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THIRD ENGEL GROUPS

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Dedicated to our teacher B.H. Neumann on the occasion of the eightieth anniversary of his birth

We present some new results on third Engel groups which are motivated by computer calculations but are not dependent on them. They include:

- for $n > 2$ every n-generator third Engel group is nilpotent of class at most $2n - 1$;
- the fifth term of the lower central series of a third Engel group has exponent dividing 20;
- the subgroup generated by fifth powers of elements in a third Engel group is nearly centre-by-metabelian;

and a normal form theorem for freest third Engel groups without elements of order 2.

1. INTRODUCTION

Macdonald and Neumann [11, p.557] conjectured: "For each positive integer n there is a finite 5-group with the properties that every 2 elements generate a subgroup of class 3 and that the group itself has class precisely n." They observed that "This is at variance with a "folk-lore" belief that the prime 5 should not be exceptional" in this context. The context was third Engel groups; that is groups in which the (left-normed) commutator $[b, a, a, a]$ is trivial for all elements a, b. Heineken [8] had proved that third Engel groups are locally nilpotent and that the fifth term of the lower central series of a third Engel group is a torsion group in which the only primes that can occur as orders of elements are 2 and 5. It was known that the prime 2 occurs essentially in the sense that there are non-nilpotent third Engel 2-groups. An example is the wreath product of a cyclic group of order 2 by an infinite elementary abelian 2-group (see Cohn [2] for an early example). Macdonald and Neumann were led to their conjecture by finding such a group for $n = 5$. The conjecture was shown to be correct by Bachmuth and Mochizuki [1, Theorem 1] when they showed there is an insoluble third Engel group of exponent 5.

In this paper we report on an investigation into third Engel groups which sheds further light on the structure of these groups. We recall first some other results which

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have been established for third Engel groups. In third Engel groups the subgroup generated by all fifth powers is soluble (Narain Gupta [4]). For all $n \geqslant 2$ there is an n-generator third Engel group which is nilpotent of class at least $2n - 1$; such groups can be found in the Bachmuth and Mochizuki example because a group all of whose n-generator subgroups are nilpotent of class at most $2n-2$ is soluble (Heineken [9], see Remark). For groups of exponent 5 this is the highest class which can occur (Bachmuth and Mochizuki [1, Theorem 2]). In general every n-generator third Engel group is nilpotent of class at most $2n$ (Kappe and Kappe [10, Corollary]). There is a torsionfree third Engel group of class exactly 4 (*ibid.*, Example 1). The fifth term of the lower central series of a third Engel group has exponent dividing 4×20^6 (combine Heineken [8, Satz 3] and Narain Gupta [5, Corollary 2.5.4]). There is an n-generator third Engel group of exponent 4 which is nilpotent of class at least $n+2$ (C.K. Gupta [3]). On the other hand (Vaughan-Lee [15, Lemma 5]) every third Engel group of exponent 4 is centre-by-metabelian and hence (Tobin [14, Lemma 5.9C]) every n-generator subgroup is nilpotent of class at most $n+2$. Combining the result of Vaughan-Lee with an argument of Gupta (see Narain Gupta [5, Corollary 2.5.7]) shows that the soluble length of the subgroup generated by fifth powers is at most 7.

Our investigation started from the observation that because the critical torsion in third Engel groups is 2-torsion and 5-torsion and because n-generator third Engel groups have nilpotency class at most $2n$ it should be possible to get quite detailed insight into the structure of the free third Engel groups of small rank by using the Canberra Nilpotent Quotient Program (see Havas and Newman, [7]). Heineken [8, Satz 1] had got very detailed information for the free third Engel group of rank 2; it is metabelian and nilpotent of class at most 4 with the fourth term of its lower central series having order at most 2. The order is actually 2 as the example of C.K. Gupta shows. With the program we were able to establish that the free third Engel group of rank 3 has class exactly 5 and that the fifth term of its lower central series has order 10^3. In particular the example of Macdonald and Neumann [11] is the 3-generator free third Engel group with exponent 5. The free third Engel group of rank 4 has class 7 and the seventh term of its lower central series has order 5^4. Moreover we could observe, for example, that $[b, a, a, b, c]$ is trivial for all elements a, b, c in a third Engel group. While the work which follows is motivated by these results, it is independent of them.

What we show is that the 5-torsion can be fully described in terms of the example of Bachmuth and Mochizuki and the 2-torsion is close to being described in terms of the example of C.K. Gupta. Just how close remains an open question. We list some of the results we prove. The term *nearly centre-by-metabelian* is defined later; suffice it to say here that every nearly centre-by-metabelian group is second-centre-by-metabelian.

