SOME RESULTS ON ω-DERIVATIVES AND BV-ω FUNCTIONS

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1. Introduction

Let $\omega(x)$ be non-decreasing on the closed interval $[a, b]$. Outside the interval $\omega(x)$ is defined by $\omega(x) = \omega(a)$ for $x < a$ and $\omega(x) = \omega(b)$ for $x > b$. Let S denote the set of points of continuity of $\omega(x)$ and D denote the set of points of discontinuity of $\omega(x)$. R. L. Jeffery [5] has defined the class $\mathscr{U}$, of functions $f(x)$ as follows:

$f(x)$ is defined on the set $S \cdot [a, b]$ and $f(x)$ is continuous at each point of $S \cdot [a, b]$ with respect to the set S. If $x_0 \in D$ then $f(x)$ tends to a limit (finite or infinite) as x tends to x_0+ and x_0- over the points of the set S. These limits will be denoted by $f(x_0+)$ and $f(x_0-)$ respectively. When $x < a$, $f(x) = f(a+)$ and $f(x) = f(b-)$ when $x > b$. $f(x)$ may or may not be defined at points of the set D.

Let $\mathscr{U}_0$ denote the class of functions $f(x)$ of $\mathscr{U}$ for which $f(x_0+)$ and $f(x_0-)$ are finite, $x_0 \in D$.

In [5] Jeffery has also introduced the following definition:

DEFINITION 1.1: For any x and $h \neq 0$ with $x+h \in S$, the function $\psi(x, h)$ is defined by

$$\psi(x, h) = \begin{cases} \dfrac{f(x+h)-f(x-)}{\omega(x+h)-\omega(x-)}, & h > 0, \quad \omega(x+h)-\omega(x-) \neq 0 \\ \dfrac{f(x+h)-f(x+)}{\omega(x+h)-\omega(x+)}, & h < 0, \quad \omega(x+h)-\omega(x+) \neq 0 \\ 0, & \omega(x+h)-\omega(x\pm) = 0. \end{cases}$$

The upper and lower limits of $\psi(x, h)$ as $h \to 0+ (x+h \in S)$ are called respectively the Upper and Lower ω-derivatives of $f(x)$ at x on the right and are denoted by $D^+f_\omega(x)$ and $D_+f_\omega(x)$. If $D^+f_\omega(x) = D_+f_\omega(x)$, the common value is called the ω-derivative of $f(x)$ at x on the right and is denoted by $f'_{+\omega}(x)$. Similarly the left ω-derivatives $D^-f_\omega(x)$, $D_-f_\omega(x)$ and

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$f'_{-\omega}(x)$ of $f(x)$ are defined. If $f'_{+\omega}(x) = f'_{-\omega}(x)$, the common value is called the ω-derivative of $f(x)$ at x and is denoted by $f'_\omega(x)$.

Any set of points $a = x_0 < x_1 < x_2 < \cdots < x_n = b$ such that $\omega(x_{i-1}) \neq \omega(x_i)$ $(i = 1, 2, \cdots, n)$ is called an ω-subdivision ([1], [2]) of $[a, b]$. In [1] the following definition has been introduced.

DEFINITION 1.2: Let $f(x)$ be defined on $[a, b]$ and be in the class $\mathscr{U}$. The least upper bound of the sums

$$V = \sum_{i=1}^{n} |f(x_i+)-f(x_{i-1}-)|$$

for all possible ω-subdivisions $x_0, x_1, x_2, \cdots, x_n$ of $[a, b]$ is called the total ω-variation, $V_\omega(f; a, b)$, of $f(x)$ on $[a, b]$. If $V_\omega(f; a, b) < +\infty$, then $f(x)$ is said to be a function of bounded variation relative to ω, $BV-\omega$, on $[a, b]$.

The purpose of the present paper is to study some properties of ω-derivatives of a function $f(x) \in \mathscr{U}$ and to show that if $f(x)$ is $BV-\omega$ on $[a, b]$, then $f'_\omega(x)$ exists and is finite at all points of $[a, b]$ except on a set of ω-measure (§ 2) zero and that $f'_\omega(x)$ is summable (LS) (§ 2) on $[a, b]$. We require the following known results.

THEOREM 1.1. ([5], lemma 2). Let E be any set on $[a, b]$. Let each point x of E be the left hand end point of a sequence of closed intervals $[x, x+h_i]$ for which $h_i \to 0$. Let $\mathscr{F}$ denote the family of all intervals thus associated with the set E. Then for every $\varepsilon > 0$ there exists a finite family of pairwise disjoint closed intervals $\Delta_1, \Delta_2, \cdots, \Delta_N$ of $\mathscr{F}$ for which

$$\sum_{i=1}^{N} \omega^*(E\Delta_i) > \omega^*(E)-\varepsilon, \quad \sum_{i=1}^{N} |\Delta_i|_\omega < \omega^*(E)+\varepsilon,$$

where $\omega^*(E)$ denotes the outer ω-measure and $|E|_\omega$ the ω-measure (§ 2) of the set E.

THEOREM 1.2. This theorem is obtained from theorem 1.1 by replacing 'left hand' by 'right hand' and $[x, x+h_i]$ by $[x-h_i, x]$.

Throughout the paper the following notations will be used. S_0 denotes the union of pairwise disjoint open intervals (a_i, b_i) in $[a, b]$ on each of which $\omega(x)$ is constant, $S_1 = \{a_1, b_1, a_2, b_2, \cdots\}$, $S_2 = SS_1$, and $S_3 = [a, b] \cdot S-(S_0+S_2)$. Then $\omega(x_1) < \omega(x_2)$ for every pair x_1, x_2 with $x_1 < x_2$ where one of them at least is a member of S_3. If $f(x) \in \mathscr{U}$, then $f'_\omega(x) = 0$ on S_0 and $f'_\omega(x)$ exists at each point of D.

