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ON GEOMETRIC PROPERTIES OF ORLICZ-LORENTZ SPACES

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ABSTRACT. Criteria for local uniform rotundity and midpoint local uniform rotundity in Orlicz-Lorentz spaces with the Luxemburg norm are given. Strict K-monotonicity and Kadec-Klee property are also discussed.

Introduction. A function $\varphi: R_+ \longrightarrow R_+$ is said to be an *Orlicz function* if φ is convex, $\varphi(0) = 0$, and $\varphi(u) > 0$ for all $u > 0$. Let (Ω, Σ, μ) denote a complete σ-finite measure space and let $L^0 = L^0(\Omega, \Sigma, \mu)$ denote the space of all (equivalence classes of) μ-measurable real-valued functions, equipped with the topology of convergence in measure on μ-finite sets.

For any $f \in L^0$ the *nonincreasing rearrangement* of f is the function f^* defined by

$$f^*(t) = \inf\{\lambda > 0 : \mu_f(\lambda) \leq t\}$$

(by convention $\inf \emptyset = +\infty$), where μ_f is the *distribution function* defined by

$$\mu_f(t) = \mu\Big(\{\omega \in \Omega : |f(\omega)| > t\}\Big), \quad t \geq 0.$$

In the sequel we shall use the following inequalities without further references (see *e.g.* [1], p. 41)

$$f^*\big(\mu_f(\lambda)\big) \leq \lambda, \quad \big(\mu_f(\lambda) < \infty\big); \ \mu_f\big(f^*(t)\big) \leq t, \quad \big(f^*(t) < +\infty\big).$$

A function $w: [0, \gamma) \longrightarrow R_+$, $0 < \gamma \leq +\infty$, is called *weight function* if it is nonincreasing and locally integrable with respect to the Lebesgue measure m.

For an Orlicz function φ and a weight function $w: [0, \gamma) \longrightarrow R_+$, $\gamma = \mu(\Omega)$, the *Orlicz-Lorentz space* $\Lambda_{\varphi,w}$ is the set of all $x \in L^0(\mu)$ such that

$$I_\varphi(\lambda x) = \int_0^\infty \varphi(\lambda x^*) w \, dm < +\infty$$

for some $\lambda > 0$.

$\Lambda_{\varphi,w}$ is a Banach space equipped with the Luxemburg norm defined by

$$\|x\| = \inf\{\lambda > 0 : I_\varphi(x/\lambda) \leq 1\}.$$

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Note that if $w \equiv 1$, then $\Lambda_{\varphi,w}$ is the Orlicz function space L_φ. If $\varphi(t) = t$, then $\Lambda_{\varphi,w}$ is the Lorentz space Λ_w.

In [7], we studied the geometry of some Calderón-Lozanovskii spaces as well as the Orlicz-Lorentz spaces. This paper is a continuation of the previous one, where we study other geometric properties of the Orlicz-Lorentz spaces. Recently Lin and Sun have also examined several other geometric properties in Orlicz-Lorentz spaces (*e.g.* [12]).

In what follows if X is a Banach space, $B(X)$ and $S(X)$ denote its unit ball and unit sphere, respectively. X is said to be *locally uniformly rotund* (*LUR*), whenever $x_n, x \in B(X)$ and $\|x_n + x\| \longrightarrow 2$ imply $\|x_n - x\| \longrightarrow 0$. X is said to be *midpoint locally uniformly rotund* (*MLUR*) if for every $x \in S(X)$ and every sequence (x_n) in $B(X)$ if $\|x_n + x\| \longrightarrow 1$ and $\|x_n - x\| \longrightarrow 1$ then $\|x_n\| \longrightarrow 0$. We say that X has the *Kadec-Klee property* if the norm and weak convergence of sequences coincide on the unit sphere of X. It is well known that LUR implies the Kadec-Klee property.

Let E be a Banach function space over a measure space (Ω, Σ, μ). E is said to have the *Kadec-Klee property for convergence in measure*, whenever $x_n \longrightarrow x$ in measure (globally) on Ω and $\|x_n\| \longrightarrow \|x\|$ imply $\|x_n - x\| \longrightarrow 0$.

E is said to have the *Kadec-Klee property for local convergence in measure* if $x_n \longrightarrow x$ in L^0 and $\|x_n\| \longrightarrow \|x\|$ imply $\|x_n - x\| \longrightarrow 0$.

The Kadec-Klee property for the local convergence in measure was studied in [8] in some Calderón-Lozanovskii spaces. This property and also the Kadec-Klee property for convergence in measure were investigated in [3] and [13] for symmetric spaces defined on any interval $[0, \alpha)$, $0 < \alpha \leq +\infty$ and on interval $[0, 1)$, respectively. It is easy to see (*cf.* [9]) that every Banach function space E which is LUR has the Kadec-Klee property for the local convergence in measure. Note also that L^1 and Orlicz spaces generated by Orlicz functions satisfying the so-called Δ_2-condition have the Kadec-Klee property for local convergence in measure (see [8]).

In the sequel we will need the definition of *K-functional* of Peetre for the interpolation couple (L^1, L^∞). For every $x \in L^1 + L^\infty$ and $t > 0$,

$$K(t, x) = \inf\{\|x_0\|_{L^1} + t\|x_1\|_{L^\infty} : x = x_0 + x_1\}.$$

It is well known (see [1] or [11]) that

$$K(t, x) = \int_0^t x^*(s)\, ds$$

and the relation $\prec$ defined on $L^1 + L^\infty$ by

$$x \prec y \text{ if and only if } K(t, x) \leq K(t, y) \text{ for all } t > 0$$

is a pre-order (see [1]).