THEOREM 4.2. *For $n > 2$ every n-generator third Engel group is nilpotent of class at most $2n-1$.*

THEOREM 4.1. *Every n-generator third Engel Group with no element of order 5 is nilpotent of class at most $n+2$ and is nearly centre-by-metabelian.*

THEOREM 4.4. *The fifth term of the lower central series of a third Engel group has exponent dividing 20 and this is best possible.*

THEOREM 4.5. *The subgroup generated by fifth powers of elements in a third Engel group is nearly centre-by-metabelian.*

As these results show, the behaviour of the 2-torsion and the 5-torsion in third Engel groups is rather different. We find it convenient to divide our discussion into two main parts. We discuss 2-torsionfree third Engel groups in Section 3 and there prove a normal form theorem (Theorem 3.9). From this we can deduce (Corollary 3.10): every 2-torsionfree third Engel group can be represented as a section of a direct product of two third Engel groups one of exponent 5 and the other nilpotent of class at most 4; the order of a free n-generator third Engel group of exponent 5 is 5^M where $M = \sum_{k=1}^{n} f_{2k}\binom{n}{k}$ and f_{2k} is the $2k$-th Fibonacci number. We discuss 5-torsionfree groups in Section 4. General results are also given in Section 4. We begin, in Section 2, with a treatment of the 3-generator groups. The methods used are primarily commutator calculations.

2. THE 3-GENERATOR GROUPS

Our notation is fairly standard: $[a, b] = a^{-1}b^{-1}ab$, $[a, b, c] = [[a, b], c]$ and $\emptyset$ denotes the identity element. The n-th term of the lower central series of G is written $\gamma_n(G)$ and the identity subgroup E. We say u is congruent to v modulo H when uv^{-1} is in H and write $u \equiv v$ modulo H.

Our whole discussion depends on Heineken's result [**8**, Hauptsatz 2] that every finitely generated third Engel group is nilpotent. His result is based on a careful analysis of 2-generator third Engel groups which can be summed up as follows.

THEOREM 2.1. *In a third Engel group generated by $\{a, b\}$ the commutators $[b,a,a,b]^2$, $[b,a,a,b,a]$, $[b,a,a,b,b]$, $[b,a,a,[b,a]]$, and $[b,a,b,[b,a]]$ are all trivial and $[b,a,a,b] = [b,a,b,a]$.*

Note that Heineken uses right-norming for commutators rather than our left-norming. In this context the two are equivalent (Macdonald and Neumann [**11**, Lemma 1]).

We base our commutator calculations on the following well-known identities which we use without further reference except to indicate use of the last one (in any form) by

"Jacobi":

$$[a,\ b] = [b,\ a]^{-1},$$
$$[ab,\ c] = [a,\ c][a,\ c,\ b][b,\ c],$$
$$[a,\ bc] = [a,\ c][a,\ b][a,\ b,\ c],$$
$$[b,\ a^{-1}] = [b,\ a]^{-1}[a,\ b,\ a^{-1}],$$
$$[c,\ [b,\ a]] = [c,\ b,\ a][c,\ a,\ b]^{-1}u$$

where u is a product of commutators with entry set $\{a,\ b,\ c\}$ and weight at least 4.

Let G be a third Engel group generated by $\{a,\ b,\ c\}$. We know that G is nilpotent. We will prove that $\gamma_6(G) = E$. So we can assume $\gamma_7(G) = E$. We begin with some congruences modulo $\gamma_5(G)$.

LEMMA 2.2. *Modulo* $\gamma_5(G)$

(i) $[c,a,a,b][c,a,b,a][c,b,a,a][c,a,b,b][c,b,a,b][c,b,b,a] \equiv \emptyset$,

(ii) $[c,a,a,b]^2[c,a,b,a]^2[c,b,a,a]^2 \equiv \emptyset$,

(iii) $[b,a,a,c] \equiv [c,b,a,a]^{-3}[c,a,a,b]^{-3}$,

(iv) $[c,a,a,b]^4[c,b,a,a]^6 \equiv \emptyset$,

(v) $[c,b,a,a]^2 \equiv [c,a,b,a]^4$.

PROOF:

(i) Expand $[c,ab,ab,ab] = \emptyset$.

(ii) Replace a by a^{-1} in (i) and multiply the result by (i).

(iii) Use Jacobi several times to unfold:
$[c,[b,a,a]] \equiv [c,[b,a],a][c,a,[b,a]]^{-1} \equiv [c,b,a,a][c,a,b,a]^{-2}[c,a,a,b]$,
and multiply the result by (ii).

(iv) Expand $[bc,a,a,bc]^2 = \emptyset$ and use (iii).

(v) Combine (ii) and (iv).

∎

These relations tell the whole story for class at most 4. A group generated by $\{a,\ b,\ c\}$ which is nilpotent of class at most 4 and satisfies these relations and the 2-variable relations of Lemma 2.1 is a third Engel group. This can be proved by a routine expansion (which we omit) of the commutator $[y,\ x,\ x,\ x]$ where $x,\ y$ are arbitrary words. Thus in a 3-generator free third Engel group F the section $\gamma_4(F)/\gamma_5(F)$ has torsionfree rank 3 and the torsion subgroup is elementary abelian of order 2^{10}.

It should be observed that the elements of $\gamma_5(G)$ which are implicit in the above congruences are products of commutators of weight at least 5 with entry set $\{a,\ b,\ c\}$. It follows that when, as in the proof of Lemma 2.3(ii) below, an entry is replaced by a commutator of weight 2 the corresponding elements lie in $\gamma_6(G)$. Appropriate variations on this observation are often used without comment later.

