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SETS WITH ALMOST COINCIDING REPRESENTATION FUNCTIONS

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Abstract

For a given integer n and a set $\mathcal{S} \subseteq \mathbb{N}$, denote by $R^{(1)}_{h,\mathcal{S}}(n)$ the number of solutions of the equation $n = s_{i_1} + \cdots + s_{i_h}$, $s_{i_j} \in \mathcal{S}$, $j = 1, \ldots, h$. In this paper we determine all pairs $(\mathcal{A}, \mathcal{B})$, $\mathcal{A}, \mathcal{B} \subseteq \mathbb{N}$, for which $R^{(1)}_{3,\mathcal{A}}(n) = R^{(1)}_{3,\mathcal{B}}(n)$ from a certain point on. We discuss some related problems.

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1. Introduction

Let $\mathbb{N}$ be the set of nonnegative integers. For a given infinite set $\mathcal{A} \subset \mathbb{N}$ the representation functions $R^{(1)}_{h,\mathcal{A}}(n)$, $R^{(2)}_{h,\mathcal{A}}(n)$ and $R^{(3)}_{h,\mathcal{A}}(n)$ are defined in the following way:

$$R^{(1)}_{h,\mathcal{A}}(n) = \#\{(a_{i_1}, \ldots, a_{i_h}) : a_{i_1} + \cdots + a_{i_h} = n, a_{i_1}, \ldots, a_{i_h} \in \mathcal{A}\},$$
$$R^{(2)}_{h,\mathcal{A}}(n) = \#\{(a_{i_1}, \ldots, a_{i_h}) : a_{i_1} + \cdots + a_{i_h} = n, a_{i_1}, \ldots, a_{i_h} \in \mathcal{A}, a_{i_1} \le \cdots \le a_{i_h}\},$$
$$R^{(3)}_{h,\mathcal{A}}(n) = \#\{(a_{i_1}, \ldots, a_{i_h}) : a_{i_1} + \cdots + a_{i_h} = n, a_{i_1}, \ldots, a_{i_h} \in \mathcal{A}, a_{i_1} < \cdots < a_{i_h}\}.$$

Representation functions have been extensively studied by many authors and are still a fruitful area of research in additive number theory. Using generating functions, Nathanson [6] proved the following result.

Let A, B and T be finite sets of integers. If each residue class modulo m contains exactly the same number of elements of A as elements of B, then we write $A \equiv B \pmod{m}$. If the number of solutions of the congruence $a + t \equiv n \pmod{m}$ with $a \in A$, $t \in T$, equals the number of solutions of the congruence $b + t \equiv n \pmod{m}$ with $b \in B$, $t \in T$, for each residue class n modulo m, then we write $A + T \equiv B + T \pmod{m}$.

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Nathanson's Theorem. *Let $\mathcal{A}$ and $\mathcal{B}$ be infinite sets of nonnegative integers, $\mathcal{A} \neq \mathcal{B}$. Then $R^{(1)}_{2,\mathcal{A}}(n) = R^{(1)}_{2,\mathcal{B}}(n)$ from a certain point on if and only if there exist positive integers N, m and finite sets A, B, T with $A \cup B \subset \{0, 1, \ldots, N\}$ and $T \subset \{0, 1, \ldots, m-1\}$ such that $A + T \equiv B + T \pmod m$, and $\mathcal{A} = A \cup C$ and $\mathcal{B} = B \cup C$, where $C = \{c > N : c \equiv t \pmod m$ for some $t \in T\}$.*

It is clear that $R^{(2)}_{2,\mathcal{A}}(n) = \lceil R^{(1)}_{2,\mathcal{A}}(n)/2 \rceil$ and $R^{(3)}_{2,\mathcal{A}}(n) = \lfloor R^{(1)}_{2,\mathcal{A}}(n)/2 \rfloor$, so for the sets $\mathcal{A}$, $\mathcal{B}$ in Nathanson's theorem we have $R^{(2)}_{2,\mathcal{A}}(n) = R^{(2)}_{2,\mathcal{B}}(n)$ and $R^{(3)}_{2,\mathcal{A}}(n) = R^{(3)}_{2,\mathcal{B}}(n)$ from a certain point on. It is easy to see that the symmetric difference of the sets $\mathcal{A}$ and $\mathcal{B}$ in the above theorem is finite. Sárközy asked whether there exist two infinite sets of nonnegative integers $\mathcal{A}$ and $\mathcal{B}$ with infinite symmetric difference, that is,

$$|(A \cup B) \setminus (A \cap B)| = \infty$$

and

$$R^{(i)}_{2,\mathcal{A}}(n) = R^{(i)}_{2,\mathcal{B}}(n)$$

if $n \geq n_0$, for $i = 1, 2, 3$. For $i = 1$, the answer is negative (see [3]). For $i = 2$ and 3, respectively, Dombi [3] and Chen and Wang [2] proved that the set of nonnegative integers can be partitioned into two subsets $\mathcal{A}$ and $\mathcal{B}$ such that $R^{(i)}_{2,\mathcal{A}}(n) = R^{(i)}_{2,\mathcal{B}}(n)$ for all $n \geq n_0$. In [5] Lev gave a common proof of the above mentioned results of Dombi [3] and Chen and Wang [2]. Using generating functions, Sándor [7] determined the sets $\mathcal{A} \subset \mathbb{N}$ for which either

$$R^{(2)}_{2,\mathcal{A}}(n) = R^{(2)}_{2,\mathbb{N}\setminus\mathcal{A}}(n) \quad \text{for all } n \geq n_0$$

or

$$R^{(3)}_{2,\mathcal{A}}(n) = R^{(3)}_{2,\mathbb{N}\setminus\mathcal{A}}(n) \quad \text{for all } n \geq n_0.$$

In [8] Tang gave an elementary proof of Sándor's results and in [1] Chen and Tang studied related questions. We can rewrite Nathanson's theorem in equivalent form as follows.