A Banach function space E over the Lebesgue measure space $([0, \gamma), \Sigma, m)$ with $\gamma \in (0, +\infty]$ is said to be *symmetric* if for every $x \in L^0$ and $y \in E$ with $m_x = m_y$, we have $x \in E$ and $\|x\| = \|y\|$. For basic properties of symmetric spaces we refer to [1] and [11].

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A symmetric space E is said to be *K-monotone* if

$$x \prec y \text{ with } x \in L^1 + L^\infty,\ y \in E \quad \text{imply } x \in E \text{ and } \|x\| \leq \|y\|.$$

By the well known result of Calderón [2] (see also [11]) it follows that the class of all K-monotone spaces is exactly the class of all *exact interpolation* spaces between L^1 and L^∞. It is well known that symmetric spaces with the Fatou property as well as separable symmetric spaces are K-monotone (see [1], [11]). Recall that a Banach function space X has the *Fatou property* if $0 \leq x_n \uparrow$, $\sup_{n\geq 1} \|x_n\| < +\infty$ imply $x = \sup x_n \in X$ and $\|x_n\| \longrightarrow \|x\|$.

A symmetric space with the Fatou property is called *rearrangement invariant* (r.i. in short). Examples of r.i. spaces are among others Lorentz spaces, Marcinkiewicz spaces and Orlicz-Lorentz spaces.

A symmetric space E is said to be *strictly K-monotone* if

$$x, y \in E,\ x \prec y,\ x^* \neq y^* \quad \text{imply } \|x\| < \|y\|.$$

The following result has been proved in [3].

THEOREM 1. *Let E be a separable symmetric space on* $[0, +\infty)$*. If E is strictly K-monotone then the following statements are equivalent:*

(i) E has the Kadec-Klee property for convergence in measure.

(ii) E has the Kadec-Klee property for local convergence in measure.

In what follows the Δ_2-condition for the Orlicz function will play an important role. We say that φ satisfies the Δ_2-condition if there exist positive constants K and u_0 such that the inequality $\varphi(2u) \leq K\varphi(u)$ holds for all $u > 0$, whenever $\int_0^\gamma w\,dm = +\infty$ and it holds for $u \geq u_0$, whenever $\int_0^\gamma w\,dm < +\infty$. We simply indicate this by $\varphi \in \Delta_2$.

In the sequel we will need the following result.

THEOREM 2. *The Orlicz-Lorentz space $\Lambda_{\varphi,w}$ has the Kadec-Klee property for convergence in measure whenever $\varphi \in \Delta_2$.*

PROOF. The proof presented in [8] works also in the case of infinite measure. This follows by the fact that if $x_n \longrightarrow x$ in measure then $(x_n - x)^* \longrightarrow 0$ and $x_n^* \longrightarrow x^*$ m-a.e. (see [11]).

We shall now state two results that will be needed later on. The first result of Hardy, Littlewood and Pólya can be found in [1] (p. 88) and second one is a result of Szlenk [16].

THEOREM 3. *Let $x, y \in L^1(\mu) + L^\infty(\mu)$. Then $K(t, x) \leq K(t, y)$ for any $0 < t < \mu(\Omega)$ if and only if $I_\varphi(x) \leq I_\varphi(y)$ for any Orlicz function φ.*

THEOREM 4. *In $L^1(0, 1)$, every weakly convergent sequence has a subsequence whose arithmetic means are norm convergent.*

REMARK. Szlenk's theorem holds true for every abstract *AL*-Banach lattice. In particular it holds true in weighted L^1-spaces.

In what follows $L^1_0 = L^1_0(\mu)$ denotes the closure of $L^1(\mu)$ in $L^1(\mu) + L^\infty(\mu)$ equipped with the norm $K(1,\cdot)$. It is easy to show that $x \in L^1_0$ if and only if x is locally integrable and $x^*(s) \longrightarrow 0$ as $s \longrightarrow +\infty$.

Let for each $y \in L^1(\mu) \cap L^\infty(\mu)$, f_y be a linear functional defined by $f_y(x) = \int_\Omega xy d\mu$ for $x \in L^1_0$. We let $\Gamma = \{f_y : y \in L_1(\mu) \cap L^\infty(\mu)\}$. It is clear that $\langle L^1_0, \Gamma\rangle$ forms a dual system. In what follows we will consider the weak topology $\sigma(L^1_0, \Gamma)$ induced on L^1_0 by Γ.

The following result has been proved by Sedaev and for the sake of completeness we provide a proof (*cf.* Theorem 8 in [15]).

THEOREM 5. *Assume that* $x_n, x \in L^1_0$ *and* $x_n \longrightarrow x$ *in the topology* $\sigma(L^1_0, \Gamma)$. *If* $x_n \not\longrightarrow x$ *in the norm topology of* $L^1(\mu) + L^\infty(\mu)$ *then there exists* $\delta \in (0, 1]$ *such that*

$$\overline{\lim_{n\longrightarrow\infty}} \sup_{t\in[\delta,\delta^{-1}]} \frac{K(t, x_n)}{K(t, x)} > 1. \tag{1}$$

PROOF. We first note that in [14] (*cf.* [3], Proposition 1.2), it is proved that if Λ_w is any Lorentz space and $x \in \Lambda_w$, $(y_n) \subset \Lambda_w$, and if (y_n) converges in measure to 0, then

$$\|x + y_n\|_{\Lambda_w} = \|x\|_{\Lambda_w} + \|y_n\|_{\Lambda_w} + o(1). \tag{2}$$

Thus, if we assume that $y_n = x_n - x \longrightarrow 0$ in measure, it follows by (2) that

$$\int_0^1 x_n^*(s)\, ds = \int_0^1 x^*(s)\, ds + \int_0^1 y_n^*(s)\, ds + o(1).$$

This implies that (1) holds with $\delta = 1$.