Let $w_1 = [c,a,a,b,b]$, $w_2 = [c,a,b,a,b]$, $w_3 = [c,a,b,b,a]$, $w_4 = [c,b,a,a,b]$, $w_5 = [c,b,a,b,a]$, $w_6 = [c,b,b,a,a]$.

LEMMA 2.3. *Modulo* $\gamma_6(G)$

(i) $[c,a,b,a,a] \equiv \emptyset$,

(ii) $[c,a,a,b,a] \equiv \emptyset$,

(iii) $w_3 \equiv w_1^{-1}w_2^{-1}$, $w_4 \equiv w_3$, $w_5 \equiv w_2$, $w_6 \equiv w_1$,

(iv) $[b,a,a,b,c] \equiv \emptyset$,

(v) $w_2 \equiv w_1^3$, $w_3 \equiv w_1^{-4}$,

(vi) $w_1^{10} \equiv \emptyset$.

PROOF:

(i) Expand $[cb,a,a,a] = \emptyset$.

(ii) Replace c by $[c,a]$ in Lemma 2.2(iii) and use (i) to get $[b,a,a,[c,a]] \equiv \emptyset$; then use Jacobi.

(iii) Replace c by $[c,a]$ and $[c,b]$ respectively in Lemma 2.2(i) to get $w_1w_2w_3 \equiv \emptyset$ and $w_4w_5w_6 \equiv \emptyset$. Commuting Lemma 2.2(i) by b gives $w_1w_2w_4 \equiv \emptyset$. Expand $[c,ab,a,ab,ab] \equiv \emptyset$ to obtain $w_1w_4w_5 \equiv \emptyset$. The result follows.

(iv) Use Jacobi to unfold: $[c,[b,a,a,b]] \equiv w_1w_2^{-2}w_5^2w_6^{-1}$.

(v) Expand $[bc,a,a,bc,b] \equiv \emptyset$ to get $[b,a,a,c,b]w_1 \equiv \emptyset$. Use Lemma 2.2(iii) to get $w_1^{-2}w_4^{-3} \equiv \emptyset$ and hence, using (iii), the result.

(vi) Replacing c by $[c,b]$ in Lemma 2.2(v) yields $w_6^2 \equiv w_5^4$ and hence the result.

∎

It follows from the above that $\gamma_5(G)/\gamma_6(G)$ has order dividing 10^3. The examples of Macdonald and Neumann and of C.K. Gupta together show that in 3-generator free third Engel groups equality holds here. So no essentially new congruences can be found.

THEOREM 2.4. *A 3-generator third Engel group is nilpotent of class at most* 5.

PROOF: We retain the notation above. Recall we are assuming $\gamma_7(G) = E$. Clearly $[w_1,b] = \emptyset$ and, using Lemma 2.3(iii), $[w_1,a] = \emptyset$. So it remains to prove $[w_1,c] = \emptyset$. Expanding $[ad,b,b,ad,c] \equiv \emptyset$ gives $[a,b,b,d,c][d,b,b,a,c]u = \emptyset$ where u is a product of commutators of weight 6 with entry set $\{a,b,c,d\}$. Replacing d by $[c,a]$ gives $[a,b,b,[c,a],c][c,a,b,b,a,c] = \emptyset$. Unfolding $[c,a,[a,b,b],c]$ and using Lemma 2.3 gives $[c,a,a,b,b,c]^5 = \emptyset$.

Expanding $[c[c,b,b],a,a,c[c,b,b]]^2 = \emptyset$ gives $[c,b,b,a,a,c]^2[c,a,a,[c,b,b]]^2 = \emptyset$. Hence unfolding and using Lemma 2.3 gives $[c,b,b,a,a,c]^4[c,a,a,b,c,b]^{-4} = \emptyset$. Substituting b,c and $[c,a,a]$ respectively for a, b and c in Lemma 2.2(ii) and using Lemma 2.3(iv) gives $[c,a,a,b,b,c]^2[c,a,a,b,c,b]^2 = \emptyset$. Hence $[c,a,a,b,b,c]^8 = \emptyset$ and so $[c,a,a,b,b,c] = \emptyset$ as required. ∎

COROLLARY 2.5.

(i) *The statements in* Lemma 2.3 *are relations.*

(ii) $[c,a,b,b,a]^5 = \emptyset$.

(iii) (Kappe and Kappe [**10**, Corollary]) *In third Engel groups all commutators with a triple entry are trivial.*

PROOF: (i) is obvious. From Lemma 2.3 $w_3^5 = \emptyset$ which is (ii). Turn now to (iii). Let u be a product of conjugates of a, a^{-1}, then $[u,v] = u^{-1}u^v$ is also such a product. Hence every commutator with an entry a can be written as a product of conjugates of $a^{\pm 1}$. It follows that every commutator with two entries a can be written as a product of conjugates of commutators of the form $[a^{\pm u},\, a^{\pm v}]$ and every commutator with three entries a can be written as a product of conjugates of commutators of the form $[a^{\pm u},\, a^{\pm v},\, a^{\pm w}]$. Now $[a^u,\, a^v] = [a,\, a^{vu^{-1}}]^u = [a^{uv^{-1}},\, a]^v$ and $[a^{-u},\, a^{\pm v}] = [a^u,\, a^{\pm v}]^{-a^{-u}}$, so $[a^{\pm u},\, a^{\pm v},\, a^{\pm w}]$ can be written as a conjugate of $[a^b,\, a,\, a^c]^{\pm 1}$. Since $[a^b,\, a,\, a^c] = [a[a,\, b],\, a,\, a[a,\, c]]$, expansion in a third Engel group gives $[a^b,\, a,\, a^c] = \emptyset$ and the result follows. ∎

