COMPLETIONS OF SEMILATTICES OF CANCELLATIVE SEMIGROUPS

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Introduction. A semilattice of cancellative semigroups S is a p.o. semigroup with the order relation $a \leq b$ iff $ab = a^2$. If S is a strong semilattice of cancellative semigroups (i.e., multiplication in S is given by structure maps $\phi_{e,f}$ $(f \leq e$ in $E))$, for each supremum-preserving completion $\bar{E}$ of the semilattice E there is a strong semilattice of cancellative semigroups T over $\bar{E}$ which is a supremum-preserving completion of S in $\leq$. Given $\bar{E}$, T is constructed directly. In this paper it is shown that multiplication by an element of S distributes over suprema in $\leq$ if E has this property (called strong distributivity). Next it is shown that the completion construction also applies to a semilattice of cancellative semigroups which is not strong if S is commutative and $\bar{E}$ is strongly distributive. Finally, it is shown that for semilattices of cancellative monoids a completion is completely determined, up to isomorphism over S, by completions of E.

We begin by noting that if S is a semilattice of cancellative semigroups $S_e (e \in E)$ then there are three particular ways of defining an order relation on S, namely

$$a \leq_1 b \Leftrightarrow ab = a^2, \qquad a \leq_2 b \Leftrightarrow ba = a^2$$

and

$$a \leq_3 b \Leftrightarrow asb = bsa = asa \quad \text{for all} \quad s \in S$$

(see [**5**] and [**10**]). These all coincide in this case. For if $a \in S_e$, $b \in S_f$ and $a \leq_1 b$ then $ab = a^2$ (giving $e \leq f$), so that $aba = a^3$ and $ba = a^2$, since S_e is cancellative. Hence $\leq_1$ and $\leq_2$ coincide. If $a \leq_3 b$ then $a^2b = a^3$ giving $a \leq_1 b$, while if $a \leq_1 b$ and $s \in S$, the equation

$$asbasa = asaasa$$

and cancellation give the remaining equivalence. Necessary and sufficient conditions for these relations to be order relations are found in [**5**] and [**10**]. This is the case for semilattices of cancellative semigroups.

In the case of inverse semigroups whose idempotents are central, this order coincides with the natural order for an inverse semigroup [**6**, p. 40]. In particular this applies to semilattices.

The order relation on S makes S into a p.o. semigroup [**5**, Proposition 3] and the relation is called Abian's order. A subset X of S can have an upper bound in S only if it is *boundable*, i.e., for $x, y \in X$, $xy^2 = x^2y$. A semigroup S is *complete* if every boundable set in S has a supremum. An embedding $S \subset T$ of semigroups is a *completion* if (i) T is a semilattice of cancellative semigroups, (ii) T is complete and (iii) every element of T is the supremum of some boundable set in S. We shall be dealing with completions such that the inclusion $S \subset T$ preserves suprema which exist in S.

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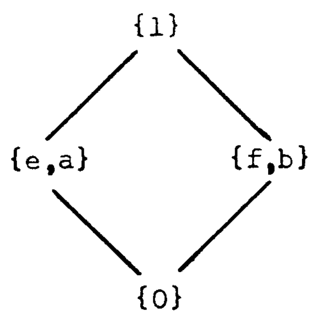
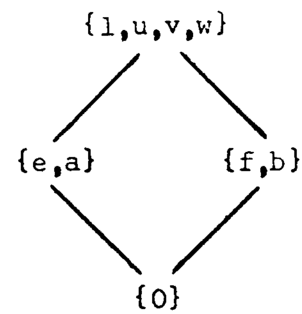
Finally, if $S=\bigcup_E S_e$ is a strong semilattice of cancellative semigroups (i.e., for $f\le e$ in E there are homomorphisms $\phi_{e,f}:S_e\to S_f$ such that for $a\in S_e$, $b\in S_f$, $ab=\phi_{e,ef}(a)\phi_{f,ef}(b)$) then a set $X\subset S$ is boundable if and only if for $x, y\in X$, $x\in S_e$, $y\in S_f$ then $\phi_{e,ef}(x)=\phi_{f,ef}(y)$. Note that distinct elements of a boundable set are in distinct cancellative parts of S.