2. ω-measure of a bounded set and Lebesgue-Stieltjes integral

The ω-measure $|(\alpha, \beta)|_\omega$ ([5], § 1) of an open interval (α, β) is defined by $|(\alpha, \beta)|_\omega = \omega(\beta-)-\omega(\alpha+)$. The ω-measure $|G|_\omega$ of a bounded open set $G = \sum_i(\alpha_i, \beta_i)$, where the open intervals (α_i, β_i) are pairwise disjoint, is

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defined by $|G|_\omega = \sum_i |(\alpha_i, \beta_i)|_\omega$. If G is void, then $|G|_\omega = 0$. The ω-measure $|I|_\omega$ of a closed interval $I = [\alpha, \beta]$ is defined by $|I|_\omega = \omega(\beta+)-\omega(\alpha-)$. The ω-measure $|F|_\omega$ of a bounded closed set F is defined by

$$|F|_\omega = |I|_\omega - |C_I F|_\omega,$$

where I is the smallest closed interval containing F and $C_I F$ denotes the complement of F with respect to I. The outer ω-measure $\omega^*(E)$ of a bounded set E is the infimum of the ω-measures of all bounded open sets containing E and the inner ω-measure $\omega_*(E)$ is the supremum of the ω-measures of all closed sets contained in E. If $\omega^*(E) = \omega_*(E)$, the set E is said to be ω-measurable and the common value is denoted by $|E|_\omega$. Two sets A_1 and A_2 are said to be separated relative to ω-measure or ω-separated if corresponding to every $\varepsilon > 0$ there exist open sets G_1, G_2 with $G_1 \supset A_1$, $G_2 \supset A_2$ such that $|G_1 G_2|_\omega < \varepsilon$. A function $f(x)$ defined on the ω-measurable set E is said to be ω-measurable ([5], def. 2) if for every real number r, the set $E(f > r) = \{x; x \in E \text{ and } f(x) < r\}$ is ω-measurable.

Let $f(x)$ be ω-measurable on the bounded set E and $A < f(x) < B$ on E. Let $A = y_0 < y_1 < y_2 < \cdots < y_n = B$ be a subdivision of $[A, B]$ and $e_i = E(y_i \leqq f < y_{i+1})$ $(i = 0, 1, 2, \cdots, n-1)$. The limit of $\sum_{i=0}^{n-1} y_i |e_i|_\omega$ as $\max |y_i - y_{i-1}| \to 0$ is called the Lebesgue-Stieltjes integral ([5], def. 3) of $f(x)$ over E and is written as $\int_E f d\omega$. This definition may be extended to unbounded functions in the usual way.

One can verify that (i) the results of the sections 1—4 ([6]), Ch. III) and the theorems 2.7, 2.17—2.20 ([4], Ch. II) corresponding to ω-measures, (ii) the results of the sections 1, 2 ([6], Ch. IV) and the theorems 3.9—3.11 ([4], Ch. III) corresponding to ω-measurable functions (iii) the results of the sections 2, 3 ([6], Ch. V) and 1, 2 ([6], Ch. VI) corresponding to Lebesgue-Stieltjes integral, are true. Whenever necessary we shall refer these results with a star for the corresponding results of ω-measures, ω-measurable functions and Lebesgue-Stieltjes integral.

If a property P is satisfied at all points of a set A except a set of ω-measure zero, then it will be said that P is satisfied almost everywhere (ω) in A or at ω-almost all points of A.

3. ω-density of sets

DEFINITION 3.1. (cf. [4], def. 5.2, p. 114).

Let A be any subset of S_3, x be any point and

$$v = [x, x+h] \ (h > 0, x+h \in S).$$

Then

$$\limsup_{h\to 0} \frac{\omega^*(Av)}{|v|_\omega}, \quad \liminf_{h\to 0} \frac{\omega^*(Av)}{|v|_\omega}$$

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are respectively called the right upper and lower ω-densities of A at x. If these limits are equal, their common value is the right ω-density of A at x. Similar difinitions are given for left ω-densities of A. If the left and right ω-densities of A at x are equal, their common value is the ω-density of A at x. Since $\omega^*(Av) \leqq |v|_\omega$ for any interval v it follows that none of the four ω-densities can exceed unity.

DEFINITION 3.2. Let A be a subset of S_3 and x be any point. A is said to be ω-dense at x if $\omega^*(Av) > 0$ for any open interval v containing x. A is said to be ω-dense in itself if A is ω-dense at each point of A.

THEOREM 3.1. Let A be a subset of S_3. Then at almost all points (ω) of A the ω-density of A is unity.

COROLLARY 3.1.1. If $A \subset S_3$ then A is ω-dense at almost all points (ω) of A.

THEOREM 3.2. Let A and B be two subsets of S_3. If A and B are ω-separated, then at almost all points (ω) of one set the ω-density of the other is zero.

THEOREM 3.3. Let A and B be two subsets of S_3. If at almost all points (ω) of A the ω-density of B is zero, then A and B are ω-separated.

The above theorems can be proved in a way analogous to that used in proving the results of the section 5.2 ([4], Ch. V) by making use of the theorems 1.1 and 1.2.

Let A and B be any two subsets of S_3. Let A_B and B_A denote the parts of A, B respectively where at least one of the four ω-densities of B, A is different from zero.

THEOREM 3.4. If A and B are not ω-separated, then $\omega^*(A_B) > 0$ and $\omega^*(B_A) > 0$; also no part of A_B with positive outer ω-measure is ω-separated from B_A and no part of B_A with positive outer ω-measure is ω-separated from A_B.

PROOF. From theorem 3.3 it follows that $\omega^*(A_B) > 0$ and $\omega^*(B_A) > 0$. Let $E \subset A_B$ with $\omega^*(E) > 0$. If possible, let E be ω-separated from B_A. Write $B' = B - B_A$. Then $B = B' + B_A$. At each point of B' the ω-density of A and therefore of E is zero. By theorem 3.3 the sets E and B' are ω-separated. So the sets E and $B = B' + B_A$ are ω-separated. Then by theorem 3.2 at almost all points (ω) of E the ω-density of B is zero. This contradicts the definition of A_B. If $E \subset B_A$ and $\omega^*(E) > 0$ then as above we can show that E and A_B are not ω-separated.

THEOREM 3.5. For any two sets A and B and any interval v, we have

$$\omega^*(vA_B) = \omega^*(vB_A).$$

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PROOF. If A and B are ω-separated then by Theorem 3.2, $\omega^*(A_B) = 0$ and $\omega^*(B_A) = 0$. Therefore $\omega^*(vA_B) = \omega^*(vB_A)$.