3. THE 2-TORSIONFREE GROUPS

Let G be a 2-torsionfree third Engel group with a generating set A. We begin by determining the exponent of $\gamma_5(G)$. Every 2-generator subgroup of G is nilpotent of class at most 3. So the (P.) Hall collection process (see, for example, M. Hall [**6**, Theorem 12.3.1]) yields $(ab)^5 = a^5b^5[b,a]^{10}[b,a,a]^{10}[b,a,b]^{30}$. From this it is routine to deduce a "regularity" result

$$(1) \qquad (c_1 \dots c_t)^5 = c_1^5 \dots c_{t+q}^5$$

where $c_{t+1}, \dots, c_{t+q}$ are suitable powers of commutators in $c_1, \dots, c_t$. This is the key to the following theorem.

THEOREM 3.1. *The fifth term of the lower central series of a 2-torsionfree third Engel group has exponent dividing 5.*

PROOF: We begin by working modulo $\gamma_6(G)^5$. Expanding $[a,\, bcd,\, bcd,\, a] = \emptyset$ gives

$$[a,b,c,d,a]^2[a,b,d,c,a][a,c,b,d,a][a,c,d,b,a][a,d,b,c,a]u = \emptyset$$

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where u is a product of commutators of weight at least 6. Taking fifth powers and using (1) gives

$$[a,b,c,d,a]^{10}[a,b,d,c,a]^5[a,c,b,d,a]^5[a,c,d,b,a]^5[a,d,b,c,a]^5 \equiv \emptyset.$$

Expanding $[a, bc, d, bc, a]^5 \equiv \emptyset$ gives $[a,b,d,c,a]^5[a,c,d,b,a]^5 \equiv \emptyset$, and similarly $[a,c,b,d,a]^5[a,d,b,c,a]^5 \equiv \emptyset$. Therefore $[a,b,c,d,a]^{10} \in \gamma_6(G)^5$. By a theorem of Heineken and Macdonald (see Hanna Neumann [**12**, Theorem 34.33])

$$[a,\, b,\, c,\, d,\, e] = \prod_i [a_i,\, b_i,\, c_i,\, d_i,\, a_i]^{\pm 1}$$

for some a_i, b_i, c_i, d_i in the subgroup generated by $\{a, b, c, d, e\}$. It follows that $[a, b, c, d, e]^{10} \in \gamma_6(G)^5$. Hence $[a_1,\ldots,a_i]^{10} \in \gamma_{i+1}(G)^5$ for all $i \geqslant 5$. Every element of G when written as a word in A involves only finitely many elements of A. So, without loss of generality, A can be taken to be finite and G to be nilpotent. We prove by reverse induction on $i \geqslant 5$ that $\gamma_i(G)^5 = E$. Since G is nilpotent there is a k such that $\gamma_k(G)$ is trivial. For $5 \leqslant i < k$ suppose inductively that $\gamma_{i+1}(G)^5 = E$. Therefore $[a_1,\ldots,a_i]^{10} = \emptyset$. Since G is 2-torsionfree $[a_1,\ldots,a_i]^5 = \emptyset$. Hence $\gamma_i(G)^5 = E$, using (1). It follows that $\gamma_5(G)^5 = E$ as required. ∎

COROLLARY 3.2.

(i) *In a third Engel group* $[a,\, b,\, c,\, d,\, e]$ *has order dividing* 5.2^k *for some* k.

(ii) (Heineken [**8**, Hauptsatz 1]) *A third Engel group with no elements of order 2 or 5 is nilpotent of class at most 4.*

(iii) G^5 *is a subgroup of the fourth centre of* G.

PROOF: Parts (i) and (ii) are immediate consequences of the theorem. Part (iii) comes from expanding $[a^5,\, b,\, c,\, d,\, e]$ repeatedly using the collection process. ∎

We turn now to the commutator structure of 2-torsionfree third Engel groups. Since all commutators of weight at least 5 have order dividing 5 we can reduce all exponents modulo 5 and in particular avoid inverses.

LEMMA 3.3. *Modulo* $\gamma_6(G)$

(i) $[a,d,d,b,c] \equiv [a,b,d,d,c] \equiv [a,b,c,d,d]$,

(ii) $[a,d,b,d,c] \equiv [a,b,d,c,d] \equiv [a,b,c,d,d]^3$.

PROOF:

(i) This follows from Lemma 2.2(iv).

(ii) This follows from Lemma 2.2(v) and (i).

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LEMMA 3.4.