1. The completion of a strong semilattice of cancellative semigroups. Throughout this part let S be a semigroup which is a semilattice E of cancellative semigroups S_e $(e\in E)$, where multiplication in S is given by structure maps $\phi_{e,f}:S_e\to S_f$ for $e, f\in E$, $f\le e$.

The construction of a completion T for S is done by directly constructing T as a lattice of cancellative semigroups. This construction owes something to the construction of semigroups of quotients of semilattices of groups as found in **[9]** and **[11]**, but here a completion of E is at the base of it all and the cancellative components need not be groups.

Examples. Consider the following lattices of groups.

S: 

T: 

where, in both, $e^2=e$, $f^2=f$. In T, ker $\phi_{1,e}=\{1, u\}$, ker $\phi_{1,f}=\{1, v\}$. The boundable sets of S are: the singletons, $\{e, f\}$, $\{e, b\}$ $\{a, f\}$, $\{a, b\}$ and these four with 0 added and $\{1, e, f, 0\}$. We have that S is not complete since, for example, $\{a, b\}$ has no upper bound. However T is a completion of S with the obvious embedding. This is a model for the general construction.

The semilattice E can be completed in various ways, $E\subseteq \bar{E}$, $\bar{E}$ a complete lattice. In particular we may take $\bar{E}$ to be the Dedekind–MacNeille completion where the element $f\in\bar{E}$ corresponds to the subset $A=\{e\in E \mid e\le f\}$ of E (see **[12**, p. 44]). The embedding $E\subseteq\bar{E}$ preserves all suprema which exist in E. The completion to be constructed will be a lattice of cancellative semigroups $T=\bigcup_{\bar{E}} T_f$. (In order to obtain the theorem below, any supremum-preserving completion of E will suffice, the particular one being mentioned only for concreteness.) The construction of T and the verification of its properties will be done in six steps. In order to establish notation for the remainder of the article, to $f\in\bar{E}$ we make correspond a subset of E as follows: if $f=\sup\{e\in E \mid e<f\}$ then we let $A=\{e\in E \mid e<f\}$ (this occurs if $f\in\bar{E}\setminus E$ and for some elements of E); if $f\ne\sup\{e\in E \mid e<f\}$ we let $A=\{e\in E \mid e\le f\}$ (this can only occur for some elements of E, for

example 0, e, f in the preceding example, but not 1). Whichever case occurs A will be called the subset of E *corresponding to* f.

STEP 1. For $f \in \bar{E}$, let A be the corresponding subset of E. Define T_f to be the inverse limit of the system

$$\{S_e\,;\,\phi_{e,e'},\, e,\, e' \in A,\, e' \leq e\}$$

(see [**8**, p. 291]); that is, T_f is the subsemigroup of $\prod_A S_e$ consisting of the elements $(x_e)_A$ such that if $e' \leq e$ $(e, e' \in A)$, then $\phi_{e,e'}(x_e) = x_{e'}$. The result is clearly cancellative and if the S_e $(e \in A)$ are groups, so is T_f. Note that T_f could be empty although not in the case where each S_e $(e \in A)$ contains an idempotent. Also if $f \in E$ and $A = \{e \in E \mid e \leq f\}$ then $T_f = S_f$.

The elements of T_f are precisely the boundable sets X of S such that

(i) if $s \leq x$ for some $s \in S$, $x \in X$, then $s \in X$,

and

(ii) $\{e \in E \mid x \in S_e$ for some $x \in X\} = A$.

STEP 2. Put $T = \bigcup_{\bar{E}} T_f$ and define multiplication via structure maps as follows. If $f, f' \in \bar{E}$, $f' \leq f$ with corresponding subsets of $E, B \subseteq A$, then define $\psi_{f,f'}: T_f \rightarrow T_{f'}$ by $\psi_{f,f'}((x_e)_A) = (x_e)_B$, the restriction of $(x_e)_A \in T_f \subseteq \prod_A S_e$ to B. Then in general if $f, f' \in \bar{E}$ have corresponding subsets A and B of E, respectively,

$$(x_e)_A(y_{e'})_B = \psi_{f',ff'}((y_{e'})_B).$$

Abian's order is defined on T.

