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ON THE FIXED POINTS OF A FINITE GROUP ACTING ON A RELATIVELY FREE LIE ALGEBRA

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Abstract. We show that if F is a free Lie algebra of rank at least 2 and if G is a non-trivial finite group of automorphisms of F then the fixed point subalgebra F^G is not finitely generated. Some similar results are proved for relatively free Lie algebras.

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1. Introduction. Well known results in commutative and non-commutative invariant theory concern the action of a finite group on a free algebra (such as a polynomial algebra or a free associative algebra) and give conditions under which the fixed point subalgebra is finitely generated—see [**6**] for a survey. The corresponding question for free Lie algebras was partly answered in [**2**] and [**5**]. The main purpose of the present paper is to complete this answer. In [**2**], the first author showed that if F is a finitely generated free Lie algebra over a field K, where the rank of F is at least 2, and if G is a non-trivial finite group of graded Lie algebra automorphisms of F, then the fixed point subalgebra F^G is not finitely generated. A similar result was later (and independently) proved by Drensky ([**5**]) for an arbitrary non-trivial finite subgroup G of $\mathrm{Aut}(F)$, but under the additional assumption that $|G|$ is not divisible by the characteristic of K. The first main result of the present paper is a common extension of these two results (which also applies to free Lie algebras which are not finitely generated).

Theorem A. *Let F be a free Lie algebra of rank greater than* 1 *over a field K and let G be a non-trivial finite subgroup of* $\mathrm{Aut}(F)$. *Then F^G is not finitely generated.*

Drensky ([**5**]) also obtained an analogous result for free metabelian Lie algebras but again under the assumption that $|G|$ is not divisible by the characteristic of K. Our second main result removes this restriction.

Theorem B. *Let M be a free metabelian Lie algebra of rank greater than* 1 *over a field K and let G be a non-trivial finite subgroup of* $\mathrm{Aut}(M)$. *Then M^G is not finitely generated.*

Our third main result is a closely-related one for arbitrary finitely generated relatively free Lie algebras, under some additional mild restrictions on K and G.

THEOREM C. *Let R be a finitely generated relatively free Lie algebra over an infinite field K and let G be a non-trivial finite subgroup of* $\mathrm{Aut}(R)$ *which acts faithfully on the derived factor algebra R/R', where $R' = [R, R]$. Then R^G is finitely generated if and only if R is nilpotent.*

It is hoped that the methods used in the proofs of these results will be of independent interest. In particular, we give a simple but useful necessary condition for a subalgebra of a free Lie algebra to be finitely generated (see Lemma 2.3).

Section 2 of this paper contains some definitions, notation and preliminary results, and we continue in Section 3 with a key result about polynomial algebras. Theorems B and C will be proved in Section 4, and Theorem A will be proved in Section 5.

2. Preliminaries.

Let K be a field and let G be a group. For any (right) KG-module U we write

$$U^G = \{u \in U : ug = u \text{ for all } g \in G\}.$$

If E is a K-algebra (associative or non-associative) and if G is a subgroup of the group of algebra automorphisms $\mathrm{Aut}(E)$ then we write the action of G on the right. Thus E may be regarded as a KG-module and E^G is a subalgebra of E, the fixed point subalgebra of E.

For any subset S of a K-space (vector space over K) we write $\langle S\rangle$ for the K-subspace spanned by S.

For background material on Lie algebras we refer to [**1**] and [**9**]. For any Lie algebra L we use commutator notation $[u, v]$ to denote the product of elements u and v of L, while $[u_1, u_2, \dots, u_n]$ denotes the left-normed product of elements $u_1, \dots, u_n$ of L. The derived algebra $[L, L]$ and the second derived algebra $[[L, L], [L, L]]$ of L will usually be denoted by L' and L'', respectively. For each positive integer m, $\gamma_m(L)$ denotes the m-th term of the lower central series of L: thus $\gamma_1(L) = L$, $\gamma_2(L) = L'$ and $\gamma_m(L) = [\gamma_{m-1}(L), L]$ for all $m \geqslant 2$.

As usual we say that L is *residually nilpotent* if $\bigcap_{m=1}^{\infty} \gamma_m(L) = \{0\}$. We write $\mathrm{IA}(L)$ for the normal subgroup of $\mathrm{Aut}(L)$ consisting of all automorphisms of L which induce the identity automorphism on L/L'; these are the so-called IA-automorphisms.

LEMMA 2.1. *Let L be a residually nilpotent Lie algebra over a field K and let G be a non-trivial finite subgroup of* $\mathrm{IA}(L)$. *Then K has prime characteristic p and G is a p-group.*

Proof. Let g be a non-trivial element of G and let n be the order of g. Since g is non-trivial there exists an element a of L such that $ag \neq a$. Write $ag = a + b$, where $b \neq 0$. Thus, since $g \in \mathrm{IA}(L)$, we have $b \in \gamma_2(L)$. Since L is residually nilpotent, there exists a positive integer m such that $b \in \gamma_m(L)$ but $b \notin \gamma_{m+1}(L)$. Since $g \in \mathrm{IA}(L)$, we find that $bg - b \in \gamma_{m+1}(L)$.

An easy calculation shows that $a = ag^n = a + nb + c$ where $c \in \gamma_{m+1}(L)$. Thus $nb \in \gamma_{m+1}(L)$. Since $b \notin \gamma_{m+1}(L)$ we find that K has non-zero characteristic p and n is divisible by p. Arguing by induction on n, we can assume that g^p has p-power order. Hence g has p-power order, and so G is a p-group.

LEMMA 2.2. *Let G be a non-trivial group of automorphisms of a residually nilpotent Lie algebra L. Then $L^G + L' \neq L$.*

Proof. It is sufficient to prove the result in the case where G is cyclic. Suppose then that g is a generator of G. Since $g \neq 1$ there exists $a \in L$ such that $ag - a \neq 0$, and since L is residually nilpotent there exists a positive integer m such that $ag - a \notin \gamma_{m+1}(L)$. Hence, by taking such a pair (a, m) where m is minimal, we can assume that $ag - a \notin \gamma_{m+1}(L)$ but $ug - u \in \gamma_m(L)$ for all $u \in L$. Note then that $ug - u \in \gamma_{m+1}(L)$ for all $u \in L'$.

We claim that $a \notin L^G + L'$. Suppose to the contrary that $a = b + c$ where $b \in L^G$ and $c \in L'$. Then

$$ag = bg + cg = b + c + d$$

where $d \in \gamma_{m+1}(L)$. Thus $ag - a = d \in \gamma_{m+1}(L)$. This is the required contradiction.