(i) (Shifting) *In a left-normed commutator a pair of neighbouring equal entries can be moved past any other entry without changing the value of the commutator modulo commutators of higher weight:* $[\ldots, a, a, b, \ldots] \equiv [\ldots, b, a, a, \ldots]$.

(ii) (Closing) *For* $s \geqslant 3$, $[a, b_1, \ldots, b_s, a]$ *can be written as a product of commutators of weight* $s+2$ *with the same entry set and neighbouring* a*'s and of commutators of higher weight.*

PROOF:

(i) This follows from Lemma 3.3(i).

(ii) We use induction on s. Expanding $[ab_s a, b_1, \ldots, b_{s-1}, ab_s a, ab_s a] = \emptyset$, replacing a by a^{-1} in the result, multiplying together the two congruences so obtained and using the 2-torsionfreeness gives $[a, b_1, \ldots, b_s, a]$ $[b_s, b_1, \ldots, a, a][a, b_1, \ldots, a, b_s] \equiv \emptyset$ modulo higher weight and the result follows using Lemma 3.3(ii) to start the induction. ∎

We pointed out in the introduction that there are n-generator third Engel groups with class at least $2n-1$. We now show this is best possible for 2-torsionfree third Engel groups (and in Theorem 4.2 for all third Engel groups).

THEOREM 3.5. *Every* n*-generator 2-torsionfree third Engel group is nilpotent of class at most* $2n-1$.

PROOF: We use induction on n. For $n \leqslant 3$ the result has been established. Let A be an n-element generating set for a 2-torsionfree third Engel group. We show that every left-normed commutator u of weight $2n$ with entries from A is trivial. By Corollary 2.5(iii) there is nothing to prove unless each generator occurs exactly twice as an entry of u. Let a be an element of A, then closing the entries a by Lemma 3.4 shows that u can be written as a product of commutators $[v, a, a]$ where v is a commutator of weight $2n-2$ in $n-1$ generators. By induction, v is trivial and the result follows. ∎

Note that it also follows by closing that in an n-generator 2-torsionfree third Engel group $\gamma_{2n-1}(G)$ is generated by the n commutators $[a_i, a_1, a_1, \ldots, a_n, a_n]$.

We move now to a more detailed analysis of the commutator structure of 2-torsionfree third Engel groups which is summed up in Theorem 3.8 below. The main technical issue is to find a spanning set for the subgroup which is generated by the commutators of weight n with entry set $\{a_1, \ldots, a_n\}$ (that is with no repeats).

LEMMA 3.6. *Every commutator of weight* 5 *in* G *with entry set* $\{a_1, a_2, a_3, a_4, a_5\}$ *can be written as a product of the commutators*

$$[a_1, a_2, a_3, a_4, a_5],\ [a_1, a_2, a_3, a_5, a_4],\ [a_1, a_2, a_4, a_3, a_5],$$
$$[a_1, a_3, a_2, a_4, a_5],\ [a_1, a_3, a_2, a_5, a_4],$$

and commutators of weight at least 6 *with the same entry set.*

PROOF: We work modulo $\gamma_6(G)$. Using Jacobi shows that it suffices to consider left-normed commutators with first entry a_1. We deal first with commutators of the form $[a_1, a_2, \ldots]$. By Lemma 3.3 $[a_1, a_2, a_3a_4, a_5, a_3a_4][a_1, a_2, a_3a_4, a_3a_4, a_5]^2 \equiv \emptyset$. Expanding this shows that $[a_1, a_2, a_4, a_5, a_3]$ can be written as claimed. Similarly $[a_1, a_2, a_5, a_3, a_4]$ can be so written. Expanding $[a_1, a_2, a_3a_4a_5, a_3a_4a_5, a_3a_4a_5] = \emptyset$ shows that $[a_1, a_2, a_5, a_4, a_3]$ can be written as claimed.

By Lemma 3.4 $[a_1, a_2a_3, \ldots, a_2a_3, \ldots]$ can be written modulo commutators of higher weight as a product of commutators of the form $[a_1, a_2a_3, a_2a_3, \ldots]$. Expanding these congruences shows that every commutator of the form $[a_1, a_3, \ldots]$ can be written as claimed.

Next we deal with commutators of the form $[a_1, a_i, a_2, \ldots]$ where $i = 4, 5$. Expanding $[a_1, a_3a_4, a_2, a_3a_4, a_5][a_1, a_2, a_3a_4, a_3a_4, a_5]^2 \equiv \emptyset$ shows that $[a_1, a_4, a_2, a_3, a_5]$ can be written as claimed. Similarly $[a_1, a_5, a_2, a_3, a_4]$ can be written as claimed. By Lemma 2.2(v) $[a_1, a_3[a_2, a_4], a_5, a_3[a_2, a_4]]^{-4}[a_1, a_3[a_2, a_4], a_3[a_2, a_4], a_5]^2 \in \gamma_5(G)$. Therefore $[a_1, a_3, a_5, [a_2, a_4]][a_1, [a_2, a_4], a_5, a_3][a_1, a_3, [a_2, a_4], a_5]^2\ [a_1, [a_2, a_4], a_3, a_5]^2 \in \gamma_6(G)$ and thus $[a_1, a_4, a_2, a_5, a_3]$ can be written as claimed. Similarly $[a_1, a_5, a_2, a_4, a_3]$ can be so written. Finally replacing 3 by 4 and 5 respectively in the second paragraph completes the proof. ∎

