Glasgow Math. J. **50** (2008) 433–435. © 2008 Glasgow Mathematical Journal Trust.
doi:10.1017/S001708950800431X. Printed in the United Kingdom

AN INEQUALITY RELATED TO THE GEHRING–HALLENBECK THEOREM ON RADIAL LIMITS OF FUNCTIONS IN THE HARMONIC BERGMAN SPACES*

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(Received 20 October 2006; accepted 26 February 2008)

Abstract. For a function u harmonic in the unit disk $\mathbb{D}$, there holds the inequality

$$\int_0^{2\pi} M_{p,\beta}u(e^{i\theta})\,d\theta \le C_{p,\beta}\int_{\mathbb{D}} |u(z)|^p(1-|z|)^\beta\,dm(z),$$

where $p > 0$ and $\beta > -1$, and

$$M_{p,\beta}u(e^{i\theta}) = \sup_{0<r<1} |u(re^{i\theta})|^p(1-r)^{\beta+1}.$$

2000 *Mathematics Subject Classification.* 30D55, 46E15.

Let $\mathbb{D}$ be the open unit disk of the complex plane. The following theorem was proved by Gehring [**2**] for $p > 1$ and by Hallenbeck [**3**] for $0 < p \le 1$.

THEOREM A. *If u is a function harmonic in $\mathbb{D}$ such that*

$$I(u) := \int_{\mathbb{D}} |u(z)|^p(1-|z|)^\beta\,dm(z) \ < \infty \tag{1}$$

where $p > 0$, $\beta > -1$, then

$$\lim_{r\to 1^-} |u(re^{i\theta})|^p(1-r)^{\beta+1} = 0, \quad \textit{for almost all } \theta \in [0, 2\pi]. \tag{2}$$

Here dm stands for the Lebesgue measure in the plane. The class of harmonic functions satisfying (1) is called the harmonic Bergman space a^p_β. Various generalizations of this result can be found in [**5**–**8**].

Here we prove that the convergence in (2) is dominated. In order to state the result we define the maximal function $M_{p,\beta}u$ by

$$M_{p,\beta}u(e^{i\theta}) = \sup_{0<r<1} |u(re^{i\theta})|^p(1-r)^{\beta+1}.$$

THEOREM 1. *If u is a function harmonic in $\mathbb{D}$ satisfying* (1)*, where $p > 0$, $\beta > -1$, then*

$$J(u) := \int_0^{2\pi} M_{p,\beta}u(e^{i\theta})\,d\theta \ < \infty. \tag{3}$$

*Supported by MNZŽS Serbia, Project No. ON144010.

Moreover, there is a constant $C = C_{p,\beta}$ such that $J(u) \le CI(u)$.

Before proving this theorem, we show how it can be used to prove Theorem A. Namely, let u satisfy (1), and let

$$Tu(e^{i\theta}) = \limsup_{r\to 1^-} |u(re^{i\theta})|(1-r)^{(\beta+1)/p},$$

and $u_\rho(e^{i\theta}) = u(\rho e^{i\theta})$, $0 < \rho < 1$. Since $T(u_\rho)(e^{i\theta}) = 0$ for all θ, and $Tu \le T(u_\rho) + T(u - u_\rho)$, we have, by Theorem 1,

$$\begin{aligned}\int_0^{2\pi} \{Tu(e^{i\theta})\}^p \, d\theta &\le \int_0^{2\pi} \{T(u-u_\rho)(e^{i\theta})\}^p \, d\theta \\ &\le CI(u-u_\rho), \qquad 0<\rho<1.\end{aligned}$$

Since $\lim_{\rho\to 1} I(u-u_\rho) = 0$ (this is well known and easy to see), we have $Tu(e^{i\theta}) = 0$ for almost all θ.

For the proof of Theorem 1, we need the inequality

$$\sup_{|z-a|<\varepsilon} |u(z)|^p \le \frac{C_p}{\varepsilon^2} \int_{|z-a|<2\varepsilon} |u(z)|^p \, dm(z) \tag{4}$$

due to Hardy and Littlewood [**4**] and Fefferman and Stein [**1**], in the case $0 < p < 1$. In the case $p \ge 1$, this is a consequence of the sub-harmonicity of $|u|^p$.