For a field K and a non-empty set X we write P for the free commutative associative K-algebra freely generated by X (in other words, P is the polynomial algebra $K[X]$). Also, we write A for the free associative K-algebra freely generated by X. Furthermore, F denotes the free Lie algebra over K freely generated by X and M denotes the free metabelian Lie algebra over K freely generated by X. As usual, we may regard A as a Lie algebra under the operation defined by $[u, v] = uv - vu$ for all $u, v \in A$ and then F is identified with the Lie subalgebra of A (freely) generated by X. Furthermore, M is isomorphic to the factor algebra F/F''. Our convention is that P and A have an identity element and that subalgebras of P and A are taken to contain this element. Monomials of P, A, F and M are defined in the usual way as non-zero (iterated) products of elements of X (in the case of F and M, such a product is a Lie product which is not necessarily left-normed). The degree of a monomial is the length of this product. In the cases of P and A, the identity element is the only monomial of degree 0, whereas F and M have no monomials of degree 0.

If E is any of P, A, F or M then for each non-negative integer n we write E_n for the K-subspace spanned by the monomials of degree n. Thus E is a K-space direct sum

$$E = E_0 \oplus E_1 \oplus E_2 \oplus \dots .$$

This decomposition is a grading of E in the sense that, for all $i, j \geqslant 0$, every product of an element of E_i and an element of E_j belongs to E_{i+j}. The degree of an arbitrary element u of E, denoted by $\deg(u)$, is the smallest value of n such that $u \in E_0 \oplus E_1 \oplus \dots \oplus E_n$. Note that P_0 and A_0 are spanned by the identity elements of P and A, respectively, while $F_0 = \{0\}$ and $M_0 = \{0\}$. For each positive integer m, we have $\gamma_m(F) = F_m \oplus F_{m+1} \oplus \dots$ and $\gamma_m(M) = M_m \oplus M_{m+1} \oplus \dots$. Thus, in connection with Lemmas 2.1 and 2.2, we note that both F and M are residually nilpotent.

Let $x \in X$. Then, for each $n \geqslant 0$, we can write

$$E_n = E_{0,n} \oplus \dots \oplus E_{n,n},$$

where, for $i = 1, \dots, n$, $E_{i,n}$ is the K-subspace spanned by all monomials of degree n which have x-degree i (that is, monomials of degree n with exactly i factors equal to x). Note that, for all $n \geqslant 2$, we have $F_{n,n} = \{0\}$ and $M_{n,n} = \{0\}$.

Let $E(x)$ denote the subspace of E spanned by E_0 and all monomials which have at least one factor from $X\backslash\{x\}$. Thus

$$E(x) = E_0 \oplus E_{0,1} \oplus (E_{0,2} \oplus E_{1,2}) \oplus \ldots \oplus (E_{0,n} \oplus \ldots \oplus E_{n-1,n}) \oplus \ldots .$$

Note that $F(x) = \langle X\backslash\{x\}\rangle \oplus F'$ and $M(x) = \langle X\backslash\{x\}\rangle \oplus M'$.

Let q be any real number satisfying $0 \leqslant q \leqslant 1$. We write $E(x, q)$ for the subspace of E spanned by all subspaces $E_{i,n}$ with $n \geqslant 0$ and $i \leqslant qn$. In this notation, $E = E(x, 1)$ and

$$E(x) = \bigcup_{0 \leqslant q < 1} E(x, q). \tag{2.1}$$

LEMMA 2.3. *Let E be P, A, F or M.*

(i) *For each q with $0 \leqslant q \leqslant 1$, $E(x, q)$ is a subalgebra of E, and $E(x)$ is a subalgebra of E.*

(ii) *Let S be a finitely generated subalgebra of E such that $S \subseteq E(x)$. Then $S \subseteq E(x, q)$ for some q with $0 \leqslant q < 1$.*

Proof. (i) Let $0 \leqslant q \leqslant 1$. Suppose that $u \in E_{i,n}$ and $v \in E_{i',n'}$ where $i \leqslant qn$ and $i' \leqslant qn'$. Then clearly the product of u and v belongs to $E_{i+i',n+n'}$. But $i + i' \leqslant qn + qn' = q(n + n')$. Both parts of (i) now follow.

(ii) This follows easily from (i) and (2.1).

Let E be P, A, F or M, as above, and let K_1 be an extension field of K. Then $K_1 \otimes E$ (tensor product taken over K) may be identified with the corresponding free algebra over K_1 and we may regard E as embedded in $K_1 \otimes E$. Each algebra automorphism of E extends, uniquely, to an algebra automorphism of $K_1 \otimes E$.

LEMMA 2.4. *Let E be P, A, F or M and let K_1 be an extension field of K. Let G be a group of automorphisms of E and view G as a group of automorphisms of $K_1 \otimes E$. Then $(K_1 \otimes E)^G = K_1 \otimes E^G$.*

Proof. Clearly $K_1 \otimes E^G \subseteq (K_1 \otimes E)^G$. Let Λ be a K-basis of K_1. Then $K_1 \otimes E = \bigoplus_{\lambda \in \Lambda} \lambda \otimes E$, where, for each λ, the map $E \to \lambda \otimes E$ given by $a \mapsto \lambda \otimes a$ (for $a \in E$) is a K-space isomorphism. Suppose that $\sum \lambda \otimes a_\lambda \in (K_1 \otimes E)^G$, where $a_\lambda \in E$ for each λ. Then we obtain $a_\lambda g = a_\lambda$ for each element g of G and each λ; thus $(K_1 \otimes E)^G \subseteq K_1 \otimes E^G$.

The following result is elementary and well-known, at least in the finite-dimensional case.

LEMMA 2.5. *Let U be a non-zero KG-module, where K is a field of prime characteristic p and G is a finite p-group. Then $U^G \neq \{0\}$.*

Proof. Let I be a right ideal of KG which is minimal subject to $UI \neq \{0\}$ and let J be a right ideal of KG which is maximal in I. Thus $UJ = \{0\}$. By the conditions on K and G, every irreducible KG-module is trivial. Thus $I(g - 1) \subseteq J$ for all $g \in G$. Hence $UI(g - 1) = \{0\}$ for all $g \in G$, and so $UI \subseteq U^G$.

In Section 5 we shall require the following simple result.

LEMMA 2.6. *Let K be a field of prime characteristic p and let $\mu_1, \ldots, \mu_{p-1}$ be elements of K which are not all zero. Then there exists $k \in \{1, \ldots, p-1\}$ such that* $\mu_1^k + \mu_2^k + \ldots + \mu_{p-1}^k \neq 0$.