Equivalent form of Nathanson's Theorem. *Let $\mathcal{A}$ and $\mathcal{B}$ be infinite sets of nonnegative integers, $\mathcal{A} \neq \mathcal{B}$. Then $R^{(1)}_{2,\mathcal{A}}(n) = R^{(1)}_{2,\mathcal{B}}(n)$ from a certain point on if and only if there exist positive integers n_0, M and finite sets $F_{\mathcal{A}}$, $F_{\mathcal{B}}$, T with $F_{\mathcal{A}} \cup F_{\mathcal{B}} \subset \{0, 1, \ldots, Mn_0 - 1\}$ and $T \subset \{0, 1, \ldots, M-1\}$ such that*

$$\mathcal{A} = F_{\mathcal{A}} \cup \{kM + t : k \geq n_0, t \in T\},$$
$$\mathcal{B} = F_{\mathcal{B}} \cup \{kM + t : k \geq n_0, t \in T\},$$

and

$$(1 - z^M) \mid (F_{\mathcal{A}}(z) - F_{\mathcal{B}}(z))T(z).$$

We conjecture that Nathanson's theorem can be generalised in the following way.

Conjecture. *Let $h \geq 2$, $\mathcal{A}$ and $\mathcal{B}$ be infinite sets of nonnegative integers, $\mathcal{A} \neq \mathcal{B}$. Then $R^{(1)}_{h,\mathcal{A}}(n) = R^{(1)}_{h,\mathcal{B}}(n)$ from a certain point on if and only if there exist positive* *integers* n_0, M *and sets* $F_{\mathcal{A}}$, $F_{\mathcal{B}}$ *and* T *such that* $F_{\mathcal{A}} \cup F_{\mathcal{B}} \subset \{0, 1, \ldots, Mn_0 - 1\}$, $T \subset \{0, 1. \ldots, M - 1\}$,

$$\mathcal{A} = F_{\mathcal{A}} \cup \{kM + t : k \geq n_0, t \in T\},$$
$$\mathcal{B} = F_{\mathcal{B}} \cup \{kM + t : k \geq n_0, t \in T\},$$

and

$$(1 - z^M)^{h-1} \mid (F_{\mathcal{A}}(z) - F_{\mathcal{B}}(z))T(z)^{h-1}.$$

The next theorem shows the sufficiency of the conjecture.

Theorem 1.1. *Let* $\mathcal{A}$ *and* $\mathcal{B}$ *be infinite sets of nonnegative integers,* $\mathcal{A} \neq \mathcal{B}$. *If there exist positive integers* n_0, M *and finite sets* $F_{\mathcal{A}}$, $F_{\mathcal{B}}$ *and* T *with* $F_{\mathcal{A}} \cup F_{\mathcal{B}} \subset \{0, 1, \ldots, Mn_0 - 1\}$, $T \subset \{0, 1, \ldots, M - 1\}$ *such that*

$$\mathcal{A} = F_{\mathcal{A}} \cup \{kM + t : k \geq n_0, t \in T\},$$
$$\mathcal{B} = F_{\mathcal{B}} \cup \{kM + t : k \geq n_0, t \in T\},$$

and

$$(1 - z^M)^{h-1} \mid (F_{\mathcal{A}}(z) - F_{\mathcal{B}}(z))T(z)^{h-1}$$

then $R^{(1)}_{h,\mathcal{A}}(n) = R^{(1)}_{h,\mathcal{B}}(n)$ *from a certain point on.*

However, we can only prove the conjecture in full in the case $h = 3$.

Theorem 1.2. *Let* $\mathcal{A}$ *and* $\mathcal{B}$ *be infinite sets of nonnegative integers,* $\mathcal{A} \neq \mathcal{B}$. *Then* $R^{(1)}_{3,\mathcal{A}}(n) = R^{(1)}_{3,\mathcal{B}}(n)$ *from a certain point on if and only if there exist positive integers* n_0, M *and sets* $F_{\mathcal{A}}$, $F_{\mathcal{B}}$ *and* T *with* $F_{\mathcal{A}} \cup F_{\mathcal{B}} \subset \{0, 1, \ldots, Mn_0 - 1\}$, $T \subset \{0, 1, \ldots, M - 1\}$ *such that*

$$\mathcal{A} = F_{\mathcal{A}} \cup \{kM + t : k \geq n_0, t \in T\}, \tag{1.1}$$
$$\mathcal{B} = F_{\mathcal{B}} \cup \{kM + t : k \geq n_0, t \in T\}, \tag{1.2}$$

and

$$(1 - z^M)^2 \mid (F_{\mathcal{A}}(z) - F_{\mathcal{B}}(z))T(z)^2. \tag{1.3}$$

In 2011, Yang [9] gave another proof of Nathanson's theorem without using generating functions. In his paper he posed the following problem.

Problem. *If* $p \geq 3$ *is a prime and* $\mathcal{A}$ *is an infinite set of nonnegative integers, then does there exist an infinite set of nonnegative integers* $\mathcal{B}$ *with* $\mathcal{A} \neq \mathcal{B}$ *such that* $R^{(1)}_{p,\mathcal{A}}(n) = R^{(1)}_{p,\mathcal{B}}(n)$ *for all sufficiently large* n?

In this paper we show that the answer to Yang's question is negative.

Theorem 1.3. *For every prime* p *there exists an infinite set of nonnegative integers* $\mathcal{A}$ *such that for any infinite set of integers* $\mathcal{B}$, $\mathcal{A} \neq \mathcal{B}$, *we have* $R^{(1)}_{p,\mathcal{A}}(n) \neq R^{(1)}_{p,\mathcal{B}}(n)$ *for infinitely many positive integer* n.

We studied some similar problems for the following results.

THEOREM 1.4. *For every positive integer $H \geq 2$ there exist infinite sets of nonnegative integers $\mathcal{A}$, $\mathcal{B}$, $\mathcal{A} \neq \mathcal{B}$ such that $R^{(l)}_{h,\mathcal{A}}(n) = R^{(l)}_{h,\mathcal{B}}(n)$, for every $l = 1, 2, 3$ and $2 \leq h \leq H$ from a certain point on.*

In the special case $l = 1$, Theorem 1.4 cannot be extended for infinitely many h.