We now generalise. We begin by defining suitable sets of left-normed commutators. Let A be an ordered set of elements of a group. Define $B_1(A)$ to be A and $B_2(A)$ to be the set of commutators $[a_1, a_2]$ with $a_1 < a_2$ in A. For $n \geqslant 3$ define $B_n(A)$ to be the set of commutators $[u, a]$, $[v, a'', a']$ where $u \in B_{n-1}(A)$ and a exceeds every entry of u and where $v \in B_{n-2}(A)$, a' exceeds every entry of v and $a' < a''$. Thus, for example, $B_5(\{a_1, \ldots, a_5\})$ is the set of commutators given in Lemma 3.6. As usual we write $|B|$ for the cardinality of B. Note that

$$|B_n(\{a_1, \ldots, a_n\})| = |B_{n-1}(\{a_1, \ldots, a_{n-1}\})| + |B_{n-2}(\{a_1, \ldots, a_{n-2}\})|,$$

so that $|B_n(\{a_1, \ldots, a_n\})|$ is the n-th Fibonacci number f_n.

LEMMA 3.7. *For every* $n \geqslant 5$ *every commutator in* G *with entry set* $\{a_1, \ldots, a_n\}$ *can be written as a product of commutators in* $B_n(\{a_1, \ldots, a_n\})$ *and commutators of weight at least* $n+1$ *with the same entry set.*

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PROOF: The case $n = 5$ has been proved above. Consider $n > 5$ and suppose inductively that every commutator of weight $n-1$ with no repeated entry can be written in the required form. By Jacobi it suffices to consider left-normed commutators with first entry a_1. Let w be a left-normed commutator of weight n with entry set $\{a_1, \ldots, a_n\}$. If the last entry of w is a_n the result follows at once from the inductive hypothesis. We proceed, by reverse induction on $i \in \{2, \ldots, n\}$, to prove the result when the last entry of w is a_i. Suppose $w = [u, a_i]$. There are two cases.

(a) a_{i+1} occurs before a_{i+2} in u.

In this case $u = [v, a_{i+1}, \ldots, a_i]$ will $v \in B_{i-1}(\{a_1, \ldots, a_{i-1}\})$. By Lemma 3.4 $[v, a_ia_{i+1}, \ldots, a_ia_{i+1}]$ can be written as a product of commutators of the form $[v, a_ia_{i+1}, a_ia_{i+1}, \ldots]$ and commutators of higher weight. Expanding this congruence gives that $[u, a_i]$ can be written, modulo higher weight commutators, as a product of commutators of weight n with last entry among $a_{i+1}, \ldots, a_n$ and so, inductively, is expressible in the required form.

(b) a_{i+2} occurs before a_{i+1} in u.

Here $u = [v, a_{i+2}, a_{i+1}, \ldots, a_i]$. Expanding $[v, a_ia_{i+1}a_{i+2}, a_ia_{i+1}a_{i+2}, \ldots, a_ia_{i+1}a_{i+2}]$ gives the result as before. ∎

Let G be generated by an ordered set A. We define a set C of left-normed commutators with entries from A. Let r and s be integers with $r \geqslant 0$, $s \geqslant 1$, $s + 2r \geqslant 5$. For every disjoint pair of finite subsets A_1, A_2 of A with cardinality s and r respectively C contains the f_s commutators of weight $s + 2r$ whose initial segment of length s is one of the f_s commutators in $B_s(A_1)$ and this is followed by r neighbouring pairs of entries of elements of A_2 in order. Thus if $A_1 = \{a_2, a_3, a_5\}$ and $A_2 = \{a_1, a_4\}$ the two commutators are $[a_2, a_3, a_5, a_1, a_1, a_4, a_4]$, $[a_2, a_5, a_3, a_1, a_1, a_4, a_4]$. There are no other elements in C. The elements in C are ordered first by weight, then by the number of single entries and then lexicographically.

THEOREM 3.8. *Every element of $\gamma_5(G)$ can be written*

$$\prod_{i=1}^{t} c_i^{\beta(i)} \tag{2}$$

where $c_i \in C$ and $c_1 < c_2 < \ldots < c_t$ and $\beta(i) \in \{1, 2, 3, 4\}$.

This can not be extended comfortably to the whole of G because of a perturbation at weight 4.

PROOF: Every element of G when written as a word in A involves only finitely many elements of A. So we can assume without loss of generality that A is finite and hence that G is nilpotent. Every element of $\gamma_5(G)$ can be written as a product of left-normed commutators of weight at least 5 with entries from A. Collect the commutators of lowest weight in such an expression to the left (using Jacobi to replace complex commutators by left-normed commutators). Use Lemma 3.7 to replace the commutators of lowest weight not in C by ones in C and ones of higher weight. Collect these commutators from C into order and reduce powers modulo 5. Repeat for the next lowest weight and so on. This process stops because G is nilpotent. □

We now show this result is best possible. Let F be the free 2-torsionfree third Engel group freely generated by A; that is, $F = F^*/T$ where F^* is the free third Engel group freely generated by A and T is its Sylow 2-subgroup. We do this by showing that (2) is a normal form for elements of $\gamma_5(F)$.