STEP 3. The embedding of $S \subseteq T$ is as follows. If $e \in E$ then $e \in \bar{E}$ and we assign to $x \in S_e$ the element $(x_{e'})_A$ where A is the subset of E corresponding to e and for $e' \in A$, $x_{e'} = \phi_{e,e'}(x)$. This embedding is clearly order-preserving.

STEP 4. Every element of T is the supremum of a subset of S. Consider $x = (x_e)_A \in T_f$ then $(x_e)_A = \sup\{x_e \mid e \in A\}$. Indeed, for $e' \in A$,

$$xx_{e'} = \psi_{f,e'}((x_e)_A)x_{e'} = x_{e'}^2.$$

Thus y is an upper bound of $X = \{x_e \mid e \in A\}$. Suppose that $y = (y_e)_B \in T_{f'}$ is an upper bound of X. Then $(y_e)_B x_{e'} = x_{e'}^2$, showing that $e' \leq f'$ for all $e' \in A$. Hence, $A \subseteq B$ and $f \leq f'$. By the cancellation property, for $e \in A$, $y_e = x_e$ so that $\psi_{f',f}(y) = x$ and so $xy = x^2$, giving the result.

STEP 5. The semigroup T is complete. Let $X = \{(x_e^\alpha)_{A_\alpha} \mid \alpha \in \Lambda\}$ be a boundable set in T. We may assume that if $t \in T$, $x \in X$ and $t \leq x$ then $t \in X$. We have that for $(x_e^\alpha)_{A_\alpha}$ and $(x_e^\beta)_{A_\beta}$ in X,

$$(x_e^\alpha)_{A_\alpha}^2(x_e^\beta)_{A_\beta} = (x_e^\alpha)_{A_\alpha}(x_e^\beta)_{A_\beta}^2.$$

Calculating we get

$$(x_e^\alpha)_{A_\alpha}^2(x_e^\beta)_{A_\beta} = ((x_e^\alpha)^2 x_e^\beta)_{A_\gamma} = (x_e^\alpha(x_e^\beta)^2)_{A_\gamma},$$

where if A_α corresponds to f_α and A_β to f_β then A_γ corresponds to $f_\alpha f_\beta$; thus $A_\gamma = A_\alpha \cap A_\beta$. Hence for all $e \in A_\gamma$, $x_e^\alpha = x_e^\beta$. Put

$$U = \{x_e^\alpha \mid e \in A_\alpha, \alpha \in \Lambda\} \subseteq S.$$

Then

$$\begin{aligned}(x_e^\alpha)^2 x_{e'}^\beta &= \phi_{e,ee'}((x_e^\alpha)^2)\phi_{e',ee'}(x_{e'}^\beta)\\ &= (x_{ee'}^\alpha)^2 x_{ee'}^\beta = x_{ee'}^\alpha (x_{ee'}^\beta)^2\\ &= x_e^\alpha (x_{e'}^\beta)^2.\end{aligned}$$

Hence U is boundable.

Put $A = \{e \in E \mid x_e^\alpha \in S_e$ for some $x_e^\alpha \in U\}$. This set has a supremum f in $\bar{E}$ and $f = \sup(\sup A_\alpha)$ (see [**1**, p. 53]); hence if $e \in E$, $e \leq f$ then $e \leq \sup A_\alpha$ for some $e \in A$ and thus $e \in A_\alpha$. Now put $x = (x_e)_A$ where x_e is any x_e^α $(\alpha \in \Lambda)$. This is well-defined, since if $e \in A_\alpha \cap A_\beta$ then $x_e^\alpha = x_e^\beta$. We claim that $x = \sup X$.

For $(x_{e'}^\alpha)_{A_\alpha} \in X$ we have

$$(x_e)_A (x_{e'}^\alpha)_{A_\alpha} = (x_{e'} x_{e'}^\alpha)_{A_\alpha},$$

since $A_\alpha \subseteq A$; hence for $e' \in A_\alpha$, $x_{e'} = x_{e'}^\alpha$ and we conclude that

$$x(x_{e'}^\alpha)_{A_\alpha} = (x_{e'}^\alpha)^2_{A_\alpha}.$$

Hence x is an upper bound. But $x = \sup U$ and every $x_e^\alpha \in U$ is below an element of X (indeed if $x_e^\alpha \in \{x_{e'}^\alpha\}_{A_\alpha}$ then $x_e^\alpha \leq (x_{e'}^\alpha)_{A_\alpha}$ by the nature of the embedding $S \subseteq T$). Hence $x = \sup X$.