THEOREM 1.5. *If for some infinite sets of nonnegative integers $\mathcal{A}$ and $\mathcal{B}$ the representation function $R^{(1)}_{h,\mathcal{A}}(n) = R^{(1)}_{h,\mathcal{B}}(n)$, for $n \geq n_0(h)$, for infinitely many positive integers $h \geq 2$, then $\mathcal{A} = \mathcal{B}$.*

In this paper let $A(z)$, $B(z)$, $F_{\mathcal{A}}(z)$, $F_{\mathcal{B}}$, $T(z)$, $S(z)$ denote the generating functions of the sets $\mathcal{A}$, $\mathcal{B}$, $F_{\mathcal{A}}$, $F_{\mathcal{B}}$, T and $S \subseteq \mathbb{N}$ (that is, $A(z) = \sum_{a \in \mathcal{A}} z^a$, where z is a complex number, $z = r \cdot e^{2\pi i\theta}$, and so on, and these functions converge in the open unit disc).

2. Proof of Theorem 1.1

In order to prove Theorem 1.1 we need to show that $A(z)^h - B(z)^h = P(z)$, where $P(z)$ is a polynomial. By definition of $\mathcal{A}$ and $\mathcal{B}$,

$$A(z) = F_{\mathcal{A}}(z) + \frac{z^{n_0 M} T(z)}{1 - z^M}$$

and

$$B(z) = F_{\mathcal{B}}(z) + \frac{z^{n_0 M} T(z)}{1 - z^M}.$$

Therefore, using the binomial theorem,

$$\begin{aligned} A(z)^h - B(z)^h &= \Big(F_{\mathcal{A}}(z) + \frac{z^{n_0 M} T(z)}{1 - z^M}\Big)^h - \Big(F_{\mathcal{B}}(z) + \frac{z^{n_0 M} T(z)}{1 - z^M}\Big)^h \\ &= \sum_{k=1}^{h} \binom{h}{k} \Big(\frac{z^{n_0 M} T(z)}{1 - z^M}\Big)^{h-k} (F_{\mathcal{A}}(z)^k - F_{\mathcal{B}}(z)^k). \end{aligned}$$

Now we verify that, for $1 \leq k \leq h - 1$,

$$(1 - z^M)^{h-k} \mid T(z)^{h-k}(F_{\mathcal{A}}(z)^k - F_{\mathcal{B}}(z)^k).$$

Since

$$F_{\mathcal{A}}(z) - F_{\mathcal{B}}(z) \mid F_{\mathcal{A}}(z)^k - F_{\mathcal{B}}(z)^k,$$

it is enough to show that

$$(1 - z^M)^{h-k} \mid T(z)^{h-k}(F_{\mathcal{A}}(z) - F_{\mathcal{B}}(z)).$$

For a given integer m, where $m \mid M$, denote by $\Phi_m(z)$ the mth cyclotomic polynomial. It remains to prove that

$$\Phi_m(z)^{h-k} \mid T(z)^{h-k}(F_{\mathcal{A}}(z) - F_{\mathcal{B}}(z)).$$

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Let $T(z) = \Phi_m(z)^{k_1} u(z)$ and $F_{\mathcal{A}}(z) - F_{\mathcal{B}}(z) = \Phi_m(z)^{k_2} v(z)$, where $u(z)$ and $v(z)$ are polynomials with the property $\Phi_m(z) \nmid u(z)v(z)$. By assumption of Theorem 1.1 we know that $(h-1)k_1 + k_2 \geq h-1$. Thus either $k_1 = 0$, so $k_2 \geq h-1$ and therefore

$$\Phi_m(z)^{h-k} \mid F_{\mathcal{A}}(z) - F_{\mathcal{B}}(z),$$

or $k_1 \geq 1$ and therefore

$$\Phi_m(z)^{h-k} \mid T(z)^{h-k},$$

which completes the proof.

3. Proof of Theorem 1.2

First we would like to prove that if $R^{(1)}_{3,\mathcal{A}}(n) = R^{(1)}_{3,\mathcal{B}}(n)$ from a certain point on then we have nonnegative integers n_0, M and finite sets of nonnegative integers $F_{\mathcal{A}}$, $F_{\mathcal{B}}$, T with $F_{\mathcal{A}} \cup F_{\mathcal{B}} \subset \{0, 1, \ldots, Mn_0 - 1\}$, $T \subset \{0, 1, \ldots, M-1\}$ such that (1.1)–(1.3) hold. It is easy to see that there exists a positive integer N_0 such that $\mathcal{A} \cap [N_0, +\infty) = \mathcal{B} \cap [N_0, +\infty)$, because $R^{(1)}_{3,\mathcal{A}}(n) \equiv 0 \pmod 3$ if $n/3 \notin \mathcal{A}$, and $R^{(1)}_{3,\mathcal{A}}(n) \equiv 1 \pmod 3$ if $n/3 \in \mathcal{A}$. Similarly, $R^{(1)}_{3,\mathcal{B}}(n) \equiv 0 \pmod 3$ if $n/3 \notin \mathcal{B}$, and $R^{(1)}_{3,\mathcal{B}}(n) \equiv 1 \pmod 3$ if $n/3 \in \mathcal{B}$. Thus there exist an integer N_1, finite sets of nonnegative integers $F_{\mathcal{A}}$, $F_{\mathcal{B}}$ and an infinite set of nonnegative integers S with $F_{\mathcal{A}}, F_{\mathcal{B}} \subset \{0, 1, \ldots, N_1\}$, $S \subset \{N_1+1, N_1+2, \ldots\}$ such that

$$\mathcal{A} = F_{\mathcal{A}} \cup S \tag{3.1}$$

and

$$\mathcal{B} = F_{\mathcal{B}} \cup S. \tag{3.2}$$

Since $A(z)$ and $B(z)$ are the generating functions of the sets $\mathcal{A}$ and $\mathcal{B}$,

$$A^3(z) = \sum_{n=0}^{\infty} R^{(1)}_{3,\mathcal{A}}(n) z^n$$

and

$$B^3(z) = \sum_{n=0}^{\infty} R^{(1)}_{3,\mathcal{B}}(n) z^n.$$

Since $R^{(1)}_{3,\mathcal{A}}(n) = R^{(1)}_{3,\mathcal{B}}(n)$, for $n \geq N_2$, it is clear that there is a polynomial $Q(z)$ such that

$$\sum_{n=1}^{\infty} R^{(1)}_{3,\mathcal{A}}(n) z^n - \sum_{n=1}^{\infty} R^{(1)}_{3,\mathcal{B}}(n) z^n = Q(z). \tag{3.3}$$