Proof. We can write $\mu_1^k + \mu_2^k + \ldots + \mu_{p-1}^k$ as $s_1\nu_1^k + \ldots + s_m\nu_m^k$, with $1 \leqslant m \leqslant p-1$, where $\nu_1, \ldots, \nu_m$ are the distinct non-zero elements of $\{\mu_1, \ldots, \mu_{p-1}\}$ and where $1 \leqslant s_i \leqslant p-1$ for $i = 1, \ldots, m$. The van der Monde matrix

$$\begin{pmatrix} \nu_1 & \nu_2 & \ldots & \nu_m \\ \nu_1^2 & \nu_2^2 & \ldots & \nu_m^2 \\ \vdots & \vdots & & \vdots \\ \nu_1^m & \nu_2^m & \ldots & \nu_m^m \end{pmatrix}$$

is non-singular: hence its columns are linearly independent. Thus

$$s_1(\nu_1, \ldots, \nu_1^m) + \ldots + s_m(\nu_m, \ldots, \nu_m^m) \neq (0, \ldots, 0).$$

Hence $s_1\nu_1^k + \ldots + s_m\nu_m^k \neq 0$ for some $k \in \{1, \ldots, m\}$.

3. Polynomial algebras. The purpose of this section is to derive a result about polynomial algebras which will be used in Section 4 in our study of free metabelian Lie algebras.

Let K be a field. As in Section 2, let X be a non-empty set and let P be the polynomial algebra $K[X]$. Let V denote the subspace of P spanned by X. If h is any element of the general linear group $\mathrm{GL}(V)$ then the action of h may be extended (uniquely) to P so that h acts as an algebra automorphism of P. Each subspace P_n, for $n \geqslant 0$, is invariant under the action of h. The automorphisms of P of this type will be called the graded automorphisms of P. If H is a group of graded automorphisms then we may, of course, regard P as a KH-module.

LEMMA 3.1. *Let $P = K[X]$ where $|X| > 1$. Let H be a finite group of graded automorphisms of P. Let $x \in X$, let q be a real number such that $0 \leqslant q < 1$, and let r be a positive integer. Then there exists a positive integer s, with $s \geqslant r$, and an element a of P_s such that $\sum_{h \in H} ah \notin P(x, q)$.*

Proof. If K_1 is an extension field of K and if $\sum_{h \in H} ah \in P(x, q)$ for all $a \in P_s$ then it follows that $\sum_{h \in H} ah \in (K_1 \otimes P)(x, q)$ for all $a \in K_1 \otimes P_s$. Thus we may assume that K is infinite. Clearly we may also assume that $|H| > 1$.

As before we write $V = \langle X \rangle = P_1$. Let $H = \{h_0, h_1, \ldots, h_{n-1}\}$ where $h_0 = 1$ and $n = |H|$. For $i = 1, \ldots, n-1$ let $V_i = \{v \in V : vh_i = v\}$. Then each of $V_1, \ldots, V_{n-1}$ and $\langle X \backslash \{x\} \rangle$ is a proper subspace of V. But it is well-known and easy to see that a non-zero vector space over an infinite field is not equal to the (set-theoretic) union of any finite collection of proper subspaces. Hence there exists $v \in V$ such that

$$v \notin V_1 \cup \ldots \cup V_{n-1} \cup \langle X \backslash \{x\} \rangle.$$

It follows that the elements $vh_0, \ldots, vh_{n-1}$ are distinct.

The matrix

$$\begin{pmatrix} 1 & 1 & \ldots & 1 \\ vh_0 & vh_1 & \ldots & vh_{n-1} \\ \vdots & \vdots & & \vdots \\ (vh_0)^{n-1} & (vh_1)^{n-1} & \ldots & (vh_{n-1})^{n-1} \end{pmatrix}$$

with entries from the integral domain P is a non-singular (van der Monde) matrix over Q, the field of quotients of P. Thus the vectors

$$(1, vh_0, \ldots, (vh_0)^{n-1}), \ldots, (1, vh_{n-1}, \ldots, (vh_{n-1})^{n-1})$$

are linearly independent over Q, and so linearly independent over K. By considering the components $P_0, \ldots, P_{n-1}$, we see that the elements

$$1 + (vh_0) + \ldots + (vh_0)^{n-1}, \ldots, 1 + (vh_{n-1}) + \ldots + (vh_{n-1})^{n-1}$$

are linearly independent over K. (The argument we have used is basically the same as the proof of Proposition 3.1 of [**4**].)

For each non-negative integer m, write $v(m) = v^m + v^{m+1} + \ldots + v^{m+n-1}$. We shall show that there exists m with $m \geqslant r$ such that $\sum_{i=0}^{n-1} v(m)h_i \notin P(x, q)$. It follows that $\sum_{i=0}^{n-1} v^{m+j}h_i \notin P(x, q)$ for some $j \in \{0, \ldots, n-1\}$. This will give the required result.

Note that

$$\sum_{i=0}^{n-1} v(m)h_i = \sum_{i=0}^{n-1}((vh_i)^m + (vh_i)^{m+1} + \ldots + (vh_i)^{m+n-1}) \tag{3.1}$$

$$= \sum_{i=0}^{n-1}(1 + (vh_i) + \ldots + (vh_i)^{n-1})(vh_i)^m. \tag{3.2}$$

For $i = 0, \ldots, n-1$, write $vh_i = \lambda_i x + w_i$ where $\lambda_i \in K$ and $w_i \in \langle X \backslash \{x\} \rangle$. Since $v \notin \langle X \backslash \{x\} \rangle$ we have $\lambda_0 \neq 0$.

We deal separately with the cases where K has non-zero characteristic and where it has characteristic 0. Suppose first that K has prime characteristic p. Take m to be a power of p such that $m \geqslant r$, $m \geqslant n$ and $m > q(m + n - 1)$. Suppose, in order to get a contradiction, that $\sum_i v(m)h_i \in P(x, q)$. Since m is a power of p, we have $(vh_i)^m = \lambda_i^m x^m + w_i^m$ for each i. Hence, by (3.2),

$$\sum_i v(m)h_i = \sum_i \lambda_i^m (1 + (vh_i) + \ldots + (vh_i)^{n-1})x^m + \sum_i (1 + (vh_i) + \ldots + (vh_i)^{n-1})w_i^m.$$

The monomials occurring in $\sum_i (1 + (vh_i) + \ldots + (vh_i)^{n-1})w_i^m$ have x-degree which does not exceed $n - 1$. But, since $m \geqslant n$, the monomials occurring in $\sum_i \lambda_i^m (1 + (vh_i) + \ldots + (vh_i)^{n-1})x^m$ have x-degree which exceeds $n - 1$. Hence these monomials must also occur in $\sum_i v(m)h_i$. Since $\sum_i v(m)h_i \in P(x, q)$, we obtain

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$$\sum_i \lambda_i^m(1+(vh_i)+\ldots+(vh_i)^{n-1})x^m \in P(x,q).$$

But every monomial occurring in $(1+(vh_i)+\ldots+(vh_i)^{n-1})x^m$ has degree at most $m+n-1$ and x-degree at least m. Since $m > q(m+n-1)$ we deduce that

$$\sum_i \lambda_i^m(1+(vh_i)+\ldots+(vh_i)^{n-1})x^m = 0.$$

Thus

$$\sum_i \lambda_i^m(1+(vh_i)+\ldots+(vh_i)^{n-1}) = 0.$$

Since $\lambda_0 \neq 0$, this contradicts the linear independence of the elements $1+(vh_i)+\ldots+(vh_i)^{n-1}$.