Now assume that for some $\tau > 0$, $\overline{\lim}_{n\longrightarrow\infty} \mu\big(Q_n(\tau)\big) > 0$, where $Q_n(\tau) = \{\omega \in \Omega : |y_n(\omega)| > \tau\}$. In [14], Lemma 3, it is proved that formula (1) holds with some $\delta \in (0, 1]$ whenever for some $0 < 2\tau_1 < \tau$ the following holds

$$\overline{\lim_{n\longrightarrow\infty}} \mu\big(M(\tau_1) \cap Q_n(2\tau_1)\big) > 0, \tag{3}$$

where $M(s) = \{\omega \in \Omega : |x(\omega)| > s\}$ for any $s > 0$.

In order to finish the proof we only need to consider the case $\lim_{n\longrightarrow\infty} \mu\big(M(\tau_1) \cap Q_n(2\tau_1)\big) = 0$. This implies that for some decreasing sequence (τ_n) with $\tau_n \longrightarrow 0$, we have

$$\mu\big(M(\tau_n) \cap Q_n(2\tau_n)\big) \longrightarrow 0. \tag{4}$$

Consider a sequence $(y_n\chi_{M(\tau_n)})$, where $y_n = x_n - x$. It follows by (3) that this sequence tends to 0 in measure. Thus if we assume that $y_n\chi_{M(\tau_n)} \not\longrightarrow 0$ in the norm topology of $L^1(\mu) + L^\infty(\mu)$, then we obtain by (2) that

$$\begin{aligned}\overline{\lim_{n\longrightarrow\infty}} \int_0^1 (x + y_n)^*(s)\, ds &\geq \overline{\lim_{n\longrightarrow\infty}} \int_0^1 (x + y_n\chi_{M(\tau_n)})^*(s)\, ds \\ &= \int_0^1 x^*(s)\, ds + \overline{\lim_{n\longrightarrow\infty}} \int_0^1 y_n\chi_{M(\tau_n)}\, ds.\end{aligned}$$

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This shows that (1) holds with $\delta = 1$.

Suppose now that $y_n\chi_{M(\tau_n)} \longrightarrow 0$ in the norm of $L^1(\mu) + L^\infty(\mu)$. Then, for any $t > 0$, we have $K(t, x+y_n) = K(t, x+z_n)+o(1)$, where $z_n = y_n\chi_{\Omega\setminus M(\tau_n)}$. In consequence the proof will be finished if we can show that for some $0 < \delta \le 1$

$$\overline{\lim_{n\to\infty}} \sup_{t\in[\delta,\delta^{-1}]} \frac{K(t, x+z_n)}{K(t,x)} > 1.$$

Without loss of generality, we may assume that for some τ_0, we have $\{\omega \in M(\tau_0) : |x_n(\omega)| > 3\tau_0\} = \emptyset$ and

$$\mu\big(\{\omega \in \Omega : |z_n(\omega)| > 3\tau_0\}\big) > p > 0.$$

Hence by the definition of the set $M(\tau_0)$, it follows that for any $s \ge \tau_0$, we have

$$q_n(s) = \mu\big(\{\omega \in \Omega : |x(\omega) + z_n(\omega)| > s\}\big) \ge \mu\big(\{\omega \in \Omega : |x(\omega)| > s\}\big) = q(s),$$

and for $\tau_0 \le s \le 2\tau_0$,

$$q_n(s) \ge \mu\big(\{\omega \in \Omega : |x(\omega)| > s\} \cup \{\omega \notin M(\tau_0) : |z_n(\omega)| > 3\tau_0\}\big) \ge q(s) + p.$$

The integration by parts yields

$$\begin{aligned} K\big(q(\tau_0)+p, x_n\big) &= \int_0^{q(\tau_0)+p} x_n^*(s)\,ds = x_n^*(s)s\big|_0^{q(\tau_0)+p} - \int_0^{q(\tau_0)+p} s\,dx_n^*(s) \\ &= \big(q(\tau_0)+p\big)x_n^*\big(q(\tau_0)+p\big) + \int_{x_n{}^*(q(\tau_0)+p)}^\infty q_n(s)\,ds \\ &= \int_0^\infty \tilde{q}_n(s)\,ds, \end{aligned}$$

and

$$\begin{aligned} K\big(q(\tau_0)+p, x\big) &= \int_0^{q(\tau_0)+p} x^*(s)\,ds \\ &= \big(q(\tau_0)+p\big)x^*\big(q(\tau_0)+p\big) + \int_{x^*(q(\tau_0)+p)}^\infty q(s)\,ds \\ &= \int_0^\infty \tilde{q}(s)\,ds, \end{aligned}$$

where

$$\tilde{q}_n(s) = \begin{cases} q(\tau_0)+p, & 0 \le s \le x_n^*\big(q(\tau_0)+p\big), \\ q_n(s), & x_n^*\big(q(\tau_0)+p\big) < s < \infty, \end{cases}$$

and

$$\tilde{q}(s) = \begin{cases} q(\tau_0)+p, & 0 \le s \le x^*\big(q(\tau_0)+p\big), \\ q(s), & x^*\big(q(\tau_0)+p\big) < s < \infty. \end{cases}$$

Since $q_n(\tau_0) \ge q(\tau_0) + p$, it follows by the properties of rearrangement that

$$x^*\big(q(\tau_0)+p\big) \le \tau_0 \le x_n^*\big(q_n(\tau_0)\big) \le x_n^*(q(\tau_0)+p).$$

Finally, combining the above with $\tilde{q}_n(s) \ge \tilde{q}(s)$ for all $s > 0$ and $\tilde{q}_n(s) > \tilde{q}(s) + p$ for $\tau_0 \le s \le 2\tau_0$, we obtain

$$K\big(q(\tau_0)+p, x\big) \le \tau_0 p + K\big(q(\tau_0)+p, x_n\big),$$

which completes the proof.