Proof of Theorem 1. Let $r_j = 1 - 2^{-j}$, $j \ge 0$. Then

$$J(u) \le \int_0^{2\pi} d\theta \sum_{j=0}^{\infty} 2^{-j(\beta+1)} \sup_{r_j\le r\le r_{j+1}} |u(re^{i\theta})|^p. \tag{5}$$

For a fixed θ, let $a_j = (r_j + r_{j+1})e^{i\theta}/2$ and $\varepsilon_j = (r_{j+1} - r_j)/2 = 2^{-j-2}$. From (4), we conclude that

$$2^{-j(\beta+1)} \sup_{r_j\le r\le r_{j+1}} |u(re^{i\theta})|^p \le C2^{-j(\beta+1)}2^{2j} \int_{|z-a_j|<2^{-j-1}} |u(z)|^p \, dm(z). \tag{6}$$

On the other hand, simple calculation shows that $|z - a_j| \le 2^{-j-1}$ implies

$$2^{-j-2} \le 1 - |z| \quad \text{and} \quad |z - e^{i\theta}| \le 3 \times 2^{-j-2} \le 2^{-j-1}.$$

Hence,

$$2^{-j}2^{2j} \le 2^4 P(z, e^{i\theta}), \quad \text{for } |z - a_j| < 2^{-j-1},$$

where $P(z, e^{i\theta})$ denotes the Poisson's kernel

$$P(z, e^{i\theta}) = \frac{1-|z|^2}{|z-e^{i\theta}|^2}.$$

From this and (6), we get

$$\begin{aligned}2^{-j(\beta+1)} \sup_{r_j\le r\le r_{j+1}} |u(re^{i\theta})|^p &\le C2^{-j\beta} \int_{r_{j-1}\le|z|\le r_{j+2}} P(z, e^{i\theta})\, |u(z)|^p \, dm(z) \\ &\le C \int_{r_{j-1}\le|z|\le r_{j+2}} (1-|z|)^\beta P(z, e^{i\theta})\, |u(z)|^p \, dm(z)\end{aligned}$$

https://doi.org/10.1017/S001708950800431X Published online by Cambridge University Press

where $r_{-1} = 0$ and we have used the inclusion

$$\{z\colon |z - a_j| \le 2^{-j-1}\} \subset \{z\colon r_{j-1} \le |z| \le r_{j+2}\}.$$

Hence, by summation from

$j = 0$ to ∞, we get

$$\sum_{j=0}^{\infty} 2^{-j(\beta+1)} \sup_{r_j \le r \le r_{j+1}} |u(re^{i\theta})|^p \le C \int_{\mathbb{D}} (1 - |z|)^{\beta} P(z, e^{i\theta})\, |u(z)|^p \, dm(z).$$

Now we integrate this inequality over $\theta \in [0, 2\pi]$ and use the formula

$$\int_0^{2\pi} P(z, e^{i\theta})\, d\theta = 2\pi$$

together with (5) to get $J(u) \le CI(u)$, which was to be proved. □

REMARK 1. If $p > 1$ or if $p > 0$ and u is holomorphic, then the proof can be made shorter. Namely, we can apply the Hardy–Littlewood maximal theorem to get

$$\int_0^{2\pi} \sup_{r_j \le r \le r_{j+1}} |u(re^{i\theta})|^p \, d\theta \le C_p \int_0^{2\pi} |u(r_{j+1} e^{i\theta})|^p \, d\theta.$$

From this and (5), it follows that

$$\begin{aligned} J(u) &\le C_p \sum_{j=0}^{\infty} 2^{-j(\beta+1)} \int_0^{2\pi} |u(r_{j+1} e^{i\theta})|^p \, d\theta, \\ &\le C_p \int_0^1 (1 - r)^{\beta} r \, dr \int_0^{2\pi} |u(re^{i\theta})|^p \, d\theta, \end{aligned}$$

where we have used the 'increasing' property of the integral means.

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