Step 6. The embedding $S \subseteq T$ preserves all suprema which exist in S. Let $s = \sup X$ (X a subset of S, $s \in S$). Then if $A = \{e' \mid x \in S_{e'}$ for some $x \in X\}$ and $s \in S_e$, it follows that $e = \sup A$. If $g \leq e$ is another upper bound of A in E, it is readily seen that $y = \phi_{e,g}(x)$ is an upper bound of X and that $y \leq x$. Hence $y = x$ and $e = g$. But $e = \sup A$ in $\bar{E}$ as well and the boundable set X has a supremum t in T_e, and hence $t \leq s$. But T_e is cancellative so $s = t$.

This completes the construction, giving the following theorem; its corollary follows from it and the remark in Step 1.

Theorem 1. *Let S be a semigroup with decomposition $S = \bigcup_E S_e$, where E is a semilattice and the S_e are cancellative. Suppose further that multiplication in S is given by structure maps $\phi_{e,e'}: S_e \to S_{e'}$ for $e' \leq e$ in E. Then S has a completion in Abian's order T where T is a semigroup of the same type as S and the inclusion $S \subseteq T$ preserves suprema from S.*

Corollary 2. *Let S be a semilattice of groups. Then S has a completion T in Abian order, where T is a lattice of groups.*

The completion of a semigroup is not unique (unlike the case for rings [**2**, Theorem 12]) since even a lattice may be completed in several non-isomorphic ways. Uniqueness will be discussed further in part 4 below. Theorem 1 does yield an internal characterization of complete semigroups (of the type being studied here). The proof is clear from the proof of Theorem 1.

PROPOSITION 3. *Let S be a semigroup with decomposition $S=\bigcup_E S_e$, where E is a semilattice, the S_e are cancellative and the multiplication in S is given by structure maps $\phi_{e,e'}: S_e \to S_{e'}$ $(e' \le e$ in $E)$. Then S is complete if and only if* (i) *E is a complete lattice,* (ii) *if $f \in E$ is such that $f=\sup A$ where $A=\{e\in E \mid e<f\}$ then $S_f=\varprojlim_A \{S_e;\phi_{e,e'}\}$ and* (iii) *if $e'\le e$ in E then $\phi_{e,e'}$ is the homomorphism induced by the universal property of inverse limits.*

EXAMPLE. Let E be a semilattice with 0 such that $ef=0$ for all $e\neq f$. Then with $S_0=\{0\}$ and S_e arbitrary $(e\neq 0)$, a semigroup $S=\bigcup_E S_e$ can be formed. By adjoining an element 1 to E we get a completion $\bar{E}$. Clearly $T_1=\prod_E S_e$ and $T_e=S_e$ for all $e\in E$. Here $\bar{E}$ is the Dedekind–MacNeille completion of E. Using the same E we can also form the ideal completion F of E, which in this case is supremum-preserving (it is not always [**7**]); F is the set of all subsets of E which contain 0. For $U\in F$, $T_U=\prod_U S_e$. These two completions are clearly not isomorphic.

2. Distributivity. In the case of semiprime rings, Abian's order and Conrad's order satisfy an infinite distributivity: if R is a semiprime ring and if $x=\sup X$, $a\in R$ then $\sup aX=ax$ and $\sup Xa=xa$ ([**4**, Corollary 3]). For semigroups this is false since there are lattices which are not distributive. However, for the type of semigroups we have been studying, distributivity will be seen to be a property of the underlying semilattice. Let us say that a semilattice L is *strongly distributive* if for any subset X of L and $e\in L$, if $\sup X$ exists then $\sup eX=e(\sup X)$.

