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A non-cyclic, locally free, free-by-cyclic group all of whose finite factor groups are cyclic

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We construct here a group G with the properties described in the title of this note. The properties of G should be viewed in the context of the following theorem:

A finitely generated cyclic extension of a free group is residually finite [1].

We construct G as a direct limit of free groups $G_i = \langle a_i, b_i \rangle$ of rank two. To this end let $\phi_i : G_i \to G_{i+1}$ be defined as follows:

$$a_i\phi_i = b_{i+1}^{-(i+1)!}a_{i+1}^{-1}b_{i+1}^{(i+1)!}a_{i+1} , \quad b_i\phi_i = b_{i+1} , \quad i = 1, 2, \ldots .$$

It follows easily that ϕ_i is a monomorphism. The groups G_i together with the monomorphisms ϕ_i constitute a direct system. The direct limit of this system is the desired group G. As usual we identify, for example, a_i with its image $a_i\phi_i$. Then G is the union of its subgroups G_i. Moreover, if we put $b_1 = b$, then $b_i = b$ for all i.

In order to see that G has the desired properties, let N_i be the normal closure in G_i of a_i. Then it is not hard to show that N_i is a

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free factor of N_{i+1} . Therefore the union N of these subgroups N_i is a free group. But N is the normal subgroup of G generated by the elements $a_1, a_2, \ldots$. It follows easily that G/N is infinite cyclic on bN , which means that G is free-by-cyclic as claimed. In addition G is locally free because it is an ascending union of free groups. It remains only to show that every finite factor group of G is cyclic.

Let K be a normal subgroup of G of finite index, n say. Then, working modulo K , we have

$$a_n \equiv b_{n+1}^{-(n+1)!} a_{n+1}^{-1} b_{n+1}^{(n+1)!} a_{n+1} \equiv a_{n+1}^{-1} a_{n+1} = 1 \quad (K) .$$

This implies that $N_n \leq K$. Since $N_i \leq N_n$ when $i \leq n$, it follows that $N_i \leq K$, for $i \leq n$. However a similar inductive argument shows that $N_i \leq K$ for every i . So $N \leq K$. Therefore G/K is cyclic, as claimed. This completes the proof that G is a group of the desired type.

Reference

[1] Gilbert Baumslag, "A non-cyclic one-relator group all of whose finite quotients are cyclic", *J. Austral. Math. Soc.* 10 (1969), 497-498.

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