Now suppose that K has characteristic 0. Take m so that $m \geqslant r$ and $m > q(m+n-1)$. Suppose, to get a contradiction, that $\sum_i v(m)h_i \in P(x,q)$. Since $\sum_i v(m)h_i$ has degree at most $m+n-1$, where $m > q(m+n-1)$, it follows that every monomial occurring in $\sum_i v(m)h_i$ has x-degree which is at most $m-1$. Thus $\sum_i v(m)h_i$ becomes 0 when differentiated m times with respect to x. Hence, by (3.1),

$$\sum_i \Big((m!/0!)\lambda_i^m + ((m+1)!/1!)\lambda_i^m(vh_i) + \ldots \\ \ldots + ((m+n-1)!/(n-1)!)\lambda_i^m(vh_i)^{n-1}\Big) = 0.$$

By comparison of the degrees we see that

$$\sum_i ((m+j)!/j!)\lambda_i^m(vh_i)^j = 0,$$

for $j = 0, \ldots, n-1$. Hence $\sum_i \lambda_i^m(vh_i)^j = 0$ for each j and so

$$\sum_i \lambda_i^m(1+(vh_i)+\ldots+(vh_i)^{n-1}) = 0.$$

We now have a contradiction as in the previous case.

4. Free metabelian Lie algebras. Let K be a field. As in Section 2, let X be a non-empty set, let P be the polynomial algebra $K[X]$, and let M be the free metabelian Lie algebra over K freely generated by X. Let V denote the subspace spanned by X: note that we use the same notation for this in both P and M. We regard $V \otimes P$ (tensor product taken over K) as a right P-module in the obvious way. Clearly it is a free P-module with $\{x \otimes 1 : x \in X\}$ as a free generating set.

It is well-known and easy to verify that the derived algebra M' of M may be viewed as a right P-module in which the image of an element u of M' under the action of a monomial $x_1 \cdots x_n$ of P (where $x_1, \ldots, x_n \in X$) is the left-normed Lie product $[u, x_1, \ldots, x_n]$. (One way to see this is to use the fact that M' is naturally a module for the Lie algebra M/M' and P may be regarded as the universal enveloping algebra of M/M'.) For $u \in M'$ and $v \in P$ we write $[u; v]$ to denote the image of u under the module action of v.

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LEMMA 4.1. (i) *There is a P-module embedding* $\varepsilon : M' \to V \otimes P$ *in which*

$$[v_1, v_2, \ldots, v_r]\varepsilon = v_1 \otimes v_2 v_3 \cdots v_r - v_2 \otimes v_1 v_3 \cdots v_r \tag{4.1}$$

for all $r \geqslant 2$ *and all* $v_1, v_2, \ldots, v_r \in V$.

(ii) *If* u *is a non-zero element of* M' *and* v *is a non-zero element of* P *then* $[u; v] \neq 0$.

Proof. (i) We first note that there is a K-space embedding $\varepsilon : M' \to V \otimes P$ satisfying (4.1): the analogous result over the integers holds by Theorem 3.1 of [**7**], and the result over K can be proved similarly or deduced from the integral result by tensoring with K. For all $v_1, v_2, \ldots, v_r, v \in V$, with $r \geqslant 2$, we have

$$\begin{aligned}([v_1, v_2, \ldots, v_r]\varepsilon)v &= v_1 \otimes v_2 v_3 \cdots v_r v - v_2 \otimes v_1 v_3 \cdots v_r v \\ &= [v_1, v_2, \ldots, v_r, v]\varepsilon \\ &= [[v_1, v_2, \ldots, v_r]; v]\varepsilon.\end{aligned}$$

It follows that ε is a P-module homomorphism.

(ii) Suppose $u \in M'$ and $v \in P$ where $u \neq 0$ and $v \neq 0$. By (i), $[u; v]\varepsilon = (u\varepsilon)v$ and $u\varepsilon \neq 0$. Since $V \otimes P$ is a free P-module and P is an integral domain, $V \otimes P$ is torsion-free as a P-module. Thus $(u\varepsilon)v \neq 0$, and so $[u; v] \neq 0$.

Let Q be the field of quotients of P. Since $V \otimes P$ is a free right P-module it may be embedded in $V \otimes Q$, which is a vector space over Q (with Q acting on the right) with basis $\{x \otimes 1 : x \in X\}$.

Suppose that G is a subgroup of $\mathrm{Aut}(M)$ and write $N = G \cap \mathrm{IA}(M)$. Thus N is a normal subgroup of G: it is the kernel of the action of G on M/M'. Write $\overline{G} = G/N$ and, for each $g \in G$, write $\overline{g}$ for the element gN of G/N. Since N acts trivially on M/M', we may regard M/M' as a $K\overline{G}$-module, and $\overline{G}$ acts faithfully on this module. There is a K-space isomorphism from M/M' to V such that $x + M'$ is mapped to x for all $x \in X$. Using this isomorphism we may regard $\overline{G}$ as a subgroup of $\mathrm{GL}(V)$ and so $\overline{G}$ may be regarded as a group of graded automorphisms of P (see Section 3). In particular, P is a $K\overline{G}$-module.

LEMMA 4.2. *With* G *and* $\overline{G}$ *as above, let* $u \in M'$ *and* $v \in P$. *Then, for all* $g \in G$,

$$[u; v]g = [ug; v\overline{g}].$$

Proof. This is straightforward.

With G, N and $\overline{G}$ as above, M^N is a KG-submodule of M. But since N acts trivially on this module we may regard it as a $K\overline{G}$-module. Thus, for $g \in G$ and $u \in M^N$, we have $u\overline{g} = ug$. The same considerations apply to the submodule $M^N \cap M'$, and we note that $M^N \cap M' = (M')^N$. It is easily verified that $(M')^N$ is a P-submodule of M' (in fact, $(M')^N$ is an ideal of M).