Thus $A^3(z) - B^3(z) = Q(z)$. In view of (3.1) and (3.2) it follows that

$$A(z) = F_{\mathcal{A}}(z) + S(z)$$

and

$$B(z) = F_{\mathcal{B}}(z) + S(z).$$

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Hence

$$(S(z)+F_{\mathcal{A}}(z))^3-(S(z)+F_{\mathcal{B}}(z))^3 = 3S^2(z)F_{\mathcal{A}}(z)+3S(z)F_{\mathcal{A}}^2(z)-3S^2(z)F_{\mathcal{B}}(z) - 3S(z)F_{\mathcal{B}}^2(z)+F_{\mathcal{A}}^3(z)-F_{\mathcal{B}}^3(z)=Q(z).$$

Since $F_{\mathcal{A}}$ and $F_{\mathcal{B}}$ are finite sets there is a polynomial $P(z)$ such that

$$3S(z)(S(z)+F_{\mathcal{A}}(z)+F_{\mathcal{B}}(z))(F_{\mathcal{A}}(z)-F_{\mathcal{B}}(z))=P(z).$$

It follows that there are relatively prime polynomials $P_1(z)$ and $P_2(z)$ such that

$$3S(z)(S(z)+F_{\mathcal{A}}(z)+F_{\mathcal{B}}(z))=\frac{P(z)}{F_{\mathcal{A}}(z)-F_{\mathcal{B}}(z)}=\frac{P_1(z)}{P_2(z)}. \tag{3.4}$$

The left-hand side of (3.4) converges in the open unit disc. Then

$$F_{\mathcal{A}}(z)-F_{\mathcal{B}}(z)=z^l(c_0+c_1z+\cdots+c_qz^q),$$

where $|c_0|=1$ and $|c_q|=1$. Thus

$$P_2(z)=z^k(d_0+d_1z+\cdots+d_wz^w),$$

where $|d_0|=1$ and $|d_w|=1$. Assume that $k\neq 0$. Then the right-hand side of (3.4) tends to infinity in absolute value and the left-hand side of (3.4) converges in absolute value when $z\to 0$, which is absurd. So $k=0$. Thus

$$P_2(z)=d_0+d_1z+\cdots+d_wz^w,$$

and

$$F_{\mathcal{A}}(z)-F_{\mathcal{B}}(z)=\sum_{n=0}^{N_1} f_nz^n,$$

where all the f_n are integers and $|f_n|\le 1$.

We now prove the following lemma.

Lemma 3.1. *If $P_2(z_0)=0$ for some complex number z_0, then $|z_0|\ge 1$.*

Proof. We prove this by contradiction. Assume that there exists $z_0\in\mathbb{C}$ such that $P_2(z_0)=0$ and $|z_0|<1$. Take the limit as $z\to z_0$ in (3.4). Then

$$3S(z)(S(z)+F_{\mathcal{A}}(z)+F_{\mathcal{B}}(z))\to 3S(z_0)(S(z_0)+F_{\mathcal{A}}(z_0)+F_{\mathcal{B}}(z_0))$$

and

$$|3S(z)(S(z)+F_{\mathcal{A}}(z)+F_{\mathcal{B}}(z))|\to |3S(z_0)(S(z_0)+F_{\mathcal{A}}(z_0)+F_{\mathcal{B}}(z_0))|\in\mathbb{R}.$$

Since $P_1(z)$ and $P_2(z)$ are relatively prime, $P_1(z_0)\neq 0$,

$$\left|\frac{P_1(z)}{P_2(z)}\right|\to\infty,$$

as $z\to z_0$, which is absurd. □

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We may suppose that $d_w = 1$. This means that the roots of $P_2(z)$ are algebraic integers. In this case the product of the roots of the polynomial $P_2(z)$ is d_0 and $|d_0| = 1$. It follows from Lemma 3.1 that the absolute value of each root is 1. Since $d_w = 1$ it is well known that the roots lie with their conjugates in the closed unit disc. It follows from a well-known theorem of Kronecker [4] that every root is a root of unity. Thus

$$P_2(z) = \prod_{j=1}^{u} (z - \varepsilon_j)^{m_j},$$

where ε_j is a root of unity and has multiplicity m_j.

We prove that for every j, $m_j \le 2$. Assume that there exists an $m_j \ge 3$. Then, from (3.4),

$$3S(z)(S(z) + F_{\mathcal{A}}(z) + F_{\mathcal{B}}(z))(z - \varepsilon_j)^2 = \frac{P_1(z)}{R(z)(z - \varepsilon_j)^{m_j - 2}}, \tag{3.5}$$

where $R(z)$ is a polynomial, $R(\varepsilon_j) \neq 0$ and $P_1(\varepsilon_j) \neq 0$. Then

$$\left| \frac{P_1(r\varepsilon_j)}{R(r\varepsilon_j)(r\varepsilon_j - \varepsilon_j)^{m_j - 2}} \right| \to \infty,$$

as $r \to 1^-$. For $z = r\varepsilon_j$, we have $|z - \varepsilon_j|^2 = \left| r\varepsilon_j - \varepsilon_j \right|^2 = (1 - r)^2$ and

$$S(z) = \sum_{n=0}^{\infty} \chi_S(n) z^n,$$

where $\chi_S(n)$ is the characteristic function of the set S (that is, $\chi_S(n) = 1$, if $n \in S$ and $\chi_S = 0$, if $n \notin S$). Then we have the following estimation of the left-hand side of (3.5) for $r < 1$:

$$\begin{aligned} &|3S(r\varepsilon_j)| \cdot |(S(r\varepsilon_j) + F_{\mathcal{A}}(r\varepsilon_j) + F_{\mathcal{B}}(r\varepsilon_j))| \cdot |r\varepsilon_j - \varepsilon_j|^2 \\ &\le 3\Big(\sum_{n=0}^{\infty} \chi(n)|r|^n\Big)\Big(\sum_{n=0}^{\infty} \chi(n)|r|^n + C_1\Big) \cdot (1 - r)^2 \\ &< \frac{C_2}{(1 - r)^2} \cdot (1 - r)^2 = C_2, \end{aligned}$$

which is absurd.