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1. **Results.** We start by proving some general results. Some of them will be applied in order to prove criteria for LUR and MLUR of Orlicz-Lorentz spaces. Recall that the *fundamental function* ψ_E of a symmetric space E on $[0,\gamma)$ is defined by $\psi_E(t) = \|\chi_{(0,t)}\|_E$ for $t \in (0,\gamma)$. It is well known that the Köthe dual E' of a symmetric space E, defined by

$$E' = \Big\{ y \in L^0 : \|y\|_{E'} = \sup_{\|x\|_E \le 1} \int |xy| \, dm < +\infty \Big\},$$

is a r.i. space on $[0,\gamma)$ with the fundamental function $\psi_{E'}(t) = t/\psi_E(t)$ for $t \in (0,\gamma)$ (see [1], [11]).

LEMMA 6. *Let E be a* r.i. *space on $[0,\gamma)$ such that $\psi_{E'}(0+) = 0$. Then $B(E)$ is sequentially compact in the weak topology $\sigma(E,\Gamma)$ induced by $\Gamma = \{f_y : y \in L^1 \cap L^\infty\}$.*

PROOF. By the Fatou property of E, we have $E = E''$ with equality of norms. Thus it follows by the order density of $\Delta = L^1 \cap L^\infty$ in E and the Beppo-Levi theorem that

$$E = (\bar{\Delta}^{E'})' = X'$$

with equality of norms, where $X = \bar{\Delta}^{E'}$ is the closure of Δ in E'. Since X is a symmetric space with $\psi_X = \psi_{E'}$, it follows by $\psi_{E'}(0+) = 0$ that X is separable (see [11]). Thus, we obtain that X' is order isometric to the dual space X^* of X. In consequence E is isometric to X^*, which yields in particular that the unit ball of E is a compact for the $\sigma(E,\Gamma)$ topology. Since Γ contains a countable total subset of functionals, $B(E)$ is metrizable in the $\sigma(E,\Gamma)$ topology and this completes the proof.

REMARK. It is well known that a Banach space X is a dual space if and only if $B(X)$ is compact in some locally convex Hausdorff topology on X. Note also that the well known Pełczyński's result says that separable dual spaces do not contain copies of c_0. Using the above and taking for example $E = L^1_0(R_+)$ we conclude that in general the assumptions that E has the Fatou property and $\psi_{E'}(0+) = 0$ are essential in Lemma 6.

PROPOSITION 7. *Let E be a* r.i. *space with $\psi_{E'}(0+) = 0$ and let $E \subset L^1_0$. Then $B^*(E) := \{x^* : x \in B(E)\}$ is a compact subset of $L^1 + L^\infty$.*

PROOF. Let $(x_n^*) \subset B^*(E)$. By Lemma 6 we may assume, by passing to a subsequence that $x_n^* \longrightarrow x$ in the topology $\sigma(E,\Gamma)$ with $x \in B(E)$. Assume that $x_n^* \not\longrightarrow x$ in $L^1 + L^\infty$. Then, by Theorem 5, it easily follows by concavity of the K-functional of Peetre and a compactness argument that there are $\delta \in (0,1)$, $t \in [\delta, \delta^{-1}]$ and a subsequence (y_n) of (x_n^*) such that

$$\lim_{n\to\infty} K(t,y_n) > K(t,x).$$

This yields

$$\lim_{n\to\infty} \int_0^t y_n(s)\,ds > \int_0^t x^*(s)\,ds \ge \int_0^t x(s)\,ds.$$

Since $x_n^* \longrightarrow x$ in $\sigma(E,\Gamma)$, we obtain a contradiction. Thus we have that $\|x_n^* - x\|_{L^1+L^\infty} \longrightarrow 0$. This implies by the inequality

$$\|x_n^* - x^*\|_{L^1+L^\infty} \le \|x_n^* - x\|_{L^1+L^\infty}$$

that $x_n^* \longrightarrow x^*$ in $L^1 + L^\infty$ and thus $x \in B^*(E)$.

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PROPOSITION 8. *Let X be a Banach function space over a measure space (Ω, Σ, μ) with $\operatorname{supp} X = \Omega$. If $x_n \longrightarrow x$ weakly in X and $x_n \longrightarrow y$ in L^0 then $y = x$.*

PROOF. By the σ-finiteness of μ, we have $\operatorname{supp} X' = \operatorname{supp} X$. Thus there is a strictly positive function h on Ω with $\|h\|_{X'} = 1$. Since $X \subset X''$ and $\|x\|_{X''} \leq \|x\|_X$ for each $x \in X$, we obtain that $X \hookrightarrow L^1(h\,d\mu)$ with continuous inclusion. Assume that $x_n \longrightarrow x$ weakly in X. Then by the Szlenk's theorem we can suppose, by passing to a subsequence, that

$$\|s_n - x\|_{L^1(h\,d\mu)} \longrightarrow 0, \tag{5}$$

where $s_n = \frac{1}{n}\sum_{k=1}^{n} x_k$. It follows, by (5) that $s_n \longrightarrow x$ in L^0. Since $x_n \longrightarrow y$ in L^0, we get $s_n \longrightarrow y$ in L^0, and thus $x = y$.

By a modification of the proof of Proposition I.1 in [4] in the case of a probability measure space, we obtain the following result.