Next we suppose that A and B are not ω-separated. Write $A_0 = vA_B$ and $B_0 = vB_A$. Assume that $\omega^*(A_0) < \omega^*(B_0)$. Let Δ be any open interval containing the sets v, A_B, B_A. Choose an open set $G \subset \Delta$ such that $A_0 \subset G$ and $|G|_\omega < \omega^*(B_0)$. Let F denote the complement of G relative to Δ. Then $\omega^*(FB_0) > 0$. Since the sets F and G are ω-separated, the same is true for the sets FB_0 and GA_B. Again since $FB_0 \subset v$ and $FA_B \subset \Delta - v$ the sets FB_0 and FA_B are ω-separated. Hence FB_0 is ω-separated from $A_B = G \cdot A_B + F \cdot A_B$. Since $FB_0 \subset B_A$ and $\omega^*(FB_0) > 0$ this contradicts the Theorem 3.4. Similarly we can show that the assumption $\omega^*(B_0) < \omega^*(A_0)$ leads to a contradiction. Hence $\omega^*(A_0) = \omega^*(B_0)$.

COROLLARY 3.5.1. If A and B are not ω-separated, then

$$\omega^*(A_B) = \omega^*(B_A) > 0.$$

THEOREM 3.6. If A and B are not ω-separated, then at almost all points (ω) of A_B the ω-density of B is unity and at almost all points (ω) of B_A the ω-density of A is unity.

PROOF. Let $0 < \tau_1 < \tau_2 < \cdots$ be a sequence of real numbers with $\tau_i \to 1$ and let E_i denote the set of points of A_B where the right lower ω-density of B is less than τ_i. Consider the set E_n. If $x \in E_n$ there exists a null sequence $\{h_i\}$ ($h_i > 0$, $x+h_i \in S$) such that for all i

$$\frac{\omega^*(Bv_i)}{|v_i|_\omega} < \tau_n,$$

where $v_i = [x, x+h_i]$. Since $B_A \subset B$ we have for all i

$$\omega^*(v_i B_A) < \tau_n |v_i|_\omega. \tag{1}$$

Let $\mathscr{F}$ denote the family of all closed intervals v_i thus associated with the set E_n. Choose $\varepsilon > 0$ arbitrarily. Then by theorem 1.1 there exists a finite family of pairwise disjoint closed intervals $\Delta_1, \Delta_2, \cdots, \Delta_N$ of $\mathscr{F}$ for which

$$\sum_{i=1}^{N} \omega^*(\Delta_i E_n) > \omega^*(E_n) - \varepsilon, \quad \sum_{i=1}^{N} |\Delta_i|_\omega < \omega^*(E_n) + \varepsilon. \tag{2}$$

So,

$$\begin{aligned} \omega^*(E_n) - \varepsilon &< \sum_{i=1}^{N} \omega^*(\Delta_i A_B) = \sum_{i=1}^{N} \omega^*(\Delta_i B_A) \qquad \text{[by theorem 3.5]} \\ &< \tau_n \sum_{i=1}^{N} |\Delta_i|_\omega < \tau_n[\omega^*(E_n)+\varepsilon] \qquad \text{[by (1) and (2)]} \end{aligned} \tag{3}$$

Since $\varepsilon > 0$ is arbitrary, (3) leads to a contradiction unless $\omega^*(E_n) = 0$. If E' denotes the set of points of A_B where the right lower ω-density of B

is less than unity, then $E' = \sum_{i=1}^{\infty} E_i$. Since $\omega^*(E_i) = 0$ for all i, $\omega^*(E') = 0$. If E'' denotes the set of points of A_B where the left lower ω-density of B is less than unity, then as above we can show that $\omega^*(E'') = 0$. Write $E = E' + E''$. Then $\omega^*(E) = 0$. Clearly at each point of $A_B - E$ the ω-density of B is unity.

Similarly we can show that at almost all points (ω) of B_A the ω-density of A is unity. This completes the proof.

THEOREM 3.7. If A be a closed set contained in S_3, then A can be expressed as $A = P+H$, where P is perfect and ω-dense in itself and where the ω-measure of H is zero.

PROOF. Denote by P the set of points of A where A is ω-dense and write $H = A - P$. Then $A = P+H$. By corollary 3.1.1, the ω-measure of H is zero. Let α be a limiting point of P and v be any open interval containing α. Then v contains a point $\xi (\neq \alpha)$ of P which gives that $\omega^*(Av) > 0$. Since v is arbitrary it follows that A is ω-dense at α and therefore $\alpha \in P$. So the set P is closed. Again let $\alpha \in P$ and v be any open interval containing α. Then $\omega^*(vA) > 0$. But $\omega^*(Av) = \omega^*(Pv)$. Since $vP \subset S_3$, v contains infinity of points of P; so α is a limiting point of P. Thus the set P is perfect. Clearly P is ω-dense at each point of P. This completes the proof.

4. Results on ω-derivatives of $f(x) \in \mathscr{U}$

THEOREM 4.1. If $f(x)$ is in the class $\mathscr{U}$, then all the four ω-derivatives of $f(x)$ are ω-measurable on $[a, b]$.

PROOF. We prove the theorem for the derivative $D^+f_\omega(x)$. The proofs in the other cases are analogous. We have $[a, b] = S_0 + S_2 + S_3 + D$, where the sets S_0, S_2, S_3, D are pairwise disjoint and ω-measurable. $D^+f_\omega(x) = 0$ at each point of S_0. Since $|S_2|_\omega = 0$ and D is at most enumerable, $D^+f_\omega(x)$ is ω-measurable on each of the sets S_0, S_2 and D. The theorem will be proved if we can show that $D^+f_\omega(x)$ is ω-measurable on the set S_3.

For any real number r write $A_r = \{x; x \in S_3 \text{ and } D^+f_\omega(x) < r\}$ and $B_r = \{x; x \in S_3 \text{ and } D^+f_\omega(x) \geqq r\}$. Suppose that $D^+f_\omega(x)$ is not ω-measurable on S_3. There is then a real number r for which the sets A_r and B_r are not ω-measurable. So by theorem 2.20* ([4], p. 59) the sets A_r and B_r are not ω-separated. Let $c_1 < c_2 < c_3 < \cdots$ be a sequence of real numbers with $c_i \to r$. Let E_{ik} be the set of points ξ of A_r for which

$$\frac{f(\xi+h)-f(\xi)}{\omega(\xi+h)-\omega(\xi)} < c_i \tag{4}$$

whenever $0 < h < 1/k$ and $\xi+h \in S$. If $i_1 \leqq i_2$ and $k_1 \leqq k_2$ then