Thus for some positive integer M we have $P_2(z) \mid (1 - z^M)^2$, so there is a polynomial $P_3(z)$ such that

$$3S(z)(S(z) + F_{\mathcal{A}}(z) + F_{\mathcal{B}}(z)) = \frac{P_3(z)}{(1 - z^M)^2}. \tag{3.6}$$

Multiplying (3.6) by 12 and adding $9(F_{\mathcal{A}}(z) + F_{\mathcal{B}}(z))^2$ to it gives us

$$(6S(z) + 3F_{\mathcal{A}}(z) + 3F_{\mathcal{B}}(z))^2 = \frac{P_4(z)}{(1 - z^M)^2}.$$

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So

$$(6S(z) + 3F_{\mathcal{A}}(z) + 3F_{\mathcal{B}}(z))^2(1 - z^M)^2 = P_4(z).$$

We prove that $P_4(z) = (u(z))^2$, where $u(z)$ is a polynomial with integer coefficients.

Let

$$|(6S(z) + F_{\mathcal{A}}(z) + F_{\mathcal{B}}(z))^2| \cdot |(1 - z^M)^2| = \left|\sum_{n=0}^{\infty} g_n z^n\right|^2 = |P_4(z)|, \tag{3.7}$$

where $g_n \in \mathbb{Z}$. Since $P_4(z)$ is a polynomial, the integral $\int_0^{2\pi} |P_4(z)|\, d\theta$ is bounded for $r \leq 1$. On the other hand, if there exist infinitely many n such that $g_n \neq 0$, that is, $g_n^2 \geq 1$, then, using the Parseval formula,

$$\int_0^{2\pi} \left|\sum_{n=0}^{\infty} g_n z^n\right|^2 d\theta = \sum_{n=0}^{\infty} g_n^2 r^{2n} \to \infty,$$

as $r \to 1^-$, which is absurd. Thus the series $\sum_{n=0}^{\infty} g_n z^n = u(z)$ is a polynomial.

This means that there is an integer K such that if $n \geq K$ then $g_n = 0$, and according to (3.7) if $n \geq N_3$ then $g_n = 6(\chi(n) - \chi(n + M)) = 0$. So χ is periodic in M. Therefore, there exist a positive integer n_0 and finite sets $F_{\mathcal{A}}$, $F_{\mathcal{B}}$, T with $F_{\mathcal{A}} \cup F_{\mathcal{B}} \subset \{0, 1, \ldots, Mn_0 - 1\}$ and $T \subset \{0, 1, \ldots, M - 1\}$ such that

$$A = F_{\mathcal{A}} \cup \{kM + t : k \geq n_0, t \in T\},$$

and

$$B = F_{\mathcal{B}} \cup \{kM + t : k \geq n_0, t \in T\}.$$

Hence the generating functions of $\mathcal{A}$ and $\mathcal{B}$ are

$$A(z) = F_{\mathcal{A}}(z) + \frac{T(z)z^{n_0 M}}{1 - z^M}$$

and

$$B(z) = F_{\mathcal{B}}(z) + \frac{T(z)z^{n_0 M}}{1 - z^M}.$$

Then, from (3.3),

$$A^3(z) - B^3(z) = \left(\frac{T(z)z^{n_0 M}}{1 - z^M} + F_{\mathcal{A}}(z)\right)^3 - \left(\frac{T(z)z^{n_0 M}}{1 - z^M} + F_{\mathcal{B}}(z)\right)^3 = Q(z).$$

Thus

$$\frac{3T(z)z^{n_0 M}}{1 - z^M}\left(\frac{T(z)z^{n_0 M}}{1 - z^M} + F_{\mathcal{A}}(z) + F_{\mathcal{B}}(z)\right)(F_{\mathcal{A}}(z) - F_{\mathcal{B}}(z)) = P(z),$$

that is,

$$\frac{T(z)z^{n_0 M}(T(z)z^{n_0 M} + (F_{\mathcal{A}}(z) + F_{\mathcal{B}}(z))(1 - z^M))(F_{\mathcal{A}}(z) - F_{\mathcal{B}}(z))}{(1 - z^M)^2} = R(z),$$

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where $R(z)$ is also a polynomial. Since $(1 - z^M, z^{n_0 M}) = 1$,

$$(1 - z^M)^2 \mid T(z)(T(z)z^{n_0 M} + (F_{\mathcal{A}}(z) + F_{\mathcal{B}}(z))(1 - z^M))(F_{\mathcal{A}}(z) - F_{\mathcal{B}}(z)), \tag{3.8}$$

that is,

$$\begin{aligned}(1 - z^M)^2 \mid z^{n_0 M}&(F_{\mathcal{A}}(z) - F_B(z))T(z)^2 \\ &+ (1 - z^M)(F_{\mathcal{A}}(z) + F_B(z))(F_{\mathcal{A}}(z) - F_B(z))T(z).\end{aligned} \tag{3.9}$$

We prove that $1 - z^M \mid (F_{\mathcal{A}}(z) - F_{\mathcal{B}}(z))T(z)$. By way of contradiction, assume that

$$1 - z^M \nmid (F_{\mathcal{A}}(z) - F_{\mathcal{B}}(z))T(z).$$

This means that there exists an integer k such that $k \mid M$ and

$$\Phi_k(z) \nmid (F_{\mathcal{A}}(z) - F_{\mathcal{B}}(z))T(z).$$

Then, by (3.8),

$$\Phi_k(z) \mid T(z)z^{n_0 M} + (F_{\mathcal{A}}(z) + F_{\mathcal{B}}(z))(1 - z^M).$$

Thus $\Phi_k(z) \mid T(z)z^{n_0 M}$, but since $(\Phi_k(z), z^{n_0 M}) = 1$ we get $\Phi_k(z) \mid T(z)$, which is absurd. Then

$$(1 - z^M)^2 \mid (1 - z^M)(F_{\mathcal{A}}(z) + F_B(z))(F_{\mathcal{A}}(z) - F_B(z))T(z),$$

so, by (3.9),

$$(1 - z^M)^2 \mid z^{n_0 M}(F_{\mathcal{A}}(z) - F_B(z))T(z)^2.$$

But, using the fact that $((1 - z^M)^2, z^{n_0 M}) = 1$, this means that (1.3) holds, as desired.