PROPOSITION 9. *Let (Ω, Σ, μ) be a nonatomic measure space. If $x \in L_0^1$ and $(x_n) \subset L_0^1$ are such that $K(t, x_n) \longrightarrow K(t, x)$ and $K(t, x_n + x) \longrightarrow 2K(t, x)$ for every $t > 0$ then $x_n \longrightarrow x$ in L^0.*

Before the proof of the next result we introduce the notion of (LUR*)-property for symmetric spaces. A symmetric space E is said to have (LUR*)-property if $x^* \in B^*(E)$ and $(x_n^*) \subset B^*(E)$ with $\|x_n^* + x^*\| \longrightarrow 2$ imply $\|x_n^* - x^*\| \longrightarrow 0$.

PROPOSITION 10. *If a* r.i. *space $E \subset L_0^1$ has* (LUR*)*-property then $x \in B(E)$, $(x_n) \subset B(E)$, and $\|x_n + x\| \longrightarrow 2$ imply $x_n \longrightarrow x$ in $L^1 + L^\infty$.*

PROOF. Let $x \in B(E)$, $(x_n) \subset B(E)$ and $\|x_n + x\| \longrightarrow 2$. We have $(x_n + x) \prec (x_n^* + x^*)$. Since r.i. spaces are K-monotone, it follows that

$$\|x_n + x\| \leq \|x_n^* + x^*\| \leq \|x_n^*\| + \|x^*\| \leq 2.$$

Hence $\|x_n^* + x^*\| \longrightarrow 2$ and thus, by the assumption that E has (LUR*)-property, $\|x_n^* - x^*\| \longrightarrow 0$. This implies

$$\lim_{n\to\infty} K(t, x_n) = K(t, x) \quad \text{for every } t > 0.$$

Moreover, we have

$$\|(x_n + x)^*/2 + x^*\| \leq \|x_n + x\|/2 + \|x\| \leq 2,$$
$$\lim_{n\to\infty} \|(x_n + x)^*/2 + (x_n^* + x^*)/2\| = \lim_{n\to\infty} \|(x_n + x)^*/2 + x^*\|$$

and

$$(x_n + x) \prec (x_n + x)^*/2 + (x_n^* + x^*)/2.$$

Hence $\|(x_n + x)^*/2 + x^*\| \longrightarrow 2$. Thus $\|(x_n + x)^*/2 - x^*\| \longrightarrow 0$, by (LUR*)-property. Consequently

$$\lim_{n\to\infty} K(t, x_n + x) = 2K(t, x) \quad \text{for every } t > 0.$$

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From Proposition 9, it follows that $x_n \longrightarrow x$ in L^0. Since $\|x_n^* - x^*\| \longrightarrow 0$, we have $x_n^* \longrightarrow x^*$ in the topology $\sigma(E, E' \cap L_0^1)$. It follows from [6], Corollary 29, that there exists a subsequence (y_n) of (x_n) such that (y_n) converges to some $y \in E$ in the topology $\sigma(E, E' \cap L_0^1)$ and thus $y_n \longrightarrow y$ weakly in $L^1 + L^\infty$. Now, by Proposition 8

$$x_n \longrightarrow x \quad \text{weakly in } L^1 + L^\infty.$$

Since $K(t, x_n) \longrightarrow K(t, x)$ for any $t > 0$, it then easily follows by Theorem 5 (*cf.* the proof of the Proposition 7) that $x_n \longrightarrow x$ in $L^1 + L^\infty$.

Now we are in a position to prove criteria for LUR and MLUR of Orlicz-Lorentz spaces. Recall that a Banach space X is *rotund* if every $x \in S(X)$ is an extreme point. In what follows we will need the following simple technical lemma.

LEMMA 11. *Every symmetric rotund and K-monotone space E is strictly K-monotone.*

PROOF. Let $x, y \in E$ satisfy $x \prec y$ and $x^* \neq y^*$. We need to prove that $\|x\| < \|y\|$. Assume for the contrary that $\|x\| = \|y\|$. We have for every $t \in (0, \gamma)$,

$$\int_0^t 2x^*(s)\, ds \leq \int_0^t x^*(s)\, ds + \int_0^t y^*(s)\, ds = \int_0^t (x^* + y^*)^*(s)\, ds.$$

Thus $2x \prec x^* + y^*$ and by the K-monotonicity of E we obtain

$$\|2x\| \leq \|x^* + y^*\| \leq \|x^*\| + \|y^*\| = 2\|x\|.$$

This implies by rotundity of E that $x^* = y^*$, a contradiction.

In all further results we assume that the Orlicz-Lorentz space $\Lambda_{\varphi,w}$ is defined over the Lebesgue measure space $([0, \gamma), \Sigma, m)$, where $0 < \gamma \leq +\infty$.

THEOREM 12. *For the Orlicz-Lorentz space $\Lambda_{\varphi,w}$ the following conditions are equivalent:*

(i) φ is strictly convex on R_+, $\varphi \in \Delta_2$, $\int_0^\gamma w\, dm = +\infty$ whenever $\gamma = +\infty$ and w is positive on $(0, \gamma)$.
(ii) $\Lambda_{\varphi,w}$ is locally uniformly rotund.
(iii) $\Lambda_{\varphi,w}$ is midpoint locally uniformly rotund.
(iv) $\Lambda_{\varphi,w}$ is rotund.

PROOF. The implications (ii) $\Rightarrow$ (iii) $\Rightarrow$ (iv) are obvious and (iv) $\Rightarrow$ (i) was proved in [10]. In order to finish the proof we need to show (i) $\Rightarrow$ (ii). Assume that (i) is satisfied and let $x = x^* \in B(\Lambda_{\varphi,w})$ and $(x_n) = (x_n^*) \subset B(\Lambda_{\varphi,w})$ satisfy $\|x_n + x\| \longrightarrow 2$.