THEOREM 3.9. *Every element of $\gamma_5(F)$ can be uniquely written in the form* (2).

PROOF: Let $w = \prod_{i=1}^{t} c_i^{\beta(i)}$ be a non-trivial product of the form (2). It suffices to show that $w \neq \emptyset$. Suppose c_1 has s single entries and r double entries and label them $a_1, \ldots, a_{r+s}$ with $a_1 < \ldots < a_s$ and $a_{s+1} < \ldots < a_{s+r}$. We use induction on the total number $m(w)$ of elements of A which occur as entries in the factors $c_1, \ldots, c_t$ of w. Clearly $m(w) \geqslant r+1$. For $m(w) = r+1$ we proceed by induction on s. For $s = 1$ all of $c_2, \ldots, c_t$ have a double entry a_1. Let θ be the automorphism which inverts a_1 and fixes the other generators, then $w(w\theta)^{-1} = c_1^{2\beta(1)}$. The example of Bachmuth and Mochizuki shows that $c_1 \neq \emptyset$. It follows that $w \neq \emptyset$. For $s > 1$, Lemmas 3.3 and 3.4 can be used to write each $[c_i, a_s]$ in terms of C. It can then be seen that the leading term of $[w, a_s]$ is a non-trivial power of $[c_1, a_s]$ unless $c_1 = [u, a_{s-1}, a_s]$ and $c_2 = [u, a_s, a_{s-1}]$ and $\beta(2) \equiv 3\beta(1)$ modulo 5. In this outstanding case the leading term of $[w, a_{s-1}]$ is $[c_1, a_{s-1}]^{\beta(1)}$. So, by induction, either $[w, a_s] \neq \emptyset$ or $[w, a_{s-1}] \neq \emptyset$. It follows that $w \neq \emptyset$ as required. For $m(w) > r+1$ let δ be an endomorphism of F which maps to the identity an element of $A \setminus \{a_1, \ldots, a_{r+1}\}$ which occurs as an entry in a factor of w and fixes the other generators. Then $w\delta$ has leading factor $c_1^{\beta(1)}$ and $m(w\delta) < m(w)$. Hence, by induction, $w\delta \neq \emptyset$ and therefore $w \neq \emptyset$. □

COROLLARY 3.10.

(i) *For a (relatively) free group F of infinite rank the centre meets $\gamma_5(F)$ in E.*

(ii) *For all F the subgroup F^5 meets $\gamma_5(F)$ in E.*

(iii) *Every 2-torsionfree third Engel group can be written as a section of a direct product of two third Engel groups U and V where U has exponent 5 and V is nilpotent of class at most 4.*

(iv) *If F has finite rank n then its torsion subgroup has order 5^N where*

$$N = 3\binom{n}{3} + 18\binom{n}{4} + \sum_{k=5}^{n} f_{2k}\binom{n}{k}.$$

(v) *A free exponent 5 third Engel group K of finite rank n has order 5^M where*

(3) $$M = \sum_{k=1}^{n} f_{2k} \binom{n}{k}.$$

PROOF:

(i) This follows at once from the proof of Theorem 3.9 except when $s = 1$. In that case, and that only, we need to use another generator a_{r+2} and observe that $[c_1, a_{r+2}]$ is non-trivial.

(ii) By Corollary 3.2(iii) F^5 lies in the fourth centre and the result follows from (i).

(iii) It follows from (ii) that F is a subdirect product of a group of exponent 5 and a group of class at most 4. Hence the result.

(iv) The crux is to count the number of commutators in C with entry set a fixed k-element subset B of the generating set of F. For $k \geqslant 5$ and $j \in \{1, \dots, k\}$ the commutators in C with entry set B and weight $2k - j$ have j single entries and $k - j$ double entries. So there are $f_j\binom{k}{j}$ of them. Therefore there are $\sum_{j=1}^{k} f_j\binom{k}{j}$ commutators in C with entry set B. It is a nice exercise (which we omit) to show that $\sum_{j=1}^{k} f_j\binom{k}{j} = f_{2k}$. For $k = 3, 4$ it is routine to check using the above argument that there are, respectively, 3 and 18 commutators in C with entry set B. Since there are $\binom{n}{k}$ k-element subsets of the generating set of F the result follows.

(v) It follows from (iv) that it remains to consider commutators of weight at most 4. They are covered by the discussion in Section 2 except for commutators of weight 4 with four single entries. It can be shown, as in Lemma 3.6 and Theorem 3.9, that modulo $\gamma_5(K)$ the commutators $[a_1, a_2, a_3, a_4]$, $[a_1, a_2, a_4, a_3]$ and $[a_1, a_3, a_2, a_4]$ form a basis for the commutators with entry set $\{a_1, a_2, a_3, a_4\}$. It is then straight-forward to get the formula.