The other direction is a corollary of Theorem 1.1.

4. Proof of Theorem 1.3

Let $\mathcal{A}$ be a sparse set, which means that $\alpha(N) < N^{1/p}$ (here, $\alpha(N) = |[0, N] \cap \mathcal{A}|$). Let $\mathcal{A} = \{a_1, a_2, \ldots\}$. We prove the theorem by contradiction. Assume that $\mathcal{A}, \mathcal{B}$ are different sets and $R^{(1)}_{p,\mathcal{A}}(n) = R^{(1)}_{p,\mathcal{B}}(n)$ from a certain point on. Since $\alpha(a_k) = k < a_k^{1/p}$, it follows that $a_k > k^p$. The generating function of $\mathcal{A}$ is

$$\begin{aligned}A(r) = \sum_{a \in \mathcal{A}} r^a &= \sum_{n=0}^{\infty} \chi_{\mathcal{A}}(n) r^n = \sum_{n=0}^{\infty} (\alpha(n) - \alpha(n-1)) r^n \\ &= \sum_{n=1}^{\infty} \alpha(n)(r^n - r^{n+1}) = (1 - r) \sum_{n=0}^{\infty} \alpha(n) r^n \\ &= O((1 - r) \cdot (1 - r)^{-1/p - 1}) = O((1 - r)^{-1/p}),\end{aligned} \tag{4.1}$$

as $r \to 1^-$, where $\chi_{\mathcal{A}}(n)$ is the characteristic function of the set $\mathcal{A}$.

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Since $R^{(1)}_{p,\mathcal{A}}(n) = R^{(1)}_{p,\mathcal{B}}(n)$, it is clear that there is a polynomial $P(r)$ such that

$$A^p(r) - B^p(r) = P(r).$$

It is easy to see that there exists a positive integer N_0 such that $\mathcal{A} \cap [N_0, +\infty) = \mathcal{B} \cap [N_0, +\infty)$, because $R^{(1)}_{p,\mathcal{A}}(n) \equiv 0 \pmod p$ if $n/p \notin \mathcal{A}$, and $R^{(1)}_{p,\mathcal{A}}(n) \equiv 1 \pmod p$ if $n/p \in \mathcal{A}$. Similarly, $R^{(1)}_{p,\mathcal{B}}(n) \equiv 0 \pmod p$ if $n/p \notin \mathcal{B}$, and $R^{(1)}_{p,\mathcal{B}}(n) \equiv 1 \pmod p$ if $n/p \in \mathcal{B}$. Thus $A(r)$ differs from $B(r)$ in a polynomial, which means that

$$B(r) = O((1-r)^{-1/p}), \tag{4.2}$$

as $r \to 1^-$, as well. So

$$(A(r) - B(r))(A^{p-1}(r) + \cdots + B^{p-1}(r)) = P(r). \tag{4.3}$$

Therefore, there exist relatively prime polynomials $R(r)$ and $S(r)$ such that

$$R(r)(A^{p-1}(r) + \cdots + B^{p-1}(r)) = S(r). \tag{4.4}$$

As $r \to 1^-$ in (4.3) we get that $S(r)$ and $R(r)$ are bounded, and

$$A^{p-1}(r) + \cdots + B^{p-1}(r) \to \infty.$$

Therefore $r = 1$ must be a root of $R(r)$. Thus

$$R(r) = (1-r)Q(r).$$

Now we can write (4.4) in the form

$$(1-r)Q(r)(A^{p-1}(r) + \cdots + B^{p-1}(r)) = S(r). \tag{4.5}$$

Since $Q(r)$ is a polynomial, it is bounded. It follows from (4.1) and (4.2) that

$$A^{p-1}(r) + \cdots + B^{p-1}(r) = O((1-r)^{-(p-1)/p}).$$

So the order of the left-hand side of (4.5) is $O((1-r)^{1/p})$, as $r \to 1^-$. This means that $S(r)$ tends to zero as $r \to 1^-$. So $S(r) = (1-r)T(r)$, and this contradicts $(R(r), S(r)) = 1$.

5. Proof of Theorem 1.4

The construction of the sets $\mathcal{A}$ and $\mathcal{B}$ is as follows. Let n be a positive integer. Take the binary representation of n to be

$$n = \sum_{i=0}^{\lfloor \log_2(n) \rfloor} \beta_i 2^i,$$

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where $\beta_i = 0$ or 1. Denote by $\mathrm{Bin}(n) = \sum_{i=0}^{\lfloor \log_2(n) \rfloor} \beta_i$ the number of ones in the binary representation of n. Let

$$F_{\mathcal{A}} := \{kH! : 0 \le k < 2^H, \mathrm{Bin}(kH!) \equiv 0 \pmod 2\}$$

and

$$F_{\mathcal{B}} := \{kH! : 0 \le k < 2^H, \mathrm{Bin}(kH!) \equiv 1 \pmod 2\}.$$