PROPOSITION 4. *Let S be a semigroup with a decomposition $S=\bigcup_E S_e$, where E is a semilattice, the S_e are cancellative and multiplication in S is given by structure maps $\phi_{e,e'}: S_e\to S_{e'}$ $(e'\le e$ in $E)$. Suppose that E is strongly distributive. Then for any boundable set X of S and any $a\in S$, $\sup aX=a(\sup X)$ and $\sup Xa=(\sup X)a$ if $\sup X$ exists.*

Proof. Let $y=\sup X$. If $A=\{e\in E \mid x\in X\cap S_e$ for some $x\}$, then clearly if $y\in S_f$ we have $f=\sup A$. Let $a\in S_g$ and consider

$$\begin{aligned} ayax &= \phi_{g,ge}(a)\phi_{f,ge}(y)\phi_{g,ge}(a)\phi_{e,ge}(x)\\ &= \phi_{g,ge}(a)\phi_{e,ge}(x)\phi_{g,ge}(a)\phi_{e,ge}(x)\\ &= axax \quad \text{for} \quad x\in X\cap S_e. \end{aligned}$$

Hence ay is an upper bound for aX. Let $u\in S_h$ be another upper bound for aX. Since h is an upper bound for gA,

$$h \geq \sup gA = g(\sup A) = gf.$$

It follows that $\phi_{h,gf}(u)$ is an upper bound of aX in S_{gf}. By cancellation, $\phi_{h,gf}(u) = ay$ and $ay \leq u$.

An analogous statement for inverse semigroups is [**13**, Lemma 1.13].

Note that strong distributivity for semilattices and, more generally, for semigroups with Abian's order implies the following distributive property: if S is a strong semilattice of cancellative semigroups such that for $s \in S$ and a boundable set X, $\sup sX = s(\sup X)$ if either exists, then for boundable sets X and Y we get that $XY = \{xy \mid x \in X, y \in Y\}$ is boundable and $\sup XY = (\sup X)(\sup Y)$ if either side exists.

3. A generalization. In this section we attempt to construct a completion of a semilattice of cancellative semigroups where there are no structure maps available. It will be necessary to impose supplementary conditions on the cancellative semigroups and on the semilattice.

THEOREM 5. *Let S be a commutative semigroup which is a semilattice $\bigcup_E S_e$ of cancellative semigroups. Assume further that E has a supremum-preserving completion $\bar{E}$ which is strongly distributive. Then S has a supremum-preserving completion.*

Proof. We first construct for each $e \in E$ the group G_e of fractions of S_e. For $ab^{-1} \in G_e$ and $cd^{-1} \in G_{e'}$ define $ab^{-1} . cd^{-1} = ac(bd)^{-1} \in G_{ee'}$. Let $G = \bigcup_E G_e$ with the indicated multiplication; it is a semigroup of the type studied in Part 1. Let $T = \bigcup_{\bar{E}} T_f$ be the completion of G as constructed in Theorem 1.

For $f \in \bar{E}$ let A be the corresponding subset of E (see Part 1 for notation) and recall that an element of T_f has the form $(x_e)_A$ where if $e' \leq e$ in A then $\phi_{e,e'}(x_e) = x_{e'}$. Put

$$U_f = \{(x_e)_A \in T_f \mid \text{ for some } \quad B \subseteq A, \quad \sup B = \sup A = f, \quad x_e \in S_e \quad \text{for all} \quad e \in B\}$$

These are elements of T_f which are, in a sense, "almost everywhere" in S. We put $U = \bigcup_{\bar{E}} U_f$ and we shall show that U is the desired completion. Note that, as remarked in Part 2, if $\bar{E}$ is strongly distributive and $A, B \subseteq \bar{E}$ then $\sup AB = (\sup A)(\sup B)$; indeed

$$\begin{aligned}\sup AB &= \sup_A(\sup aB) = \sup_A(a \sup B)\\ &= \sup(A \sup B) = (\sup A)(\sup B).\end{aligned}$$

Firstly, U is a subsemigroup of T. Let $(x_e)_A \in U_f$ and $(y_e)_{A'} \in U_{f'}$ where A and A' are the subsets of E corresponding to f and f' respectively and for some $B \subseteq A$, $\sup B = f$, $x_e \in S_e$ for all $e \in B$ and for some $B' \subseteq A'$, $\sup B' = f'$, $y_e \in S_e$ for all $e \in B'$. Then

$$(x_e)_A . (y_e)_{A'} = (x_e y_e)_{AA'}$$

(of course $AA' = \{e \in E \mid e \leq ff'\}$). But $\sup BB' = (\sup B)(\sup B') = ff'$ (by hypothesis) and $x_e y_e \in S_e$ for all $e \in BB'$.