LEMMA 4.3. *Let* M *be a free metabelian Lie algebra of rank greater than* 1 *over a field* K *and let* G *be a finite subgroup of* Aut(M). *Then* $M^G \cap M' \neq \{0\}$.

Proof. We take a free generating set X of M and use the notation developed in connection with Lemmas 4.1 and 4.2. By Lemma 2.4 we may assume that K is infinite.

We first prove that $M^N \cap M' \neq \{0\}$. If $N = \{1\}$ this is clear. But if $N \neq \{1\}$ then, by Lemma 2.1, K has prime characteristic p and N is a p-group. In this case $M^N \cap M' = (M')^N \neq \{0\}$ by Lemma 2.5.

Let $g_0, g_1, \dots, g_{n-1}$ be elements of G such that $\overline{G} = \{\overline{g}_0, \dots, \overline{g}_{n-1}\}$ where $\overline{g}_0 = 1$ and $n = |\overline{G}|$. Clearly we may assume that $G \neq N$; thus $n > 1$. Since $\overline{G}$ acts faithfully on V, it follows, as in the proof of Lemma 3.1, that there exists a non-zero element v of V such that the elements $v\overline{g}_0, \dots, v\overline{g}_{n-1}$ are distinct.

Recall that $(M')^N$ may be regarded as a $K\overline{G}$-module and that $(M')^N \neq \{0\}$. Let u be a non-zero element of $(M')^N$. Thus each of $u\overline{g}_0, \dots, u\overline{g}_{n-1}$ is an element of $(M')^N$. Since $v\overline{g}_0, \dots, v\overline{g}_{n-1}$ are distinct elements of P, it is easy to verify (by considering the elements $(v\overline{g}_i)(v\overline{g}_j)^{-1}$ in the multiplicative group of the field of quotients Q) that there exist infinitely many positive integers t such that $(v\overline{g}_0)^t, \dots, (v\overline{g}_{n-1})^t$ are distinct. We choose t so that $\deg(u\overline{g}_i) \leqslant t+1$ for $i = 0, \dots, n-1$, and we write $w = v^t$. Thus $w\overline{g}_0, \dots, w\overline{g}_{n-1}$ are distinct elements of P_t.

Let Z be the matrix

$$\begin{pmatrix} 1 & w\overline{g}_0 & \dots & (w\overline{g}_0)^{n-1} \\ 1 & w\overline{g}_1 & \dots & (w\overline{g}_1)^{n-1} \\ \vdots & \vdots & & \vdots \\ 1 & w\overline{g}_{n-1} & \dots & (w\overline{g}_{n-1})^{n-1} \end{pmatrix}.$$

Thus Z is a van der Monde matrix over the field Q, and it is invertible over Q.

We claim that the element $[u; 1 + w + \dots + w^{n-1}]$ of $(M')^N$ generates a regular $K\overline{G}$-module. To prove this, suppose that

$$\sum_{i=0}^{n-1} \lambda_i([u; 1 + w + \dots + w^{n-1}]\overline{g}_i) = 0, \tag{4.2}$$

where $\lambda_0, \dots, \lambda_{n-1} \in K$. We shall prove that $\lambda_i = 0$ for $i = 0, \dots, n-1$.

By (4.2) and Lemma 4.2, we have

$$\sum_i \lambda_i[u\overline{g}_i; 1 + (w\overline{g}_i) + \dots + (w\overline{g}_i)^{n-1}] = 0. \tag{4.3}$$

For $i = 0, \dots, n-1$, write $e_i = \lambda_i(u\overline{g}_i)\varepsilon \in V \otimes P$. By Lemma 4.1, ε is a homomorphism of P-modules. Thus, applying ε to (4.3), we obtain

$$\sum_i e_i(1 + (w\overline{g}_i) + \dots + (w\overline{g}_i)^{n-1}) = 0. \tag{4.4}$$

But $e_i \in V \otimes (P_1 \oplus \dots \oplus P_t)$ for each i, by the choice of t and the definition of ε. Thus

$$e_i(w\overline{g}_i)^j \in V \otimes (P_{jt+1} \oplus \dots \oplus P_{(j+1)t})$$

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for $j = 0, \ldots, n-1$. Hence, by (4.4), $\sum_i e_i(w\overline{g}_i)^j = 0$ for $j = 0, \ldots, n-1$. In matrix notation,

$$(e_0, \ldots, e_{n-1})Z = (0, \ldots, 0).$$

We may regard each e_i as an element of the Q-space $V \otimes Q$. Thus, since Z is invertible over Q, we obtain $e_i = 0$ for all i. But, since ε is an embedding, $(u\overline{g}_i)\varepsilon \neq 0$ for all i. Thus $\lambda_i = 0$ for all i.

Therefore, as claimed, $[u;\, 1 + w + \ldots + w^{n-1}]$ generates a regular $K\overline{G}$-module. It follows that the element

$$[u;\, 1 + w + \ldots + w^{n-1}](\overline{g}_0 + \overline{g}_1 + \ldots + \overline{g}_{n-1})$$

is a non-zero element of $(M')^N$ which is fixed by $\overline{G}$. Thus we have a non-zero element of $(M')^G$.

Lemma 4.4. *Let M be the free metabelian Lie algebra over a field K on a free generating set X with $|X| > 1$. Let G be a finite subgroup of* $\mathrm{Aut}(M)$ *and write $N = G \cap \mathrm{IA}(M)$ and $\overline{G} = G/N$. Let $x \in X$ and let q be a real number with $0 \leqslant q < 1$. Then there exists $c \in M^N \cap M'$ such that*

$$\sum_{h \in \overline{G}} ch \notin M(x, q).$$

Proof. Write $\overline{G} = \{\overline{g}_0, \ldots, \overline{g}_{n-1}\}$ where $\overline{g}_0 = 1$ and $n = |\overline{G}|$. By Lemma 4.3 there exists a non-zero element u of $(M')^G$. Let t be the degree of u. Choose q' so that $q < q' < 1$ and choose a positive integer r so that $(q' - q)r > qt$. Let $P = K[X]$ and make P into a $K\overline{G}$-module as explained before the statement of Lemma 4.2. By Lemma 3.1, there exists $s \geqslant r$ and $a \in P_s$ such that $\sum_i a\overline{g}_i \notin P(x, q')$. Let $c = [u; a]$. Thus $c \in (M')^N$. Also, $\sum_i c\overline{g}_i = [u; \sum_i a\overline{g}_i]$ by Lemma 4.2. We claim that $[u; \sum_i a\overline{g}_i] \notin M(x, q)$.