We will show that the sets

$$A = F_{\mathcal{A}} \cup \{H!2^H, H!2^H + 1, \dots\}$$

and

$$B = F_{\mathcal{B}} \cup \{H!2^H, H!2^H + 1, \dots\}$$

are suitable. Let h be a fixed integer, $2 \le h \le H$. Then

$$F_{\mathcal{A}}(z) - F_{\mathcal{B}}(z) = \prod_{i=0}^{H-1} (1 - z^{H!2^i}),$$

and therefore

$$(1 - z^{h!}) \cdots (1 - z^{2^{h-1}h!}) \mid F_{\mathcal{A}}(z) - F_{\mathcal{B}}(z).$$

Hence

$$(1 - z) \cdots (1 - z^{h-1})(1 - z^h) \mid F_{\mathcal{A}}(z) - F_{\mathcal{B}}(z).$$

The generating function of $R^{(l)}_{h,\mathcal{A}}(n)$, $l = 1, 2, 3$, can be written using a sieve formula with suitable real numbers $C_{k_1,\dots,k_h}$:

$$\sum_{n=0}^{\infty} R^{(l)}_{h,\mathcal{A}}(n) z^n = \sum_{\substack{(k_1,\dots,k_h) \\ k_1+2k_2+\cdots+hk_h=h \\ k_i \ge 0,\ i=1,\dots,h}} C_{k_1,\dots,k_h} \prod_{i=1}^{h} A(z^i)^{k_i}. \tag{5.1}$$

We would like to prove that there is a polynomial $P(z)$ such that

$$\sum_{n=0}^{\infty} R^{(l)}_{h,\mathcal{A}}(n) z^n - \sum_{n=0}^{\infty} R^{(l)}_{h,\mathcal{B}}(n) z^n = P(z). \tag{5.2}$$

From (5.1), the left-hand side of (5.2) is equivalent to

$$\sum_{\substack{(k_1,\dots,k_h) \\ k_1+2k_2+\cdots+hk_h=h \\ k_i \ge 0,\ i=1,\dots,h}} C_{k_1,\dots,k_h} \Big(\prod_{i=1}^{h} A(z^i)^{k_i} - \prod_{i=1}^{h} B(z^i)^{k_i} \Big). \tag{5.3}$$

In view of

$$A(z) = F_{\mathcal{A}}(z) + \frac{z^{H!2^H}}{1 - z}$$

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and

$$B(z) = F_{\mathcal{B}}(z) + \frac{z^{H!2^H}}{1-z},$$

we get that (5.3) is equivalent to

$$\sum_{\substack{(k_1,\dots,k_h)\\ k_1+2k_2+\cdots+hk_h=h\\ k_i\ge 0,\ i=1,\dots,h}} C_{k_1,\dots,k_h}\Big(\prod_{i=1}^h\Big(F_{\mathcal{A}}(z^i)+\frac{z^{iH!2^H}}{1-z^i}\Big)^{k_i} - \prod_{i=1}^h\Big(F_{\mathcal{B}}(z^i)+\frac{z^{iH!2^H}}{1-z^i}\Big)^{k_i}\Big). \tag{5.4}$$

It is enough to show that the difference of the products in (5.4) is a polynomial for every h-tuple $(k_1,\dots,k_h)$. Let the h-tuple $(k_1,\dots,k_h)$ be fixed. Using the binomial theorem, we get that for suitable constants $D_{j_1,\dots,j_h}$ this expression is equal to

$$\Big(\prod_{i=1}^h\sum_{j_i=0}^{k_i}\binom{k_i}{j_i}(F_{\mathcal{A}}(z^i))^{j_i}\Big(\frac{z^{iH!2^H}}{1-z^i}\Big)^{k_i-j_i}\Big)-\Big(\prod_{i=1}^h\sum_{j_i=0}^{k_i}\binom{k_i}{j_i}(F_{\mathcal{B}}(z^i))^{j_i}\Big(\frac{z^{iH!2^H}}{1-z^i}\Big)^{k_i-j_i}\Big)$$

$$= \sum_{\substack{(j_1,\dots,j_h)\\ 0\le j_i\le k_i,\ i=1,\dots,h}} D_{j_1,\dots,j_h}\Big(\prod_{i=1}^h\Big(\frac{z^{iH!2^H}}{1-z^i}\Big)^{k_i-j_i}\Big)\Big(\prod_{i=1}^h(F_{\mathcal{A}}(z^i))^{j_i}-\prod_{i=1}^h(F_{\mathcal{B}}(z^i))^{j_i}\Big).$$

We will show that

$$\Big(\prod_{i=1}^h\Big(\frac{z^{iH!2^H}}{1-z^i}\Big)^{k_i-j_i}\Big)\Big(\prod_{i=1}^h(F_{\mathcal{A}}(z^i))^{j_i}-\prod_{i=1}^h(F_{\mathcal{B}}(z^i))^{j_i}\Big)$$

is a polynomial. To show this we will prove that there is a polynomial $Q(z)$ such that

$$\prod_{i=1}^h\Big(\frac{z^{iH!2^H}}{1-z^i}\Big)^{k_i-j_i} = \frac{Q(z)}{(1-z)\cdots(1-z^{h-1})(1-z^h)}, \tag{5.5}$$

and

$$(1-z)\cdots(1-z^{h-1})(1-z^h)\ \Big|\ \prod_{i=1}^h(F_{\mathcal{A}}(z^i))^{j_i}-\prod_{i=1}^h(F_{\mathcal{B}}(z^i))^{j_i}. \tag{5.6}$$

To deduce (5.5) it is enough to show that

$$\prod_{i=1}^h(1-z^i)^{k_i-j_i} \mid (1-z)\cdots(1-z^{h-1})(1-z^h).$$

A root of the product $\prod_{i=1}^h(1-z^i)^{k_i-j_i}$ is a primitive ith root of unity, for some $i\le h$. Let ε_i denote a primitive ith root of unity. The multiplicity of ε_i in the polynomial $(1-z)\cdots(1-z^{h-1})(1-z^h)$ is $\lfloor h/i\rfloor$. The multiplicity of ε_i in the polynomial $\prod_{i=1}^h(1-z^i)^{k_i-j_i}$ is

$$(k_i-j_i)+(k_{2i}-j_{2i})+\cdots\le k_i+k_{2i}+\cdots.$$

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We know that $k_1 + 2k_2 + \cdots + hk_h = h$. Therefore,

$$ik_i + ik_{2i} + \cdots \le ik_i + 2ik_{2i} + \cdots \le 1k_1 + 2k_2 + \cdots + hk_h = h.$$

This means that

$$k_i + k_{2i} + \cdots \le \left\lfloor \frac{h}{i} \right\rfloor,$$

which proves (5.5).