We first show that $x_n \longrightarrow x$ locally in measure. Take any Lebesgue measurable subset $A \subset [0, \gamma)$ of finite measure. Fix positive numbers ϵ and δ and define for all $n \in N$ measurable sets

$$A_n = \{t \in A : |x_n(t) - x(t)| \geq \delta\},$$
$$B_n = \left\{t \in A : \varphi\big(x_n(t)\big) > 2/\phi(\epsilon/2)\right\} \cup \left\{t \in A : \varphi\big(x(t)\big) > 2/\phi(\epsilon/2)\right\},$$

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where $\phi(t) := \int_0^t w\,dm$ for $t \in (0,\gamma)$. Then we have

$$2 \geq I_\varphi(x_n) + I_\varphi(x) \geq \int_0^{m(B_n)} \big(2/\phi(\epsilon/2)\big) w(t)\,dt = \big(2/\phi(\epsilon/2)\big)\phi\big(m(B_n)\big).$$

Hence $\phi\big(m(B_n)\big) \leq \phi(\epsilon/2)$, *i.e.* $m(B_n) \leq \epsilon/2$ for every $n \in N$. Define for $k \in N$

$$C_k = \{t \in A : w(t) > 1/k\}.$$

We have $C_k \uparrow$ and $\bigcup_{k=1}^\infty C_k = A$. Thus $m(A \setminus C_k) \longrightarrow 0$ as $k \longrightarrow \infty$ and in consequence $m\big((A_n \setminus B_n) \setminus C_k\big) < \epsilon/4$ for some k not depending on n. Now we shall estimate $m\big((A_n \setminus B_n) \cap C_k\big)$.

By the strict convexity of φ it follows that there exists $p \in (0,1)$ such that

$$\varphi\Big(\frac{u+v}{2}\Big) \leq \frac{1-p}{2}\big(\varphi(u) + \varphi(v)\big)$$

whenever $|u - v| \geq \delta$ and $\max\big(\varphi(u), \varphi(v)\big) \leq 2/\phi(\epsilon/2)$, $u, v \geq 0$. Thus we conclude that

$$\varphi\left(\frac{x_n(t) + x(t)}{2}\right) \leq \frac{1-p}{2}\Big(\varphi\big(x_n(t)\big) + \varphi\big(x(t)\big)\Big)$$

for every $t \in A_n \setminus B_n$, and by convexity of φ

$$\varphi\Big(\frac{x_n + x}{2}\Big) \leq \frac{1}{2}\big(\varphi(x_n) + \varphi(x)\big) - \frac{p}{2}\big(\varphi(x_n) + \varphi(x)\big)\chi_{A_n \setminus B_n},$$

on $[0,\gamma)$. This implies that for $D = (A_n \setminus B_n) \cap C_k$, we have

$$\begin{aligned} I_\varphi\Big(\frac{x_n + x}{2}\Big) &\leq \frac{1}{2}\big(I_\varphi(x_n) + I_\varphi(x)\big) - \frac{p}{2}\int_D \big(\varphi(x_n) + \varphi(x)\big) w\,dm \\ &\leq 1 - p\int_D \varphi\big((x_n - x)/2\big) w\,dm \\ &\leq 1 - \frac{p}{k}\int_D \varphi(\delta/2)\,dm = 1 - \frac{p}{k}\varphi(\delta/2) m(D). \end{aligned}$$

Since $\varphi \in \Delta_2$ and $\|x_n + x\| \longrightarrow 2$, we have

$$\lim_{n\to\infty} I_\varphi\Big(\frac{x_n + x}{2}\Big) = 1.$$

Consequently, we obtain that for n large enough, $m\big((A_n \setminus B_n) \cap C_k\big) < \epsilon/4$. It now follows that

$$m(A_n) \leq m(B_n) + m\big((A_n \setminus B_n) \setminus C_k\big) + m\big((A_n \setminus B_n) \cap C_k\big) < \epsilon$$

for n large enough, which finishes the proof that $x_n \longrightarrow x$ locally in measure.

Since (i) is satisfied, $\Lambda_{\varphi,w}$ is a separable r.i. rotund space (see [10]) and thus it is strictly K-monotone by Lemma 11. Now it follows from Theorems 1 and 2 that $x_n \longrightarrow x$ in $\Lambda_{\varphi,w}$. Thus we proved that the Orlicz-Lorentz space $\Lambda_{\varphi,w}$ has (LUR*)-property. This completes the proof by using Proposition 10, an obvious fact that the convergence in $L^1 + L^\infty$ implies the convergence in measure, and Theorem 2.

Now, we will study the strict K-monotonicity of $\Lambda_{\varphi,w}$. We will present some sufficient conditions and some necessary conditions separately. Unfortunately, criteria for strict K-monotonicity of Orlicz-Lorentz spaces are still unknown. In what follows we will need the following technical lemma which is an easy consequence of the Hardy, Littlewood and Poyla result (see Theorem 3).

LEMMA 13. *Let $x, y \in L^1 + L^\infty$ be such that $K(t, x) \leq K(t, y)$ for every $0 < t < \gamma$. Then $K\big(t, \varphi(x^*)\big) \leq K\big(t, \varphi(y^*)\big)$ for every $0 < t < \gamma$ and any Orlicz function φ.*

THEOREM 14. *Assume that the Orlicz function φ satisfies the Δ_2-condition and that $\int_0^\gamma w\, dm = +\infty$ if $\gamma = +\infty$. Then the Orlicz-Lorentz space $\Lambda_{\varphi,w}$ is strictly K-monotone if one of the following conditions holds:*

(i) φ is strictly convex.

(ii) w is strictly decreasing on $(0, \gamma)$.