Write $u = u_2 + \ldots + u_t$ where $u_j \in M_j$ for $j = 2, \ldots, t$. Since u has degree t, $u_t \neq 0$. Suppose, in order to get a contradiction, that $[u; \sum_i a\overline{g}_i] \in M(x, q)$. Since $a\overline{g}_i \in P_s$ for all i, it follows that $[u_t; \sum_i a\overline{g}_i] \in M(x, q)$. Note also that $[u_t; \sum_i a\overline{g}_i] \in M_{s+t}$.

Write $u_t = \sum_{j=0}^{t} u_{j,t}$ where $u_{j,t} \in M_{j,t}$ for $j = 0, \ldots, t$. Similarly, write $\sum_i a\overline{g}_i = \sum_{j=0}^{s} d_{j,s}$ where $d_{j,s} \in P_{j,s}$ for $j = 0, \ldots, s$ and write $[u_t; \sum_i a\overline{g}_i] = \sum_{j=0}^{s+t} e_{j,s+t}$ where $e_{j,s+t} \in M_{j,s+t}$ for $j = 0, \ldots, s+t$. Choose k maximal subject to $u_{k,t} \neq 0$ and choose l maximal subject to $d_{l,s} \neq 0$. Then $e_{k+l,s+t} \neq 0$ by Lemma 4.1(ii). But, by the choice of a, we have $l > q's$. Also, $(q' - q)s > qt$. Hence $k + l \geqslant l > q(s + t)$. Thus $[u_t; \sum_i a\overline{g}_i] \notin M(x, q)$, which is a contradiction.

Proof of Theorem B. Suppose, in order to get a contradiction, that M^G is finitely generated. By Lemma 2.2, $M^G + M' \neq M$. Let X_0 be a free generating set for M. Thus $M = \langle X_0 \rangle \oplus M'$. Take a basis X for $\langle X_0 \rangle$ so that, for some $x \in X$, we have $M^G \subseteq \langle X \backslash \{x\} \rangle \oplus M'$. It is easy to verify that X is a free generating set for M and, in the notation of Section 2, $M^G \subseteq M(x)$. Thus, by Lemma 2.3(ii), there exists q with $0 \leqslant q < 1$ such that $M^G \subseteq M(x, q)$. By Lemma 4.4, there exists $c \in (M')^N$ such that $\sum_{h \in \overline{G}} ch \notin M(x, q)$. But

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$$\sum_{h\in\overline{G}} ch \in M^G \subseteq M(x, q).$$

This is the required contradiction.

Theorem C will be derived as a corollary of the following result.

THEOREM 4.5. *Let R be a Lie algebra over a field K such that R/R'' is a free metabelian Lie algebra of rank greater than* 1. *Let G be a non-trivial finite subgroup of* $\mathrm{Aut}(R)$ *such that G acts faithfully on R/R'. Then R^G is not finitely generated.*

Proof. Write $M = R/R''$. Thus M is a free metabelian Lie algebra of rank greater than 1 and M/M' may be identified with R/R'. Since G acts faithfully on R/R' it acts faithfully on M/M' and so it acts faithfully on M. Thus we may regard G as a group of automorphisms of M.

Suppose, in order to get a contradiction, that R^G is finitely generated, and write $S = (R^G + R'')/R''$. Thus S is a finitely generated subalgebra of M. Also,

$$(S + M')/M' \subseteq (M/M')^G \neq M/M'.$$

Thus, as in the proof of Theorem B, we may choose a free generating set X of M and an element x of X such that $S \subseteq M(x)$. By Lemma 2.3(ii), there exists q with $0 \leqslant q < 1$ such that $S \subseteq M(x, q)$. Note that $N = G \cap \mathrm{IA}(M) = \{1\}$. Thus, by Lemma 4.4, there exists $c \in M'$ such that $\sum_{g\in G} cg \notin M(x, q)$.

Let w be any element of R such that $w + R'' = c$. Since $\sum_{g\in G} wg \in R^G$, we have

$$\sum_{g\in G} wg + R'' \in S \subseteq M(x, q).$$

But

$$\sum_{g\in G} wg + R'' = \sum_{g\in G} cg \notin M(x, q).$$

This is the required contradiction.

Proof of Theorem C. Under the hypotheses of Theorem C, suppose that R is relatively free in **V**, where **V** is a variety of Lie algebras over K. If R is nilpotent then R is finite-dimensional and so R^G is finitely generated.

Now assume that R is not nilpotent. Thus R has rank greater than 1. We shall show that R^G is not finitely generated. By Theorem 4.5 it is enough to show that **V** contains the variety of all metabelian Lie algebras over K. Suppose, in order to get a contradiction, that this does not hold. Then, by a well-known argument (see the proof of Corollary 5.4 of [**3**], for example), **V** satisfies an Engel identity. Hence R satisfies an Engel identity. But R is finitely generated. Therefore, by the results of Kostrikin ([**8**]) and Zel'manov ([**10**]), R is nilpotent. This is the required contradiction.

5. Free Lie algebras. Let K be a field. As in Section 2, let X be a non-empty set, let A be the free associative K-algebra on X, and let F be the free Lie K-algebra on X. As before we take $F \subseteq A$. Elements of X will sometimes be called letters.

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If a, b, c and d are monomials of A (any of which may be the identity element) such that $d = abc$ then we say that a is an initial segment of d, b is a segment of d, and c is a final segment of d. For any monomial a of A we write $\tilde{a}$ for the monomial of A obtained by writing the letters of a in reverse order: that is, if $a = x_1 x_2 \cdots x_n$ where $x_i \in X$ for $i = 1, \ldots, n$, then $\tilde{a} = x_n \cdots x_2 x_1$. Note that the monomials of A form a K-basis of A. Thus each element u of A may be uniquely expressed as a linear combination of monomials of A with coefficients in K. Every monomial a of A has a coefficient (possibly 0) in this expression: we call it the coefficient of a in u. We shall be particularly concerned with the special case where $u \in F$.

LEMMA 5.1. *Let $f \in F$, let a be a monomial of A, and let λ be the coefficient of a in f. Then the coefficient of $\tilde{a}$ in f is $(-1)^{\deg(a)+1}\lambda$.*

Proof. See Lemma 1.7 of [**9**].

If K has prime characteristic p, then for all $e, f \in A$ and any non-negative integer τ we have

$$[e, f^{p^\tau}] = [e, f, \ldots, f],$$

where there are p^τ copies of f in the second commutator (see (1.6.1) of [**9**], for example). Thus if $e, f \in F$ then $[e, f^{p^\tau}] \in F$. Much of the work towards the proof of Theorem A is done in the proof of the following technical result.