$E_{i_1k_1} \subset E_{i_2k_2}$. Also if $x \in A_r$, then $x \in E_{ik}$ for some i, k. Hence from theorems 2.18* and 2.20* ([4], p. 58—59) it follows that for sufficiently large i, k the sets E_{ik} and B_r are not ω-separated. So by theorem 3.6 there is a set $E \subset B_r$ with $\omega^*(E) > 0$ such that at each point of E the ω-density of E_{ik} is unity. Let α be any point of E and c be any real number with $c_i < c < r$. Since $D^+f_\omega(\alpha) > c$ there exists h' with $0 < h' < 1/k$, $\alpha+h' \in S$ such that

$$\frac{f(\alpha+h')-f(\alpha)}{\omega(\alpha+h')-\omega(\alpha)} > c. \tag{5}$$

Since the ω-density of E_{ik} at α is unity, every interval

$$[\alpha, \alpha+h] \ (h > 0, \alpha+h \in S)$$

contains infinity of points of the set E_{ik}. Choose any ξ of E_{ik} in $(\alpha, \alpha+h')$ and write $h = \alpha+h'-\xi$. Then $\xi+h = \alpha+h'$ and $0 < h < 1/k$. So, from (4) and (5) we have

$$f(\alpha+h')-f(\alpha) > c[\omega(\alpha+h')-\omega(\alpha)]$$

and

$$f(\xi+h)-f(\xi) < c_i[\omega(\xi+h)-\omega(\xi)]$$

from which we get

$$f(\xi)-f(\alpha) > [\omega(\alpha+h')-\omega(\alpha)]\left[c-\frac{\omega(\alpha+h')-\omega(\xi)}{\omega(\alpha+h')-\omega(\alpha)}\, c_i\right]. \tag{6}$$

Now suppose that $\xi \to \alpha+$ over the points of E_{ik}. Then $h \to h'$ and from (6) we get

$$f(\alpha+)-f(\alpha) \geqq [\omega(\alpha+h')-\omega(\alpha)]\ (c-c_i). \tag{7}$$

Since $\omega(\alpha+h')-\omega(\alpha) > 0$ and $c > c_i$, the relation (7) contradicts the fact that $f(x)$ is continuous at α with respect to the set S. This proves the theorem.

THEOREM 4.2. Let $f(x)$ belong to the class $\mathscr{U}$ and P be a non-void perfect set contained in S_3. If all the four ω-derivatives of $f(x)$ are greater than A and less than B $(> A)$, then there exists a closed interval $[c, d]$ in $[a, b]$ such that $P \cdot [c, d]$ is a non-void perfect set and for all (x, y) in $X = \{(x, y);\, x \neq y,\, x \in P \cdot [c, d] \text{ and } y \in [c, d] \cdot S\}$,

$$A \leqq \frac{f(x)-f(y)}{\omega(x)-\omega(y)} \leqq B.$$

PROOF. Consider the function $\phi(x)$ defined by $\phi(x) = f(x)-B\omega(x)$ on S and $\phi(x) = f(x+)-B\omega(x+)$ on D. If $x \in P$, then $D^+\phi_\omega(x) < 0$. So there is a positive number h_x such that $\phi(y) \leqq \phi(x)$ for all y in $[x, x+h_x]$. From § 293 ([3], p. 393) it follows that there is a closed interval $[c_1, d_1]$

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in $[a, b]$ such that $P_1 = P \cdot [c_1, d_1]$ is a non-void perfect set and $\phi(y) \leqq \phi(x)$ for all (x, y) in $X_1 = \{(x, y); x < y, x \in P_1 \text{ and } y \in [c_1, d_1] \cdot S\}$. Then for all (x, y) in X_1

$$\frac{f(x)-f(y)}{\omega(x)-\omega(y)} \leqq B. \tag{8}$$

Since $D^-\phi_\omega(x) < 0$ for $x \in P_1$, there is an $h_x > 0$ such that $\phi(y) \geqq \phi(x)$ for all y in $[x-h_x, x]$. So there is a closed interval $[c_2, d_2]$ in $[c_1, d_1]$ such that $P_2 = P_1 \cdot [c_2, d_2]$ is a non-void perfect set and $\phi(y) \geqq \phi(x)$ for all (x, y) in

$$X_2 = \{(x, y); x > y, x \in P_2 \text{ and } y \in [c_2, d_2] \cdot S\}.$$

This gives that (8) holds for all (x, y) in X_2. Hence for all (x, y) in

$$X_3 = \{(x, y); x \neq y, x \in P_2 \text{ and } y \in [c_2, d_2] \cdot S\}$$

the relation (8) is satisfied. Considering the function $F(x)$ defined by $F(x) = f(x)-A\omega(x)$ on S and $F(x) = f(x+)-A\omega(x+)$ on D we can show that there exists a closed interval $[c, d]$ in $[c_2, d_2]$ such that $P_2 \cdot [c, d]$ is a non-void perfect set, and that

$$A \leqq \frac{f(x)-f(y)}{\omega(x)-\omega(y)}, \tag{9}$$

for all (x, y) in $X = \{(x, y); x \neq y, x \in P_2 \cdot [c, d] \text{ and } y \in [c, d] \cdot S\}$. Clearly $P_2 \cdot [c, d] = P \cdot [c, d]$. Since $X \subset X_3$, both the relations (8) and (9) are satisfied for all (x, y) in X. This proves the theorem.

THEOREM 4.3. Let $f(x)$ belong to the class $\mathscr{U}$. If E denotes the set of points in $[a, b]$ where $f'_{+\omega}(x)$ and $f'_{-\omega}(x)$ exist and are finite but not equal, then E is at most enumerable and $|E|_\omega = 0$.

PROOF. It is obvious that $E \subset S-S_0$. Write $E_0 = E-S_2$. Then $E_0 \subset S_3$. Write

$$E_1 = \{x; x \in E_0 \text{ and } f'_{-\omega}(x) < f'_{+\omega}(x)\}$$

and

$$E_2 = \{x; x \in E_0 \text{ and } f'_{+\omega}(x) < f'_{-\omega}(x)\}.$$

Then $E_0 = E_1+E_2$. Let $r_1, r_2, r_3, \cdots$ be an enumeration of the rational numbers. If $x \in E_1$ there exists a smallest positive integer k such that

$$f'_{-\omega}(x) < r_k < f'_{+\omega}(x).$$

There is then a least positive integer m such that $r_m < x$ and

$$\frac{f(\xi)-f(x)}{\omega(\xi)-\omega(x)} < r_k$$

for all $\xi \in (r_m, x) \cdot S$; and a smallest positive integer n such that $r_n > x$ and

$$\frac{f(\xi)-f(x)}{\omega(\xi)-\omega(x)} > r_k$$

for all $\xi \in (x, r_n) \cdot S$. Combining these two relations we have

$$f(\xi)-f(x) > r_k\{\omega(\xi)-\omega(x)\} \tag{10}$$

for all $\xi(\neq x)$ in $(r_m, r_n) \cdot S$.