PROOF. If (i) holds then by [10], it follows that $\Lambda_{\varphi,w}$ is rotund and thus also strictly K-monotone by Lemma 11.

Suppose now that condition (ii) holds and $x \prec y$. Since $\Lambda_{\varphi,w}$ is r.i. space, we have $\|x\| \leq \|y\|$. Assume $\|x\| = \|y\|$. We need to prove that $x^* = y^*$. We may assume without loss of generality that $\|x\| = \|y\| = 1$. Since $\varphi \in \Delta_2$, we have

$$\int_0^\gamma \varphi(x^*) w\, dm = \int_0^\gamma \varphi(y^*) w\, dm = 1. \tag{6}$$

Note that for any $f \in \Lambda_w$ and every $0 < s < \gamma$ the integration by parts yields

$$\begin{aligned} \int_0^s K(t, f)\, d\big(-w(t)\big) &= -K(t, f) w(t)\big|_0^s + \int_0^s f^*(t) w(t)\, dt \\ &= -K(s, f) w(s) + \int_0^s f^*(t) w(t)\, dt. \end{aligned} \tag{7}$$

Hence, by virtue of equalities (6) and (7), it easily follows that

$$\int_0^\gamma \Big(K\big(t, \varphi(x^*)\big) - K\big(t, \varphi(y^*)\big)\Big)\, d\big(-w(t)\big) = \int_0^\gamma \big(\varphi(y^*) - \varphi(x^*)\big) w\, dm = 0.$$

Since the function $K\big(t, \varphi(y^*)\big) - K\big(t, \varphi(x^*)\big)$ is continuous and nonnegative on $(0, \gamma)$, by Lemma 13, and w being strictly decreasing, we obtain that $K\big(t, \varphi(x^*)\big) = K\big(t, \varphi(y^*)\big)$ on $(0, \gamma)$. This implies that $\varphi\big(x^*(t)\big) = \varphi\big(y^*(t)\big)$ for every $t \in (0, \gamma)$ and in consequence $x^* = y^*$ on $(0, \gamma)$.

THEOREM 15. *If $\Lambda_{\varphi,w}$ is strictly K-monotone, then $\varphi \in \Delta_2$ and $\int_0^\gamma w\, dm = +\infty$ whenever $\gamma = +\infty$.*

PROOF. If $\varphi \notin \Delta_2$, then $\Lambda_{\varphi,w}$ contains an order isometric copy of l^∞ (see [10], [7]) and thus $\Lambda_{\varphi,w}$ is not strictly K-monotone.

Assume that $\gamma = +\infty$ and $\int_0^{+\infty} w\,dm < +\infty$. Let (A_n) be a sequence of pairwise disjoint measurable sets in $(0,+\infty)$ such that $m(A_n) = +\infty$, $n \in N$. Let a be such that $\varphi(a)\int_0^{+\infty} w\,dm = 1$. Define $x_n = a\chi_{A_n}$ for $n \in N$. Then $x_n^* = a\chi_{(0,+\infty)}$ and

$$\Big(\sum_{k=1}^{n} x_k\Big)^* = \Big(\sum_{k=1}^{\infty} x_k\Big)^* = a\chi_{(0,+\infty)}$$

implies that for all $n \in N$

$$\|x_n\| = \Big\|\sum_{k=1}^{n} x_k\Big\| = \Big\|\sum_{k=1}^{\infty} x_k\Big\| = 1,$$

Define an operator $T\colon l^\infty \longrightarrow \Lambda_{\varphi,w}$ by the formula

$$T\xi = \sum_{n=1}^{\infty} \xi_n x_n \quad \text{for } \xi = (\xi_n) \in l^\infty.$$

Since (x_n) is a sequence of pairwise disjoint elements, it follows simply by the above equalities that T is an order isometry. Thus we proved that if $\gamma = +\infty$ and $\int_0^\infty w\,dm < +\infty$, then $\Lambda_{\varphi,w}$ contains an order isometric copy of l_∞, and we are done.

In order to present the next result let as define

$$\alpha(\varphi) = \inf\{u > 0 : \varphi \text{ is affine on } [u,v] \text{ for some } v > u\},$$
$$\beta(w) = \inf\{s > 0 : w \text{ is constant on } [s,t] \text{ for some } t > s\}.$$

PROPOSITION 16. *Let φ be an Orlicz function and w be a weight function such that $\int_0^\gamma w\,dm = +\infty$ if $\gamma = +\infty$. If $\varphi\big(\alpha(\varphi)\big)\int_0^{\beta(w)} w\,dm < 1$, then $\Lambda_{\varphi,w}$ is not strictly K-monotone.*

PROOF. Let $a = \alpha(\varphi)$ and $b = \beta(w)$. By continuity of φ and $\phi(s) = \int_0^s w\,dm$, there exist $u > a$ and $c > b$ such that w is constant on $[b,c]$, φ is affine on $[a,u]$ and

$$\varphi(u)\int_0^c w\,dm < 1.$$

Choose $v > u$ such that

$$\varphi(v)\int_0^b w\,dm + \varphi(u)\int_b^{(b+c)/2} w\,dm + \varphi(a)\int_{(b+c)/2}^c w\,dm = 1$$

and define

$$x = v\chi_{(0,b)} + u\chi_{(b,(b+c)/2)} + a\chi_{(b+c)/2,c)},$$
$$y = v\chi_{(0,b)} + \big((a+u)/2\big)\chi_{(b,c)}.$$

It is obvious that $K(t,x) \geq K(t,y)$ for any $t \in (0,\gamma)$ and $x = x^* \neq y^* = y$. Since φ is affine on the interval $[a,u]$ and w is constant on the interval $[b,c]$, we get $I_\varphi(x) = I_\varphi(y) = 1$, and thus $\|x\| = \|y\| = 1$. This shows that $\Lambda_{\varphi,w}$ is not strictly K-monotone.