□

4. THE 5-TORSIONFREE GROUPS

Let G be a 5-torsionfree third Engel group. By Lemma 2.3(iv) and Corollary 2.5(ii), G satisfies the identities

(1) $$[b, a, a, b, c] = \emptyset,$$

(2) $$[c, a, b, b, a] = \emptyset.$$

Let Y be the subgroup of G generated by the central elements of order 2 and let $H = G/Y$. Then H satisfies (2) and

$$[b, a, a, b] = \emptyset.$$

We now work in H using these identities. Expanding $[bc, a, a, bc] = \emptyset$ yields

$$(3) \qquad [b, a, a, c][c, a, a, b]u = \emptyset$$

where u is a product of commutators of weight 5 with entries a, b, c (because all such commutators of weight 6 are trivial). Replacing c by $[c, b]$ in (3) and using (2) gives

$$[b, a, a, [c, b]] = \emptyset,$$

which by Jacobi and (1) gives

$$(4) \qquad [b, a, a, c, b] = \emptyset.$$

Commuting (3) on the right by b and using (4) yields

$$[b, a, a, c, c] = \emptyset.$$

Thus, using Lemma 2.3, every 3-generator subgroup of H has class at most 4. It follows (Newman [**13**, Theorem B]) that every n-generator subgroup of H has class at most $n + 1$.

THEOREM 4.1. (a) *Every n-generator 5-torsionfree third Engel group is nilpotent of class at most* $n + 2$.

This combines with Theorem 3.5 to give the following sharp upper bound.

THEOREM 4.2. *For* $n > 2$ *every* n*-generator third Engel group is nilpotent of class at most* $2n - 1$.

We now continue our analysis of H beginning with 4-generator subgroups. Note that they have class at most 5. Expanding $[a, bc, bc, d, d] = \emptyset$ gives

$$[a, b, c, d, d][a, c, b, d, d] = \emptyset.$$

Hence $[a, b, c, d, d] = [c, a, b, d, d] = [b, c, a, d, d]$, and so, using Jacobi,

$$[a, b, c, d, d]^3 = \emptyset.$$

It follows from Corollary 3.2(i) that

$$[a,b,c,d,d] = \emptyset. \tag{5}$$

Expanding $[a,bd,bd,a] = \emptyset$ yields $[a,b,d,a][a,d,b,a] = \emptyset$, so that, by Jacobi,

$$[a,b,d,a]^2[b,d,a,a] = \emptyset.$$

Replacing b by bc, expanding and using (5) gives

$$[a,b,c,d,a]^2 = \emptyset. \tag{6}$$

We now turn to 5-generator subgroups and recall they have class at most 6. By the theorem of Heineken and Macdonald (quoted earlier)

$$[a,b,c,d,e] = \prod_i [a_i,b_i,c_i,d_i,a_i]^{\pm 1}$$

for some a_i, b_i, c_i, d_i in the subgroup generated by $\{a,b,c,d,e\}$. It follows from (6) that

$$[a,b,c,d,e]^2 = \emptyset. \tag{7}$$

Expanding $[a,b,c,de,de] = \emptyset$ and using $[a,b,c,d,e,e] = \emptyset$ gives

$$[a,b,c,d,e][a,b,c,e,d][a,b,c,d,e,d][a,b,c,e,d,e] = \emptyset.$$

Commuting this with d, e respectively gives $[a,b,c,d,e,d] = \emptyset$ and $[a,b,c,e,d,e] = \emptyset$, so $[a,b,c,d,e][a,b,c,e,d] = \emptyset$ and, by Jacobi,

$$[a,b,c,[d,e]] = \emptyset. \tag{8}$$

This identity implies that H is centre-by-metabelian, but is much stronger.

We now derive information about G. First

$$[a,b,c,[d,e],f] = \emptyset. \tag{9}$$

We will say that a group is *nearly centre-by-metabelian* if it satisfies this identity. So we now have the following.

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THEOREM 4.1. *(b) A 5-torsionfree third Engel group is nearly centre-by-metabelian.*

Note that, in particular, $\gamma_3(G)$ has class at most 2. From (7) we have in G

$$[a,b,c,d,e]^4 = \emptyset, \tag{10}$$

$$[[a,b,c,d,e]^2, f] = \emptyset; \tag{11}$$

and hence

$$[a,b,c,d,e,f]^2 = \emptyset. \tag{12}$$

It follows routinely from these identities that $\gamma_5(G)$ has exponent 4.

THEOREM 4.3. *In a 5-torsionfree third Engel group the fifth term of the lower central series is nilpotent of class at most 2 and has exponent dividing 4.*

The example of C.K. Gupta shows these numbers are best possible.

Combining this result with Theorem 3.1 and the example of Bachmuth and Mochizuki gives the following general result.

THEOREM 4.4. *The fifth term of the lower central series of a third Engel group has exponent dividing 20 and this is best possible.*

It follows from (9) that in every third Engel group $[a,b,c,[d,e],f]^5 = \emptyset$, and so using Corollary 2.5(iii), $[a^5, b^5, c^5, [d^5, e^5], f^5] = \emptyset$. From this we get the following theorem.

THEOREM 4.5. *The subgroup generated by fifth powers of elements in a third Engel group is nearly centre-by-metabelian.*

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