Thus to every $x \in E_1$, there corresponds a unique triad (k, m, n). If x_1, x_2 are two distinct points of E_1, then with the help of (10) it can be shown that they correspond to two different triads. Since the set of all triads (k, m, n) is enumerable it follows that E_1 is at most enumerable. Similarly we can show that E_2 is at most enumerable; hence so is the set E_0. Since E is enumerable and contained in S it follows that $|E|_\omega = 0$.

THEOREM 4.4. Let $f(x)$ belong to the class $\mathscr{U}$. If E denotes the set of points in $[a, b]$ where all the four ω-derivatives of $f(x)$ are finite but at least one of $f'_{+\omega}(x)$ and $f'_{-\omega}(x)$ does not exist, then $|E|_\omega = 0$.

PROOF. Let E' denote the set of points of E where

$$D^+f_\omega(x)-D_+f_\omega(x) > k(> 0).$$

From Theorem 4.1. it follows that the set E' is ω-measurable. Write $E_0 = E'-S_2$. Then $E_0 \subset S_3$ and $|E_0|_\omega = |E'|_\omega$. If possible, let $|E_0|_\omega > 0$. For any positive integer r let E_r denote the set of points of E_0 where all the four ω-derivatives of $f(x)$ are numerically less than r. Then $E_r \subset E_{r+1}$ for every r and $E_0 = \sum_{r=1}^\infty E_r$. We can find a positive integer N such that $|E_N|_\omega > 0$. From § 2 and theorem 3.7 it follows that there exists a perfect set $B \subset E_N$ such that B is ω-dense in itself and $|B|_\omega > 0$. Consider the function $g(x) = f(x)+N\omega(x)$. On B all the four ω-derivatives of $g(x)$ are > 0 and $< 2N$. By theorem 4.2 there is a closed interval $[c', d']$ in $[a, b]$ such that $B' = B \cdot [c', d']$ is a non-void perfect set and

$$0 \leqq \frac{g(x)-g(x)}{\omega(x)-\omega(y)} \leqq 2N, \tag{11}$$

for all (x, y) in $X = \{(x, y); x \neq y, x \in B' \text{ and } y \in [c', d'] \cdot S\}$. From definition 3.2 it follows that $|B'|_\omega > 0$. Choose the positive integer m such that $\frac{1}{2}k(m-1) \leqq 2N < \frac{1}{2}km$. Let B_i denote the set of points of B' where

$$\tfrac{1}{2}(i-1)k \leqq D_+g_\omega(x) < \tfrac{1}{2}ik \qquad (i = 1, 2, \cdots, m).$$

Then for some integer $s(1 \leqq s \leqq m)$, $|B_s|_\omega > 0$. We can choose a perfect set $P \subset B_s$ with $|P|_\omega > 0$. At each point of P

$$\tfrac{1}{2}(s-1)k \leqq D_+g_\omega(x) < \tfrac{1}{2}sk \text{ and } D^+g_\omega(x) > \tfrac{1}{2}(s+1)k.$$

Let $[c, d]$ be the smallest interval containing P. Clearly $c, d \in P$. Let $[c, d]-P = \sum_i (\alpha'_i, \beta'_j)$, where the intervals (α'_i, β'_i) are pairwise disjoint. Choose $\varepsilon > 0$ arbitrarily with

$$\varepsilon < \frac{k|P|_\omega}{(2s+1)k+8N}. \tag{12}$$

We find the positive integer n such that $\sum_{i=n+1}^{\infty} |(\alpha'_i, \beta'_j)|_\omega < \varepsilon$. Write $\Delta' = \sum_{i=1}^{n} (\alpha'_i, \beta'_i)$ and $\Delta = [c, d]-\Delta'$. Then $P \subset \Delta$. We arrange the first n intervals (α'_i, β'_i) in the order of increasing end points and rename them as $(\alpha_1, \beta_1), (\alpha_2, \beta_2), \cdots, (\alpha_n, \beta_n)$. Write $c = \beta_0$, $d = \alpha_{n+1}$. Since P is perfect we have $\beta_i < \alpha_{i+1}$ $(i = 0, 1, \cdots, n)$. Then $\Delta = \sum_{i=0}^{n} [\beta_i, \alpha_{i+1}]$. Let $P_i = P \cdot [\beta_i, \alpha_{i+1}]$ $(i = 0, 1, 2, \cdots, n)$. If $x \in P_\tau$ then there exists a null sequence $\{h_i\}$ $(h_i > 0, x+h_i \in S)$ such that

$$g(x+h_i)-g(x) < \tfrac{1}{2}sk\{\omega(x+h_i)-\omega(x)\}. \tag{13}$$

Let $\mathscr{F}$ denote the family of all closed intervals $[x, x+h_i]$ thus associated with the set P_τ. By theorem 1.1 there exists a finite family of pairwise disjoint closed intervals $\delta_1, \delta_2, \cdots, \delta_\mu$ of $\mathscr{F}$ for which

$$\sum_{i=1}^{\mu} |\delta_i P_\tau|_\omega > |P_\tau|_\omega - \varepsilon/n+1,\ \sum_{i=1}^{\mu} |\delta_i|_\omega < |P_\tau|_\omega + \varepsilon/n+1. \tag{14}$$

Write

$$\delta_i = [x_i, x_i+k_i] \qquad (i = 1, 2, \cdots, \mu).$$

We may suppose that $x_1 < x_2 < \cdots < x_\mu$ and $x_1 = \beta_\tau$, $x_\mu + k_\mu = \alpha_{\tau+1}$. Then $x_i+k_i < x_{i+1}$ $(i = 1, 2, \cdots, \mu-1)$. Now let $\Delta''_\tau = \sum_{i=1}^{\mu} \delta_i$ and $\Delta'''_\tau = \sum_{i=1}^{\mu-1} (x_i+k_i, x_{i+1})$. We proceed in this way with each of the sets $P_0, P_1, \cdots, P_n$. The interval $[c, d]$ is thus divided into a finite number of parts consisting of the sets