By applying Theorems 14, 15 and Proposition 16 we deduce the following.

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COROLLARY 17. *(i) The Orlicz space L^φ is strictly K-monotone if and only if $\varphi \in \Delta_2$ and φ is strictly convex.*

(ii) Λ_w is strictly K-monotone if and only if w is strictly decreasing on $(0,\gamma)$ and $\int_0^\gamma w\,dm = +\infty$ whenever $\gamma = +\infty$.

REMARK. In the case when $\gamma = 1$, Corollary 17(i) has been proved by Medzhitov and Sukochev in [13]. Corollary 17(ii) has been proved by Sedaev in [14] (*cf.* also [3]).

At the end of the paper, we will consider the Kadec-Klee property in Orlicz-Lorentz spaces. The Kadec-Klee property for Lorentz space Λ_w was studied by Sedaev in [14]. It is proved there that a necessary and sufficient condition for Λ_w to have the Kadec-Klee property is the condition that the weight function w is strictly decreasing. Note also that recently Dilworth and Hsu [5] have characterized the *uniform* Kadec-Klee property for Lorentz spaces.

THEOREM 18. *Let φ be an Orlicz function and w be a weight function. Then the following statements are true:*

(i) If $\varphi \in \Delta_2$, w is strictly decreasing on $(0,\gamma)$ and $\int_0^\gamma w\,dm = +\infty$ whenever $\gamma = +\infty$, then $\Lambda_{\varphi,w}$ has the Kadec-Klee property.

(ii) If $\varphi \notin \Delta_2$ or $\int_0^\gamma w\,dm < +\infty$ whenever $\gamma = +\infty$, then $\Lambda_{\varphi,w}$ does not have the Kadec-Klee property.

PROOF. Assume first that the assumptions from (i) hold. Let $x_n, x \in S(\Lambda_{\varphi,w})$ and $x_n \longrightarrow x$ weakly in $\Lambda_{\varphi,w}$. We will show that $x_n \longrightarrow x$ in measure on $(0,\gamma)$. Indeed, if this is not the case, then $x_n \not\longrightarrow x$ in $L^1 + L^\infty$. Since $\Lambda_{\varphi,w} \subset L^1_0$, by the assumption $\int_0^{+\infty} w\,dm = +\infty$, it follows by Theorem 5 that there are a subsequence (y_n) of (x_n) and $s \in (0,\gamma)$ such that

$$K(s,x) < \lim_{n\to\infty} K(s,y_n). \tag{8}$$

We have $y_n \longrightarrow x$ weakly in $L^1 + L^\infty$. It then easily follows that we have for $t \in (0,\gamma)$,

$$K(t,x) \le \liminf_{n\to\infty} K(t,y_n). \tag{9}$$

We define on $[0,\gamma)$ a sequence (ψ_n) of concave functions by

$$\psi_n(t) = \min\{K(t,y_n), 2K(t,x)\} \quad \text{for all } t \in (0,\gamma) \text{ and } \psi_n(0) = 0.$$

By Helly's Selection Theorem we may assume, by passing to a subsequence, that $\psi_n \longrightarrow f$ pointwise with f being concave and thus continuous. Now let $f_n = \inf\{\min(\psi_k, f) : k \ge n\}$. Then f_n are concave with $f_n(0) = 0$ and we have $f_n \uparrow f$ pointwise. Since f_n are continuous, it follows by the Dini's Theorem that $f_n \longrightarrow f$ uniformly on each interval $[0,a]$ with $a < \gamma$. This implies that for the derivatives the following holds:

$$f_n' \longrightarrow f' \quad \text{a.e. on } (0,\gamma). \tag{10}$$

We obtain from (8) and (9) that for every $t \in (0,\gamma)$,

$$f(t) = K(t,f') \ge K(t,x) \quad \text{and} \quad K(s,x) < K(s,f'). \tag{11}$$

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Since $K(t, f_n') = f_n(t) \leq K(t, y_n)$ for $t \in (0, \gamma)$, it follows by Lemma 13 that for any $n \in N$,

$$\varphi(f_n') \prec \varphi(y_n).$$

Thus by the K-monotonicity of the Lorentz space Λ_w, we obtain

$$\int_0^\gamma \varphi(f_n')w\,dm \leq \int_0^\gamma \varphi(y_n^*)w\,dm = I_\varphi(y_n) = 1.$$

Combining (10) and Fatou Lemma, we get

$$\int_0^\gamma \varphi(f')w\,dm \leq \liminf_{n\to\infty} \int_0^\gamma \varphi(f_n')w\,dm \leq 1.$$

This implies that $f' \in \Lambda_{\varphi,w}$ and $\|f'\| \leq 1$. Since $x^* \neq f'$, by virtue of $K(s, x) < K(s, f')$, and $\Lambda_{\varphi,w}$ being strictly K-monotone by Theorem 14, we obtain by (11)

$$1 = \|x\| < \|f'\| \leq 1.$$

This contradiction shows that $x_n \longrightarrow x$ in measure, and thus in view of Theorem 2, $x_n \longrightarrow x$ in $\Lambda_{\varphi,w}$.

Now if the assumptions from (ii) hold, then it follows by the proof of Theorem 15 that $\Lambda_{\varphi,w}$ contains an isometric copy of l_∞, so $\Lambda_{\varphi,w}$ does not have the Kadec-Klee property.

REMARK. Since LUR implies the Kadec-Klee property, it follows that under the assumptions of Theorem 12, the Orlicz-Lorentz space $\Lambda_{\varphi,w}$ has the Kadec-Klee property.

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