It remains to prove the following lemma, which verifies (5.6).

Lemma 5.1. *If $(1-z)\cdots(1-z^{h-1})(1-z^h) \mid F_{\mathcal{A}}(z) - F_{\mathcal{B}}(z)$ then, for all t-tuples $(l_1, \ldots, l_t)$,*

$$(1-z)\cdots(1-z^{h-1})(1-z^h) \,\Big|\, \prod_{i=1}^{t}(F_{\mathcal{A}}(z^i))^{l_i} - \prod_{i=1}^{t}(F_{\mathcal{B}}(z^i))^{l_i}.$$

Proof. We prove this result by induction on t. If $t = 1$ then we show that

$$(1-z)\cdots(1-z^{h-1})(1-z^h) \mid (F_{\mathcal{A}}(z))^{l_1} - (F_{\mathcal{B}}(z))^{l_1}.$$

Since

$$(F_{\mathcal{A}}(z))^{l_1} - (F_{\mathcal{B}}(z))^{l_1} = (F_{\mathcal{A}}(z) - F_{\mathcal{B}}(z))((F_{\mathcal{A}}(z))^{l_1-1} + \cdots + (F_{\mathcal{B}}(z))^{l_1-1}),$$

we get that the case $t = 1$ holds.

Now assume that the lemma holds for all t or less. For $t + 1$ we need to show that

$$(1-z)\cdots(1-z^{h-1})(1-z^h) \,\Big|\, \prod_{i=1}^{t+1}(F_{\mathcal{A}}(z^i))^{l_i} - \prod_{i=1}^{t+1}(F_{\mathcal{B}}(z^i))^{l_i}. \tag{5.7}$$

The right-hand side of (5.7) is equal to

$$\begin{aligned}
&(F_{\mathcal{A}}(z))^{l_1}\cdots(F_{\mathcal{A}}(z^{t+1})^{l_{t+1}}) - (F_{\mathcal{A}}(z))^{l_1}\cdots(F_{\mathcal{A}}(z^t))^{l_t}(F_{\mathcal{B}}(z^{t+1}))^{l_{t+1}}\\
&\quad + (F_{\mathcal{A}}(z))^{l_1}\cdots(F_{\mathcal{A}}(z^t))^{l_t}(F_{\mathcal{B}}(z^{t+1}))^{l_{t+1}} - (F_{\mathcal{B}}(z))^{l_1}\cdots(F_{\mathcal{B}}(z^{t+1}))^{l_{t+1}}\\
&\quad = (F_{\mathcal{A}}(z))^{l_1}\cdots(F_{\mathcal{A}}(z^t))^{l_t}((F_{\mathcal{A}}(z^{t+1}))^{l_{t+1}} - (F_{\mathcal{B}}(z^{t+1}))^{l_{t+1}})\\
&\qquad -(F_{\mathcal{B}}(z^{t+1}))^{l_{t+1}}((F_{\mathcal{A}}(z))^{l_1}\cdots(F_{\mathcal{A}}(z^t))^{l_t} - (F_{\mathcal{B}}(z))^{l_1}\cdots(F_{\mathcal{B}}(z^t))^{l_t}).
\end{aligned}$$

Because of our assumption, the second term is divisible by $(1-z)\cdots(1-z^{h-1})\cdot(1-z^h)$. Since

$$(1-z)\cdots(1-z^{h-1})(1-z^h) \mid (1-z^{t+1})\cdots(1-z^{h(t+1)})$$

and

$$(1-z^{t+1})\cdots(1-z^{h(t+1)}) \mid (F_{\mathcal{A}}(z^{t+1}))^{l_{t+1}} - ((F_{\mathcal{B}}(z^{t+1}))^{l_{t+1}}),$$

this completes the induction. □

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6. Proof of Theorem 1.5

We prove the theorem by contradiction. Assume that for infinite sets of nonnegative integers $\mathcal{A}$, $\mathcal{B}$, $\mathcal{A} \neq \mathcal{B}$, there is an infinite sequence of integers $2 \leq h_1 < h_2 < \cdots < h_i < \cdots$ and polynomials $P_i(r)$ such that

$$A^{h_i}(r) - B^{h_i}(r) = \sum_{n=0}^{\infty}(R^{(1)}_{h_i,\mathcal{A}}(n) - R^{(1)}_{h_i,\mathcal{B}}(n))r^n = P_i(r).$$

Then

$$P_i(r) = A^{h_i}(r) - B^{h_i}(r) = (A(r) - B(r))(A^{h_i-1}(r) + A^{h_i-2}(r)B(r) + \cdots + B^{h_i-1}(r)).$$

As $r \to 1^-$,

$$\frac{P_{i+1}(r)}{P_i(r)} = \frac{A^{h_i-1}(r) + A^{h_i-2}(r)B(r) + \cdots + B^{h_i-1}(r)}{A^{h_{i+1}-1}(r) + A^{h_{i+1}-2}(r)B(r) + \cdots + B^{h_{i+1}-1}(r)}$$
$$\leq \frac{h_i \cdot \max\{A^{h_i-1}(r), B^{h_i-1}(r)\}}{\max\{A^{h_{i+1}-1}(r), B^{h_{i+1}-1}(r)\}} \to 0.$$

Let $P_i(r) = (1 - r)^{m_i} Q_i(r)$, where m_i is a nonnegative integer, $Q_i(r)$ is a polynomial and $Q_i(1) \neq 0$. Thus

$$\frac{P_{i+1}(r)}{P_i(r)} = \frac{(1 - r)^{m_{i+1}} Q_{i+1}(r)}{(1 - r)^{m_i} Q_i(r)},$$

and $m_{i+1} < m_i$. We get that $m_1 > m_2 > \cdots$, which is absurd.

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