(i) $\Delta'' = \sum_{\tau=0}^{n} \Delta''_\tau$ (ii) $\Delta''' = \sum_{\tau=0}^{n} \Delta'''_\tau$ and (iii) Δ'. We have

$$|\Delta''|_\omega < |P|_\omega + \varepsilon,\ |\Delta''|_\omega > |P|_\omega - \varepsilon \text{ and } |\Delta'''|_\omega < 2\varepsilon.$$

Now

$$\begin{aligned} g(\alpha_{\tau+1})-g(\beta_\tau) &= \sum_{i=1}^{\mu} \{g(x_i+k_i)-g(x_i)\} + \sum_{i=1}^{\mu-1} \{g(x_{i+1})-g(x_i+k_i)\} \\ &< \tfrac{1}{2}sk(|P_\tau|_\omega + \varepsilon/n+1) + 2N|\Delta'''_\tau|_\omega. \text{ [Using (11), (13), and (14).]} \end{aligned}$$

So,

$$\begin{aligned} g(d)-g(c) &= \sum_{\tau=0}^{n} \{g(\alpha_{\tau+1})-g(\beta_\tau)\} + \sum_{\tau=1}^{n} \{g(\beta_\tau)-g(\alpha_\tau)\} \\ &< \tfrac{1}{2}sk\Big(\sum_{\tau=0}^{n} |P_\tau|_\omega + \varepsilon\Big) + 2N\sum_{\tau=0}^{n} |\Delta'''_\tau|_\omega + \sum_{\tau=1}^{n} q_\tau \qquad (15) \\ &< \tfrac{1}{2}sk(|P|_\omega + \varepsilon) + 4N\varepsilon + q \end{aligned}$$

where $q_\tau = g(\beta_\tau)-g(\alpha_\tau)$ and $q = \sum_{\tau=1}^{n} q_\tau$. Since at each point of P, $D^+g_\omega(x) > \frac{1}{2}(s+1)k$ proceeding as above we can show that

$$(16) \qquad g(d)-g(c) > \tfrac{1}{2}(s+1)k(|P|_\omega-\varepsilon)+q.$$

From (15) and (16) we get

$$\tfrac{1}{2}(s+1)k(|P|_\omega-\varepsilon) < \tfrac{1}{2}sk(|P|_\omega+\varepsilon)+4N\varepsilon, \text{ or } \varepsilon > \frac{k|P|_\omega}{(2s+1)k+8N}.$$

This contradicts (12). Hence $|E'|_\omega = 0$.

If for a positive integer n, A_n denotes the set of points of E where $D^+f_\omega(x)-D_+f_\omega(x) > 1/n$, then

$$\sum_{n=1}^{\infty} A_n = A_+ = \{x; x \in E \text{ and } D^+f_\omega(x)-D_+f_\omega(x) > 0\}.$$

Since $|A_n|_\omega = 0$ for each n, we have $|A_+|_\omega = 0$. If

$$A_- = \{x; x \in E \text{ and } D^-f_\omega(x) > D_-f_\omega(x)\},$$

then proceeding as in the previous case we can show that $|A_-|_\omega = 0$. Clearly $E = A_+ + A_-$. So $|E|_\omega = 0$.

5. Function of sets

Definition 5.1. Let A be any set contained in S_3 and the set function $\phi(e)$ be defined for sets $e \subset A$. Let $x \in A$ and $v = [x, x+h]$ $(h > 0, x+h \in S)$. The right upper and lower derivatives $D^+\phi(e, x)$ and $D_+\phi(e, x)$ of $\phi(e)$ at x are defined by

$$D^+\phi(e, x) = \limsup_{h\to 0} \frac{\phi(Av)}{|v|_\omega}, \quad D_+\phi(e, x) = \liminf_{h\to 0} \frac{\phi(Av)}{|v|_\omega}.$$

If $D^+\phi(e, x) = D_+\phi(e, x)$, the common value is called the right derivative $D\phi_+(e, x)$ of $\phi(e)$ at x. Similarly the left derivatives $D^-\phi(e, x)$, $D_-\phi(e, x)$ and $D\phi_-(e, x)$ of $\phi(e)$ are defined. If $D\phi_+(e, x) = D\phi_-(e, x)$, the common value is called the derivative $D\phi(e, x)$ of $\phi(e)$ at x.

Theorem 5.1. Let $f(x)$ be summable (LS) on the ω-measurable set $A \subset S_3$. For any ω-measurable set $e \subset A$ if

$$\phi(e) = \int_e f d\omega$$

then $D\phi(e, x) = f(x)$ at almost all points (ω) of A.

Proof. Let H denote the set of points of A where $D\phi(e, x) = f(x)$. Choose $\varepsilon > 0$ arbitrarily. Then by theorem 3.9* ([4], p. 77) there exists a closed set $F \subset A$ with $|F|_\omega > |A|_\omega-\varepsilon$ such that $f(x)$ is continuous at each point of F with respect to F. Let E denote the set of points of F where the ω-density of F is unity. Then by theorem 3.1, $|E|_\omega = |F|_\omega$. Write $B = A-F$. For $x \in E$, let $v = [x, x+h]$ $(h > 0, x+h \in S)$. We show that as $h \to 0$ (I) $\phi(vF)/|v|_\omega \to f(x)$ for all $x \in E$ and (II) $\phi(vB)/|v|_\omega \to 0$ at almost all points (ω) of E.

Let $x \in E$. Choose $\eta > 0$ arbitrarily. Then a $\delta > 0$ exists such that $|f(x')-f(x)| < \eta$ for all $x' \in (x-\delta, x+\delta) \cdot F$. Then

$$[f(x)-\eta]|vF|_\omega \leqq \int_{vF} f d\omega \leqq [f(x)+\eta]|vF|_\omega$$

or

$$(17) \qquad [f(x)-\eta] \frac{|vF|_\omega}{|v|_\omega} \leqq \frac{\phi(vF)}{|v|_\omega} \leqq [f(x)+\eta] \frac{|vF|_\omega}{|v|_\omega}.$$

Since the ω-density of F at x is unity, letting $h \to 0$ in (17) and noting that η is arbitrary we get

$$\lim_{h\to 0} \frac{\phi(vF)}{|v|_\omega} = f(x)$$

which proves (I).

