p'*-group.*

Proof. Let $\underline{\underline{N}}$ be the class of nilpotent groups, $\underline{\underline{N}}^0 = \{\{1\}\}$ and $\underline{\underline{N}}^k = \underline{\underline{N}}^{k-1}\underline{\underline{N}}$, for $k = 2, 3, \ldots$ Let H be (unique) minimal for $H \triangleleft G$, $G/H \in \underline{\underline{N}}^{h(G)-2}$, and H/K a chief factor of G . Clearly $F \subset K < H$ and H/K is a p'-group. Moreover there exists for $h(G) > 2$, a maximal subgroup M of G complementing H/K , by [1]. Let $R = C_G(H/K)$, $S = R \cap M$. Then $C_G(H/K) = C_G(R/S)$, $G/C_G(R/S) \notin N^{h(G)-3}$ and so $h(G/S) = h(G) - 1$. For $h(G) = 2$, we may take any maximal normal subgroup of G for S .

Proof of the Theorem. Let N_1 , N_2 be two different minimal normal subgroups of G . Then $h(G) = \max(h(G/N_1), h(G/N_2)) \leq \max(f(\nu(G/N_1)), f(\nu(G/N_2)))$, by induction. If N is a minimal normal subgroup of G , and $N \subset \Phi(G)$, then $h(G) = h(G/\Phi(G))$, hence $h(G) = h(G/N) \leq f(\nu(G/N))$, by induction. Thus, if we can find an increasing function $f(\nu)$ which bounds $h(G)$ for all groups G having their Fitting subgroups as unique minimal normal subgroups, $f(\nu)$ is then a general bound for $h(G)$. Hence let us assume that G has its Fitting subgroup F as its unique minimal normal subgroup, and suppose F is a p-group. There is exactly one conjugacy class of maximal nilpotent subgroups of G containing F , namely the Sylow p-subgroups of G . Hence by Lemma 3, every abelian q-subgroup, $q \neq p$, can be generated by at most $\nu(G) - 1$ elements. Lemma 4 implies that the p'-chief factors of G are of rank at most

$\frac{\nu(G)(\nu(G)-1)}{2}$. We choose $S \lhd G$ accordingly to Lemma 6, and Lemma 1 implies $\nu(G/S) \leqq \nu(G)$. Hence, all p-chief factors of G/S are of rank at most $\frac{\nu(G)(\nu(G)-1)}{2}$. Therefore, by [6], VI. Hauptsatz 6.6 c , the p-length of G/S is at most $\frac{\nu(G)(\nu(G)-1)}{2}$. Assume $\nu(G) > 2$. Let $\{p_1, p_2, \ldots, p_{r(\nu)}\}$ be the set of all odd primes less than or equal to $\frac{\nu(G)(\nu(G)-1)}{2}$, take the upper p_1-series of G/S refine each factor by a p_2-series, etc. One obtains a normal series of G/S of length at most $2(s(\nu) + 1)^{r(\nu)} - 1$ where $s(\nu) = \frac{\nu(G)(\nu(G)-1)}{2}$, consisting of p_i-factors, $i = 1, 2, \ldots, r(\nu)$, and $\{p_1, p_2, \ldots, p_{r(\nu)}\}'$-factors. By Lemma 5, these $\{p_1, p_2, \ldots, p_{r(\nu)}\}'$-factors are all of Fitting length at most 2, and there are at most $(s(\nu) + 1)^{r(\nu)}$ of them. Hence, $h(G/S) \leqq 3(s(\nu) + 1)^{r(\nu)} - 1$ and $h(G) \leqq 3(s(\nu) + 1)^{r(\nu)}$. Since $s(\nu)$ and $r(\nu)$ are increasing functions, we may take $f(\nu) = 3(s(\nu) + 1)^{r(\nu)}$. $\nu(G) = 1$ implies $h(G) = 1$, for $\nu(G) = 2$, $\{p_1, \ldots, p_{r(\nu)}\} = \emptyset$, and again Lemma 5 implies that $h(G/S) \leqq 2$ whence $h(G) \leqq 3$. Q.E.D.

COROLLARY. *If* $q \mid |G|$ *implies* $q > \frac{\nu(G)(\nu(G)-1)}{2}$, *then* $h(G) \leqq 3$.

Proof. This is an immediate consequence of Lemma 5, provided $|G|$ is odd. Now assume $\nu(G) = 2$, and $2 \mid |G|$. We may assume that G has its Fitting subgroup F as its unique minimal normal subgroup. First, let F be a 2-group. Lemma 3 implies G_2 , is cyclic, and a Hall-Higman type argument [4] shows that the' $2'$-length of G is at most 1 whence $h(G) \leqq 3$. Now let F be a $2'$-group. Lemma 3 implies that G_2 is cyclic or a generalized quaternion group. In either case G/F possesses a characteristic subgroup of order 2 which is clearly central in G/F . Assume G/F is not nilpotent and F is a p-group, $p \in 2'$. Then G/F contains an element xF of order $2p$. Let $x = yz$, $o(y) = p^\alpha$, $o(z) \in p'$. Then $y \in C_F(G_{p'})$ for some p-complement $G_{p'}$ of G , and $xF = zF$ is a p'-element, contradiction. Hence, in this case, $h(G) \leqq 2$.