LEMMA 5.2. *Let K be a field of prime characteristic p, let X be a set such that $|X| > 1$, let A be the free associative K-algebra on X, and let F be the free Lie K-algebra on X, where we take $F \subseteq A$. Let $x \in X$, let q be a real number with $0 \leqslant q < 1$, let e be a non-zero element of F', and let $f_1, \ldots, f_{p-1}$ be elements of F' which are not all zero. Then there exists a non-negative integer τ such that*

$$[e, x^{p^\tau} + (x + f_1)^{p^\tau} + \ldots + (x + f_{p-1})^{p^\tau}] \notin F(x, q).$$

Proof. For any monomial v of A we shall write $l_x(v)$ for the largest non-negative integer s such that x^s is an initial segment of v and $r_x(v)$ for the largest s such that x^s is a final segment of v.

For $i = 1, \ldots, p-1$, let Ω_i be the set of monomials of A which have non-zero coefficient in f_i, and write $\Omega = \Omega_1 \cup \ldots \cup \Omega_{p-1}$. Choose $a \in \Omega$ so that for all $v \in \Omega$ either $l_x(v) < l_x(a)$ or $l_x(v) = l_x(a)$ and $\deg(v) \leqslant \deg(a)$. By Lemma 5.1, $\tilde{a} \in \Omega$. Also, $\tilde{a}$ has the property that for all $v \in \Omega$ either $r_x(v) < r_x(\tilde{a})$ or $r_x(v) = r_x(\tilde{a})$ and $\deg(v) \leqslant \deg(\tilde{a})$. Without loss of generality we may assume that $a \in \Omega_1$. (Thus, also, $\tilde{a} \in \Omega_1$.)

For $i = 1, \ldots, p-1$, let λ_i be the coefficient of a in f_i. Thus $\lambda_1 \neq 0$ and, by Lemma 5.1, $\tilde{a}$ has coefficient $(-1)^{\deg(a)+1}\lambda_i$ in f_i. For $i = 1, \ldots, p-1$, write $\mu_i = (-1)^{\deg(a)+1}\lambda_i^2$. Thus μ_i is the product of the coefficients of a and $\tilde{a}$ in f_i. By Lemma 2.6 there exists $k \in \{1, \ldots, p-1\}$ such that $\mu_1^k + \ldots + \mu_{p-1}^k \neq 0$.

Let Γ be the set of monomials of A which have non-zero coefficient in e. Let c be a monomial of A of smallest possible degree such that $cx^n \in \Gamma$ for some $n \geqslant 0$. For this monomial c, choose n as large as possible such that $cx^n \in \Gamma$ and write $b = cx^n$. Furthermore, let ξ be the coefficient of b in e: thus $\xi \neq 0$.

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Note that, since $e, f_1, \ldots, f_{p-1} \in F'$, every element of $\Gamma \cup \Omega$ has degree at least 2, and no element of $\Gamma \cup \Omega$ is a power of x.

Choose a positive integer l so that $\deg(v) \leqslant l$ for all $v \in \Gamma \cup \Omega$. Let t be a power of p chosen so that when m is defined as $m = t - k(l+2)$ we have $m \geqslant l$ and $kl + m > q(3kl + l + m)$. Let

$$u = [e, x^t + (x + f_1)^t + \ldots + (x + f_{p-1})^t].$$

We shall show that $u \notin A(x, q)$. This will establish the required result because $F(x, q) \subseteq A(x, q)$.

Write $d = b(x^l a\tilde{a})^k x^m$. Thus d is a monomial of A. We shall prove that d appears in u with non-zero coefficient and that d does not belong to $A(x, q)$.

Let $i \in \{1, \ldots, p-1\}$. Since

$$[e, (x + f_i)^t] = e(x + f_i)^t - (x + f_i)^t e,$$

we can write $[e, (x + f_i)^t]$ as a linear combination of terms of the form $v_0 v_1 \cdots v_t$ and terms of the form $v_1 \cdots v_t v_0$ where $v_0 \in \Gamma$ and $v_1, \ldots, v_t \in \{x\} \cup \Omega_i$. No term of the form $v_1 \cdots v_t v_0$ can be equal to d because d has a final segment x^m, but $m \geqslant \deg(v_0)$ and v_0 is not a power of x.

We shall prove that if $v_0 v_1 \cdots v_t = d$ then there is an equality of $(t+1)$-tuples

$$\begin{aligned}(v_0, v_1, \ldots, v_t) = (b, x, \ldots, x, a, \tilde{a}, x, \ldots, x, a, \tilde{a}, \ldots \\ \ldots, x, \ldots, x, a, \tilde{a}, x, \ldots, x),\end{aligned} \tag{5.1}$$

where the $(t+1)$-tuple on the right is the one given by the factorisation $b(x^l a\tilde{a})^k x^m$ of d. Suppose then that $v_0 v_1 \cdots v_t = d$, where $v_0 \in \Gamma$ and $v_1, \ldots, v_t \in \{x\} \cup \Omega_i$.

Since $l \geqslant \deg(v_0)$, v_0 is an initial segment of bx^l. But v_0 cannot have the form bx^s with $s \geqslant 1$ because of the choice of b. Hence v_0 is an initial segment of b. Recall that $b = cx^n$. By the choice of c, v_0 is not an initial segment of c unless $v_0 = c$. Thus v_0 has the form $v_0 = cx^{n'}$ where $0 \leqslant n' \leqslant n$, and so $b = v_0 x^{n-n'}$. Hence

$$v_1 \cdots v_t = x^{n-n'}(x^l a\tilde{a})^k x^m.$$

Write

$$x^{n-n'}(x^l a\tilde{a})^k x^m = w_1 \cdots w_r$$

where $w_1, \ldots, w_r \in \{x, a\tilde{a}\}$, exactly as x and $a\tilde{a}$ appear in $x^{n-n'}(x^l a\tilde{a})^k x^m$. It is easily verified that $r = n - n' + t - k$. Also,

$$v_1 \cdots v_t = w_1 \cdots w_r.$$

For $j = 1, \ldots, t$, take $\alpha(j)$ and $\beta(j)$ in $\{1, \ldots, r\}$ so that when v_j is regarded as a segment of $w_1 \cdots w_r$ it has its first letter within $w_{\alpha(j)}$ and its last letter within $w_{\beta(j)}$.

We claim that if $v_j \in \Omega_i$ then $w_{\alpha(j)} = a\tilde{a}$. For suppose otherwise that $w_{\alpha(j)} = x$ for some j with $v_j \in \Omega_i$. Then v_j is an initial segment of $w_{\alpha(j)} \cdots w_r$, which is a monomial with an initial segment of the form $x^s a$ with $s \geqslant 1$. Hence $l_x(v_j) > l_x(a)$, contrary to the choice of a. Similarly, if $v_j \in \Omega_i$ then $w_{\beta(j)} = a\tilde{a}$ because no element of Ω_i can be a final segment of any monomial with a final segment of the form $\tilde{a}x^s$ with $s \geqslant 1$, because of the maximality of $r_x(\tilde{a})$.