Let n be a positive integer and E_n denote the set of points of E where

$$(18) \qquad \limsup_{h\to 0} \int_{vB} |f| d\omega / |v|_\omega > 1/n, \ (v = [x, x+h], h > 0, x+h \in S).$$

If possible, let $\omega^*(E_n) = k > 0$. Since $|f(x)|$ is summable (LS) on A by theorem 8* ([6], p. 148) we can find a positive number $\eta < \frac{1}{2}k$ such that for any ω-measurable set $e \subset A$ we have

$$(19) \qquad \int_e |f| d\omega < \frac{k}{2n} \text{ whenever } |e|_\omega < 2\,\eta k.$$

Since the ω-density of B is zero at each point of E, if $x \in E_n$ we can choose a sequence of closed intervals $v_i = [x, x+h_i]$ $(h_i > 0, h_i \to 0, x+h_i \in S)$ such that for all i

$$(20) \qquad \int_{v_i B} |f| d\omega > \frac{1}{n} |v_i|_\omega \text{ and } |v_i B|_\omega < \eta |v_i|_\omega.$$

Let $\mathscr{F}$ denote the family of all intervals v_i thus associated to the set E_n. Then by theorem 1.1 there exists a finite family of pairwise disjoint closed intervals $\Delta_1, \Delta_2, \cdots, \Delta_N$ of $\mathscr{F}$ for which

$$(21) \qquad \sum_{i=1}^{N} \omega^*(\Delta_i E_n) > \omega^*(E_n)-\eta, \quad \sum_{i=1}^{N} |\Delta_i|_\omega < \omega^*(E_n)+\eta.$$

Write $e = \sum_{i=1}^N \Delta_i B$. Then from (20) and (21) we get $|e|_\omega < 2\eta k$ and

$$\int_e |f|d\omega = \sum_{i=1}^{N} \int_{\Delta_i B} |f|d\omega > \frac{1}{n} \sum_{i=1}^{N} |\Delta_i|_\omega > \frac{1}{n} [k-\eta] > \frac{k}{2n}$$

which contradicts (19). Hence $\omega^*(E_n) = 0$. Let E_0 denotes the set of points of E where the left hand member of (18) is positive. Then $E_0 = \sum_{n=1}^{\infty} E_n$ which gives that $\omega^*(E_0) = 0$. This proves (II).

Let $x \in E' = E - E_0$ and $v = [x, x+h]$ $(h > 0, x+h \in S)$. We have

$$(22) \qquad \frac{\phi(vA)}{|v|_\omega} = \frac{\phi(vF)}{|v|_\omega} + \frac{\phi(vB)}{|v|_\omega}.$$

Letting $h \to 0$ and using (I) and (II) we get $D\phi_+(e, x) = f(x)$ from (22). Similarly we can show that $D\phi_-(e, x) = f(x)$ for all x belonging to a set $E'' \subset E$ with $|E''|_\omega = |E|_\omega$. If $C = E'E''$, then $D\phi(e, x) = f(x)$ at each point of C and $|C|_\omega = |E|_\omega = |F|_\omega$. So $H \supset C$ which gives that $A-H \subset A-C$ and $\omega^*(A-H) < \varepsilon$. Since $\varepsilon > 0$ is arbitrary we get $\omega^*(A-H) = 0$. This proves the theorem.

6. Results on $BV-\omega$ functions

THEOREM 6.1. Let $f(x)$ be $BV-\omega$ on $[a, b]$. If E denotes the set of points in $[a, b]$ where at least one of the four ω-derivatives of $f(x)$ is infinite, then $|E|_\omega = 0$.

PROOF. Since $f(x)$ is $BV-\omega$ on $[a, b]$ it follows that $f(x) \in \mathscr{U}_0$. Let $E' = E - S_2$. Then $E' \subset S_3$ and $|E'|_\omega = |E|_\omega$. Write

$$E_1 = \{x; x \in E' \text{ and } D^+f_\omega(x) = +\infty\},$$
$$E_2 = \{x; x \in E' \text{ and } D_+f_\omega(x) = -\infty\},$$
$$E_3 = \{x; x \in E' \text{ and } D^-f_\omega(x) = +\infty\},$$
$$E_4 = \{x; x \in E' \text{ and } D^-f_\omega(x) = -\infty\}.$$

Then $E' = E_1+E_2+E_3+E_4$. Let N be any positive number. If $x \in E_1$ there is a null sequence $\{h_i\}$ $(h_i > 0, x+h_i \in S)$ such that for all i

$$(23) \qquad \frac{f(x+h_i)-f(x)}{\omega(x+h_i)-\omega(x)} > N.$$

Let $\mathscr{F}$ denote the family of all intervals $[x, x+h_i]$ thus associated with the set E_1. Choose $\varepsilon > 0$ arbitrarily. Then by theorem 1.1 there exists a finite family of pairwise disjoint closed intervals $\Delta_1, \Delta_2, \cdots, \Delta_n$ $(\Delta_i = [x_i, x_i+k_i])$ of $\mathscr{F}$ for which

$$(24) \qquad \sum_{i=1}^{n} |\Delta_i E_1|_\omega > |E_1|_\omega - \varepsilon, \sum_{i=1}^{n} |\Delta_i|_\omega < |E_1|_\omega + \varepsilon.$$

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Now

$$\sum_{i=1}^{n} |\Delta_i E_1|_\omega \leqq \sum_{i=1}^{n} \{\omega(x_i+k_i)-\omega(x_i)\}.$$

So from (23) and (24) we get

$$\sum_{i=1}^{n} |f(x_i+k_i)-f(x_i)| > N(|E_1|_\omega-\varepsilon). \tag{25}$$

We may assume that

$$x_1 < x_2 < \cdots < x_n.$$

Then

$$x_i+k_i < x_{i+1} \qquad (i = 1, 2, \cdots, n-1).$$

Since $x_i \in S_3$ the points

$$a \leqq x_1, x_1+k_1, x_2, x_2+k_2, \cdots, x_n, x_n+k_n \leqq b$$

form a ω-subdivision of $[a, b]$. So from (25) we get

$$V_\omega(f; a, b) > N(|E_1|_\omega-\varepsilon). \tag{26}$$

Since N and ε are arbitrary the relation (26) cannot hold unless $|E_1|_\omega = 0$.

Similarly we can show that $|E_i|_\omega = 0$ $(i = 2, 3, 4)$. So $|E|_\omega = 0$. This proves the theorem.

