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A NEW CLASS OF SYMMETRIC WEIGHING MATRICES

H. KHARAGHANI

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Abstract

If there is a $W(n,p)$, then there is a symmetric $W(n^2,p^2)$.

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A weighing matrix of weight p and order n is an $n \times n$ $\{0,1,-1\}$-matrix A such that $AA^t = A^tA = pI_n$. We refer to such a matrix as a $W(n,p)$. A $W(n,p)$ is called a Hadamard matrix. Goethals and Seidel [2] proved that if there is a Hadamard matrix of order n, then there is a symmetric Hadamard matrix of order n^2. W. D. Wallis [3], partially answering a question of Bush, proved that if there is a Hadamard matrix of order n, then there is a symmetric Hadamard matrix of order n^2 which can be partitioned into an $n \times n$ array of $n \times n$ blocks such that (i) each diagonal block has every entry 1, and (ii) each non-diagonal block has every row and column sum zero (we will refer to such matrices as Bush-type Hadamard). In this paper, with an entirely new approach, we shall prove that if A is a $W(n,p)$, then $A^t \times A$ is Hadamard equivalent to a symmetric $W(n^2,p^2)$ (see [4, page 408]). This provides many new symmetric weighing matrices and, with a slight modification, proves the following: if H is a Hadamard matrix of order n, then $H^t \times H$ is Hadamard equivalent to a Bush-type Hadamard matrix of order n^2. Throughout the note we will follow Geramita and Seberry [1] for definitions, etc.

THEOREM 1. *Let A be a $W(n,p)$. Then there is a symmetric $W(n^2,p^2)$ which can be obtained from $A^t \times A$ by reordering the rows.*

PROOF. Let $A = [a_{ij}]$. For $k, l = 1, 2, \ldots, n$, let $C_{kl} = [a_{li} a_{kj}]$. Then

(i) $$C_{lk}^t = [a_{ki} a_{lj}]^t = [a_{kj} a_{li}] = C_{kl}, \quad \text{for } 1 \leqslant l,\ k \leqslant n.$$

(ii) $$C_{kl} C_{k'l}^t = \left[\sum_m a_{li} a_{km} a_{k'm} a_{lj}\right] = \left[a_{li}\left(\sum_m a_{km} a_{k'm}\right) a_{lj}\right]$$
$$= \begin{cases} 0 & \text{if } k \neq k', \\ pC_{ll} & \text{if } k = k'. \end{cases}$$

(iii) $$\sum_l C_{kl} C_{kl}^t = p \sum_l C_{ll} = p\left[\sum_l a_{li} a_{lj}\right] = p^2 I_n.$$

Consider the block matrix $B = [C_{kl}]$, $k, l = 1, 2, \ldots, n$. Then B is a $\{0, 1, -1\}$-matrix with the following properties:

(a) $$B^t = [C_{lk}^t] = [C_{kl}] = B \quad \text{by (i) above;}$$

(b) $$BB^t = \left[\sum_m C_{km} C_{lm}^t\right] = p^2 I_{n^2} \quad \text{by (ii) and (iii) above.}$$

Hence B is a symmetric $W(n^2, p^2)$.

Finally, it is easy to see that row $1 + (i - 1)n + j$ of $A^t \times A$ is exactly the same as row $i + nj$ of B, for $i = 1, 2, \ldots, n$ and $j = 0, 1, 2, \ldots, n - 1$. So B can be obtained from $A^t \times A$ by reordering the rows of $A^t \times A$.

The following is easier than Goethals and Seidel's original construction.

COROLLARY 2 (Goethals and Seidel [2]). *Let A be a Hadamard matrix. Then there is a symmetric Hadamard matrix with constant diagonal which could be obtained from $A^t \times A$ by reordering the rows.*

PROOF. Let $p = n$ in Theorem 1 and note that all entries on the main diagonal of B are ones.

Actually we have more than the above.

COROLLARY 3 (W. D. Wallis [3]). *Let A be a Hadamard matrix. Then $A^t \times A$ is Hadamard equivalent to a Bush-type Hadamard matrix.*

PROOF. Let B be the matrix constructed in Theorem 1. Multiply the columns of B by the corresponding entries of the first row of C_{ii}, $i = 1, 2, \ldots, n$, and apply the corresponding row operations. This is like changing C_{kl} in Theorem 1 by $D_{kl} = [a_{li} a_{lj} a_{ki} a_{kj}]$ for each $k, l = 1, 2, \ldots, n$. So the new block matrix $D = [D_{kl}]$ remains symmetric and Hadamard equivalent to $A^t \times A$. Furthermore

(i) $$D_{ll} = [a_{li} a_{lj} a_{li} a_{lj}] = [1] = J = \text{the matrix of ones, for each}$$
$$l = 1, 2, \ldots, n,$$

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(ii)
$$\sum_i a_{li}a_{kj}a_{lj}a_{ki} = a_{kj}\left(\sum_i a_{li}a_{ki}\right)a_{lj} = \begin{cases} 0 & \text{if } l \neq k, \\ n & \text{if } l = k. \end{cases}$$

Similarly,

$$\sum_j a_{li}a_{kj}a_{lj}a_{ki} = \begin{cases} 0 & \text{if } l \neq k, \\ n & \text{if } l = k. \end{cases}$$

Consequently $D_{kl}J = JD_{kl} = \delta_{lk}nJ$. Hence D is Bush-type Hadamard. Note that each of the blocks in D is symmetric.

To the best of our knowledge the existence of a symmetric $W(n^2, p^2)$, as proved here, is new. We refer the reader to [1, pages 215–217] for more information on symmetric weighing matrices and their applications.

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Department of Mathematics
University of Alberta
Edmonton, Alberta
Canada
T6G 2G1

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