3. For $\nu(G) = 2$, $h(G) = 3$ is really attained for some groups G . The symmetric group S_4 on 4 letters provides such an example. Moreover, let $H = C_q C_p$ be a (non-direct) semidirect product of a group C_q of order

q by a group C_p of order p and let $G = \overline{C}_p \text{ wr } H$ be the wreath product of a group $\overline{C}_p$ of order p by H. Then $\nu(G) = 2$, and $h(G) = 3$ as one can easily check.

For groups minimal for $\nu(G) = 2$, $h(G) = 3$ we get the following result:

PROPOSITION 7. *Let G be a finite group which is minimal with respect to the property that $\nu(G) = 2$ implies $h(G) = 3$. Then $|G| = p^{\alpha}q^{\beta}$, p, q being distinct primes, and G contains no element of order pq. In particular, if F is the Fitting subgroup of G, then $|F| = p^{\gamma}$, for some $\gamma > 0$, and $\beta = 1$, $\gamma = \alpha - 1$.*

REMARK. Since $\nu(G) = 2$, $G = V_1V_2$ where V_1V_2 are maximal nilpotent subgroups of G. In this case, the solubility of G follows from a theorem of Wielandt and Kegel [*11*].

First we prove

LEMMA 8. *Let G be a finite group minimal with the properties that $\nu(G) = 2$ implies $h(G) = 3$. Then the maximal nilpotent subgroups of G are Hall subgroups of G.*

Proof. Let V_1 and V_2 be representatives of the two conjugacy classes of maximal nilpotent subgroups of G. By hypothesis, we may assume that G has its Fitting subgroup F as its unique minimal normal subgroup. Without loss of generality, we may assume $V_1 \supset F$. Let $|F| = p^{\gamma}$, then $V_1 = G_p$, a Sylow p-subgroup of G, and $V_2 = G_{p'} \times C_{G_p}(G_{p'})$ where $G_{p'}$ is a p-complement of G. We have to show $C_{G_p}(G_{p'}) = 1$. Let F_2/F be the Fitting subgroup of G/F, then $F_2/F \cong G_{p'}$, moreover $C_{G_p}(G_{p'}) = C_F(G_{p'})$. Also $C_F(G_{p'}) \subset Z(F_2)$ since F is abelian. We claim $Z(F_2) = C_F(G_{p'})$. For $Z(F_2) = (Z(F_2))_{p'} \times C_F(G_{p'})$ where $(Z(F_2))_{p'}$ is the p'-complement of $Z(F_2)$. As F is self-centralizing it follows $(Z(F_2))_{p'} = 1$. Thus $C_F(G_{p'}) = Z(F_2)$ char $F_2 \triangleleft G$ implying $C_F(G_{p'}) = 1$ as F is a minimal normal subgroup of G.

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Proof of Proposition 7. By Lemma 8, $V_1 = G_p$, $V_2 = G_{p'}$. Suppose $q \mid |V_2|$. Therefore G contains no element of order pq, otherwise $\nu(G) > 2$, and so G/F contains no element of order pq, $q \in p'$. Certainly $V_2F = F_2$ as $h(G/F) = 2$. Suppose V_2 is not a Sylow q-subgroup for some $q \in p'$. Then, for some prime r, let R be a Sylow r-subgroup of G, $r \neq q$. Then FR char F_2 and so $FR \triangleleft G$. Consider $G_1 = (FR)V_1 = RV_1$. RV_1 contains no element of order pr, therefore $\nu(G_1) = 2$, $G_1 < G$. Minimality of G implies $h(G_1) = 2$. But then $FR = G_1$ since F is self-centralizing and so $V_1 = F$, hence $h(G) = 2$, contradiction. Thus $V_2 = G_q$, a Sylow q-subgroup of G, and $|G| = |V_1V_2| = p^\alpha q^\beta$. V_2 acts in a fixed-point-free manner on F, hence V_2 is cyclic ($q = 2$ can be excluded, for then $h(G) = 2$, by the proof of the corollary of the theorem). Let S/F be the cyclic normal subgroup of index q in F_2/F. Then $S \triangleleft G$. Using the same argument as above, we may conclude $V_1 = F$ provided $\beta > 1$. Therefore $\beta = 1$. Let M be a maximal subgroup of G containing F_2. Then $M \triangleleft G$, $V_1 \cap M$ is a Sylow p-subgroup of M, and M contains no element of order pq. Therefore $\nu(M) = 2$. Minimality of G implies $h(M) = 2$ and so $M = F_2$. Therefore $\gamma = \alpha - 1$.

4. We are going to give an exact bound for $h(G)$ in the case of $\nu(G) = 3$, $|G|$ odd.

LEMMA 9 (Thompson). *Suppose p is an odd prime, G is a p-soluble group and G has no elementary-abelian subgroup of order p^3. Then each p-chief factor of G is of order p or p^2.*

Proof. See [10].

COROLLARY. *If $|G|$ is odd, $\nu(G) = 3$, then $h(G) \leqq 3$.*

Proof. By Lemmas 3, 5, and 6.

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