LEMMA. *If* $(x_e)_A \in T_f$, A *is the subset of* E *corresponding to* f *and* $B \subseteq A$ *is such that* $\sup B = f$ *then* $\sup\{x_e \mid e \in A\} = \sup\{x_e \mid e \in B\}$.

Proof. Let $x = \sup\{x_e \mid e \in A\}$, $y = \sup\{x_e \mid e \in B\}$. Clearly both x and y are in T_f and $y \leq x$. This gives the equality.

COROLLARY. *If* $(x_e)_A$, $(y_e)_A \in T_f$ *and for some* $B \subseteq A$, *with* $\sup B = f$, $x_e = y_e$ *for all* $e \in B$ *then* $(x_e)_A = (y_e)_A$.

Proof. By the lemma, $\sup\{x_e \mid e \in B\} = (x_e)_A = (y_e)_A$.

Returning to the theorem, we must show that U is complete; it will follow that U is a completion of S, since if $(x_e)_A \in U_f$ and $B \subseteq A$ with $\sup B = f$ and $x_e \in S_e$ for all $e \in B$ then the lemma shows that $(x_e)_A = \sup\{x_e \mid e \in B\}$, the supremum of a subset of S.

Let $X = \{(x_e^\alpha)_{A_\alpha} \mid \alpha \in \Lambda\}$ be a boundable set from U, where $A_\alpha \subseteq E$ corresponds to f_α, $B_\alpha \subseteq A_\alpha$, $\sup B_\alpha = f_\alpha$ and $x_e^\alpha \in S_e$ for all $e \in B_\alpha$. Put $x = \sup X$, an element of T. It will be shown that $x \in U$. Since X is boundable, for $e \in A_\alpha \cap A_\beta = A_\alpha A_\beta$ we have $x_e^\alpha = x_e^\beta$. Let

$$Y = \{x_e \mid x_e = x_e^\alpha \quad \text{for some } e \in \bigcup_\Lambda A_\alpha \text{ and some } \alpha \in \Lambda\}.$$

As was shown in Theorem 1, Step 5, Y is boundable with the same supremum as X. Now consider $\bigcup B_\alpha \subseteq \bigcup A_\alpha$. We have

$$\begin{aligned}\sup \bigcup B_\alpha &= \sup\{\sup B_\alpha \mid \alpha \in \Lambda\} = \sup\{f_\alpha \mid \alpha \in \Lambda\} \\ &= \sup\{\sup A_\alpha \mid \alpha \in \Lambda\} = \sup \bigcup A_\alpha = f.\end{aligned}$$

Hence $x \in U_f$.

It would be desirable to weaken the conditions on Theorem 6 to those of Theorem 1.

4. Uniqueness of completions. It has already been mentioned that completions are not unique since semilattices may have non-isomorphic completions. However, in the case of a semilattice of *monoids*, it will be shown that there is, up to isomorphism over S, one supremum-preserving completion of S, which is a semilattice of cancellative semigroups, for each isomorphism class of supremum-preserving completions of the underlying semilattice E.

THEOREM 6. *Let* $S = \bigcup_E S_e$ *be a semilattice of cancellative monoids and let* $U = \bigcup_F U_f$ *be a semilattice of cancellative semigroups which is a supremum-preserving completion of* S. *Then* (i) *each* U_f *is a monoid,* (ii) F *is a supremum-preserving completion of* E, (iii) U *is isomorphic over* S *to the completion constructed over* F *in Theorem* 1.

Proof. E is contained in F as semilattices, for if $e \in E \subseteq S$ and $e \in U_f$ then for $s \in S_e$, $s = se$. It follows that $s \in U_f$. Further if $e, e' \in E$ with $e \in U_f$, $e' \in U_{f'}$ then $ee' \in U_{ff'}$. Hence if $e \in U_f$, e may be identified with f. Further, F is a supremum-preserving completion of E.