Therefore, for $j \in \{1, \ldots, t\}$, if $v_j \in \Omega_i$ then $w_{\alpha(j)} = a\tilde{a}$ and $w_{\beta(j)} = a\tilde{a}$. Since $l \geqslant \deg(v_j)$ we must have $\alpha(j) = \beta(j)$ in this case. But, clearly, if $v_j = x$ then we also have $\alpha(j) = \beta(j)$. It follows that there are integers $\sigma(0), \sigma(1), \ldots, \sigma(r)$ with

$$0 = \sigma(0) < \sigma(1) < \ldots < \sigma(r) = t$$

such that

$$w_1 = v_1 \cdots v_{\sigma(1)},\ w_2 = v_{\sigma(1)+1} \cdots v_{\sigma(2)},\ \ldots,\ w_r = v_{\sigma(r-1)+1} \cdots v_t.$$

If $w_j = a\tilde{a}$ then we cannot have $\sigma(j) - \sigma(j-1) = 1$ because this gives $a\tilde{a} = v_{\sigma(j)}$ which implies $l_x(v_{\sigma(j)}) = l_x(a)$ and $\deg(v_{\sigma(j)}) > \deg(a)$, contrary to the choice of a. Thus, if $w_j = a\tilde{a}$ we have $\sigma(j) - \sigma(j-1) \geqslant 2$. Of course, if $w_j = x$ we have $\sigma(j) - \sigma(j-1) = 1$. There are k values of j for which $w_j = a\tilde{a}$ and there are $n - n' + t - 2k$ values of j for which $w_j = x$. Since $t = \sum_j(\sigma(j) - \sigma(j-1))$, we obtain

$$t \geqslant 2k + (n - n' + t - 2k).$$

Thus $n - n' = 0$ and whenever $w_j = a\tilde{a}$ we must have $\sigma(j) - \sigma(j-1) = 2$, that is $w_j = v_{\sigma(j)-1} v_{\sigma(j)}$.

In order to examine this last equation suppose that $a\tilde{a} = vv'$ where $v, v' \in \{x\} \cup \Omega_i$. If $\deg(v) < \deg(a)$ then $v' \in \Omega_i$, $r_x(v') = r_x(\tilde{a})$ and $\deg(v') > \deg(\tilde{a})$, which is impossible. Thus $\deg(v) \geqslant \deg(a)$. Hence $v \in \Omega_i$ and $l_x(v) = l_x(a)$; thus $\deg(v) = \deg(a)$. It follows that $v = a$ and $v' = \tilde{a}$. Therefore, whenever $w_j = v_{\sigma(j)-1} v_{\sigma(j)}$ we have $v_{\sigma(j)-1} = a$ and $v_{\sigma(j)} = \tilde{a}$.

It follows that

$$(v_1, v_2, \ldots, v_t) = (x, \ldots, x, a, \tilde{a}, \ldots, x, \ldots, x),$$

where the t-tuple on the right is the one given by the factors of $x^{n-n'}(x^l a\tilde{a})^k x^m$. But $n - n' = 0$ and so $b = v_0$. Thus we obtain (5.1).

Therefore, when $[e, (x + f_i)^t]$ is written as a linear combination of terms $v_0 v_1 \cdots v_t$ and $v_1 \cdots v_t v_0$, as previously described, the only term which is equal to the monomial d is the one specified by (5.1) (and this can only occur if i has the property that $a \in \Omega_i$). This term has coefficient $\xi \mu_i^k$. It follows that the coefficient of d in u is $\xi(\mu_1^k + \ldots + \mu_{p-1}^k)$. Thus d has non-zero coefficient in u.

The x-degree of d is at least $kl + m$, whereas

$$\deg(d) \leqslant l + k(l + 2l) + m = 3kl + l + m.$$

Since $kl + m > q(3kl + l + m)$ we see that $d \notin A(x, q)$. Hence $u \notin A(x, q)$, as required.

Lemma 5.3. *Let F be a free Lie algebra of rank greater than* 1 *over a field K of prime characteristic p. Let G be a group of* IA-*automorphisms of F such that G is cyclic of order p. Then F^G is not finitely generated.*

Proof. Let g be an element of G which generates G. In order to get a contradiction, assume that F^G is finitely generated. By Lemma 2.2, $F^G + F' \neq F$. Thus (as in the proof of Theorem B) we may choose a free generating set X of F and an element x of X such that $F^G \subseteq \langle X \backslash \{x\} \rangle \oplus F'$. By Lemma 2.3, there exists q with $0 \leqslant q < 1$ such that $F^G \subseteq F(x, q)$.

Write $xg = x + f_1$, $xg^2 = x + f_2, \ldots, xg^{p-1} = x + f_{p-1}$, where $f_1, \ldots, f_{p-1} \in F'$. Note that $f_1 \neq 0$. By Lemma 2.5 there exists a non-zero element e of $(F')^G$. Let τ be as given by Lemma 5.2 and write $w = [e, x^{p^\tau}]$. Thus $w \in F$. Clearly

$$w(1 + g + \ldots + g^{p-1}) \in F^G \subseteq F(x, q).$$

But

$$w(1 + g + \ldots + g^{p-1}) = [e, x^{p^\tau} + (x + f_1)^{p^\tau} + \ldots + (x + f_{p-1})^{p^\tau}].$$

Thus, by Lemma 5.2, $w(1 + g + \ldots + g^{p-1}) \notin F(x, q)$. This is the required contradiction.

Proof of Theorem A. We first deal with the case where G is simple. Let $N = G \cap \mathrm{IA}(F)$. Thus $N = \{1\}$ or $N = G$. If $N = \{1\}$ then the result follows from Theorem 4.5. On the other hand, if $N = G$ then, by Lemma 2.1, K has prime characteristic p and G is a p-group; so it follows that G is cyclic of order p and the result is given by Lemma 5.3.

For the general case we argue by induction on $|G|$ and assume that G is not simple. Thus G has a non-trivial normal subgroup B such that G/B is simple. By the inductive hypothesis, F^B is not finitely generated. Clearly F^B is G-invariant. If G acts trivially on F^B then $F^G = F^B$ and the result follows. Thus we may assume that G acts non-trivially on F^B. Since G/B is simple it follows that G/B acts faithfully on F^B. By the theorem of Shirshov and Witt (see [**9**] for example), F^B is a free Lie algebra over K. Since F^B is not finitely generated, it is free of rank greater than 1. Hence, by the inductive hypothesis, $(F^B)^{G/B}$ is not finitely generated. In other words, F^G is not finitely generated.

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