THEOREM 6.2. If $f(x)$ is $BV-\omega$ on $[a, b]$ then $f'_\omega(x)$ exists and is finite except on a set of ω-measure zero.

PROOF. Let E_1 denote the set of points of $[a, b]$ where at least one of the four ω-derivatives of $f(x)$ is infinite, E_2 denote the set of points of $[a, b]$ where all four ω-derivatives of $f(x)$ are finite but at least of one of $f'_{+\omega}(x)$ and $f'_{-\omega}(x)$ does not exist, E_3 denote the set of points of $[a, b]$ where $f'_{+\omega}(x)$ and $f'_{-\omega}(x)$ exist finitely but are different. Then from theorems 4.3, 4.4 and 6.1. $|E_i|_\omega = 0$ $(i = 1, 2, 3)$. Write $E = E_1+E_2+E_3$. Then $|E|_\omega = 0$ and at each point of the set $[a, b]-E$, $f'_\omega(x)$ exists and is finite. This proves the theorem.

THEOREM 6.3. If $f(x)$ is $BV-\omega$ on $[a, b]$, then $f'_\omega(x)$ is summable (LS) on $[a, b]$.

PROOF. We have $[a, b] = S_0+S_2+S_3+D$ where the sets S_0, S_2, S_3, D are pairwise disjoint and ω-measurable. Since $|S_0|_\omega = 0$, $|S_2|_\omega = 0$, $f'_\omega(x)$ is summable (LS) on the sets S_0, S_2. The set D is at most enumerable. So we can take its elements as $\alpha_1, \alpha_2, \alpha_3, \cdots$. Write $D_i = \{\alpha_i\}$. Clearly

$$\int_{D_i} |f'_\omega| d\omega = |f'_\omega(\alpha_i)|\, |D_i|_\omega = |f(\alpha_i+)-f(\alpha_i-)|.$$

Since $f(x)$ is $BV-\omega$ on $[a, b]$ the series $\sum_i |f(\alpha_i+)-f(\alpha_i-)|$ is convergent.

Hence by theorem 5* ([6], p. 146) $f'_\omega(x)$ is summable (LS) on D. From theorem 3* ([6], p. 145) it follows that the theorem will be proved if we can show that $f'_\omega(x)$ is summable (LS) on S_3.

Assume that $f'_\omega(x)$ is not summable (LS) on S_3. Let E denote the set of points of S_3 where $f'_\omega(x)$ exists and is finite. Then by theorem 6.2, $|E|_\omega = |S_3|_\omega$. Write $g(x) = |f'_\omega(x)|$ and $E_n = E\ (0 \leqq g \leqq n)$ $(n = 1, 2, 3, \cdots)$. Then $\int_{E_n} gd\omega \to \infty$ as $n \to \infty$. Let N be any positive number. We fix n such that $\int_{E_n} gd\omega > N+1$. Let k be a positive number with $k > \max\{|S_3|_\omega, 1\}$. By theorem 8* ([6], p. 148) we can find a positive number $\varepsilon < 1/4k$ such that for any ω-measurable set $e \subset E_n$ with $|e|_\omega < \varepsilon$ we have $\int_e gd\omega < \frac{1}{2}$. For any ω-measurable set $e \subset E_n$ we define $\phi(e) = \int_e gd\omega$. Let

$$E_0 = \{x;\, x \in E_n \text{ and } D\phi(e, x) = g(x)\}.$$

By theorem 5.1, $|E_0|_\omega = |E_n|_\omega$. If $x \in E_0$ and $v = [x, x+h]$ $(h > 0,\ x+h \in S)$, then

$$\lim_{h\to 0} \int_{vE_n} \frac{gd\omega}{|v|_\omega} = g(x) = \lim_{h\to 0} \frac{|f(x+h)-f(x)|}{\omega(x+h)-\omega(x)}.$$

So we can choose a sequence of intervals

$$\{v_i\}\ (v_i = [x, x+h_i],\ h_i > 0,\ h_i \to 0,\ x+h_i \in S)$$

such that for all i

$$\left| \int_{v_iE_n} \frac{gd\omega}{|v_i|_\omega} - \frac{|f(x+h_i)-f(x)|}{\omega(x+h_i)-\omega(x)} \right| < \varepsilon. \tag{27}$$

Let $\mathscr{F}$ denote the family of all intervals v_i thus associated with the set E_0. By theorem 1.1 we can select a finite family of pairwise disjoint closed intervals $\Delta_1, \Delta_2, \cdots, \Delta_m$ $(\Delta_i = [x_i, x_i+k_i])$ of $\mathscr{F}$ for which

$$\sum_{i=1}^m |E_0\Delta_i|_\omega > |E_0|_\omega - \varepsilon,\ \sum_{i=1}^m |\Delta_i|_\omega < |E_0|_\omega + \varepsilon. \tag{28}$$

Write $A = \sum_{i=1}^m \Delta_i E_0$ and $B = E_n - A$. Then from (28) $|B|_\omega < \varepsilon$. Now from (27) and (28) we have

$$\left| \int_A gd\omega - \sum_{i=1}^m ||f(x_i+k_i)-f(x_i)| \right| \leqq \sum_{i=1}^m \left| \int_{\Delta_i E_0} gd\omega - |f(x_i+k_i)-f(x_i)| \right|$$
$$< \varepsilon \sum_{i=1}^m |\Delta_i|_\omega < \varepsilon(|E_0|_\omega + \varepsilon) < \tfrac{1}{2}.$$

So

$$\sum_{i=1}^m |f(x_i+k_i)-f(x_i)| > \int_A gd\omega - \tfrac{1}{2} = \int_{E_n} gd\omega - \int_B gd\omega - \tfrac{1}{2} > N. \tag{29}$$

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We may suppose that

$$x_1 < x_2 < \cdots < x_m.$$

Then $x_i+k_i < x_{i+1}$ $(i = 1\ 2\ \cdots, m-1)$. Since $x_i \in S_3$, the points

$$a \leqq x_1, x_1+k_1, x_2, x_2+k_2, \cdots, x_n, x_n+k_n \leqq b$$

form a ω-subdivision of $[a, b]$. So from (29) we have $V_\omega(f; a, b) > N$. Since N is arbitrary, it follows that $V_\omega(f; a, b) = +\infty$ which contradicts the hypothesis. Hence $f'_\omega(x)$ is summable (LS) on S_3.

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