Let $u \in U_f$, $u = \sup X$ for some $X \subseteq S$. Put

$$A = \{e \in E \mid x \in S_e \text{ for some } x \in X\}.$$

Now for $x \in X$, $x \in S_e$, $ux = x^2 \in S_e \subseteq U_e$, so that $e \leq f$. Since U is complete, the boundable set A has a supremum g in U. Now g is an idempotent, since $g^2 e = ge = e$ for all $e \in A$, which shows that g^2 is also an upper bound of A; hence $g \leq g^2$ giving $g^3 = g^2$ in a cancellative semigroup. Thus $g = g^2$. We also have

$$gux = gx^2 = gex^2 = ex^2 = x^2$$

for $x \in X \cap S_e$, and so $gu \geq u$. From this $gu^2 = u^2$, showing that $g \in U_f$. We may identify g with $f \in F$.

It follows that each U_f is a monoid. By Proposition 3, F is complete, giving (i) and (ii).

Now let $f \in F$ with corresponding set $A \subseteq E$. Each $u \in U_f$ is the supremum of some $X \subseteq S$. Let

$$B = \{e \in E \mid x \in X \cap S_e \text{ for some } x\}.$$

Clearly $B \subseteq A$ and $\sup B = \sup A = f$. Further, if $e \in A$ there is $e' \in B$ with $e' \geq e$, from which it follows that $ue = ue'e$. But if $x \in X \cap S_{e'}$, then $u \geq x$ implies that $ue' = x$. Hence $ue = xe \in S$. Thus multiplication by $e \in A$ gives a homomorphism $\tau_e : U_f \to S_e$. Let $T_f = \varprojlim_A \{S_e ; \phi_{e,e'}\}$ (as in Part 1). The homomorphisms τ_e induce a homomorphism $\tau : U_f \to T_f$ by the universal property of inverse limits. This is readily seen to be an isomorphism. Further, for $f, f' \in F$, $f' \leq f$, multiplication by f' gives $U_f \to U_{f'}$, which is precisely the induced homomorphism $\psi_{f,f'}$ of Theorem 1. Hence U is isomorphic to T constructed as in Theorem 1 over F and the isomorphism leaves elements of S fixed.

It would be desirable to be able to get this uniqueness result for any strong semilattice of cancellative semigroups.

If the semilattice E is a Boolean algebra then there is only one completion (the Dedekind–MacNeille) and it is strongly distributive. Hence if R is a strongly regular ring then the completion of its multiplicative semigroup is unique; it is based on the completion of $B(R)$, the Boolean algebra of idempotents. This completion is the multiplicative semigroup of the completion of R as a ring which is, in this case, the complete ring of quotients, $Q(R)$ (see [**2**, Theorem 14] and [**3**, Theorem 5]). More generally, if R is a reduced p.p. ring (a ring with no non-zero idempotents in which the annihilator of each element is generated by an idempotent; in a reduced ring all idempotents are central and left and right annihilators coincide) then the multiplicative semigroup is a Boolean algebra of cancellative semigroups. Indeed for $e = e^2 \in R$, put

$$R_e = \{r \in R \mid re = r \quad \text{and if for some} \quad f = f^2, rf = r \quad \text{then} \quad e \leq f\}.$$

Now if $r, s, t \in R_e$, and $rs = rt$ we get $s - t \in \operatorname{Ann} r = gR$ for some $g = g^2$. Thus $r(1-g) = r$ and $e \leq 1 - g$ giving $eg = 0$ and

$$s - t = g(s-t) = g(es - et) = 0,$$

showing that $s = t$. Further, $R = \bigcup_{B(R)} R_e$. Let $r \in R$; then Ann $r = eR$ for some $e \in B(R)$ and $r(1-e) = r$. If $rf = r$ for $f \in B(R)$ then $r(1-f) = 0$ and $1-f \in eR$, giving $1-e \leq f$. Hence $r \in R_{1-e}$.

Now if R is commutative p.p. ring, it has a completion in Abian's order, call it $C(R)$, and $B(C(R))$ is the Dedekind–MacNeille completion of $B(R)$ ([**3**, Theorem 11]). We have shown the following:

PROPOSITION 7. *Let R be a commutative p.p. ring. Then there is a unique supremum-preserving completion of the multiplicative semigroup of R. It is the multiplicative semigroup of the completion of the ring R.*

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