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ON ERGODIC EXTENSIONS OF STATIONARY MEASURES WITH MINIMAL SUPPORT

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ABSTRACT. Let T be an ergodic measure preserving transformation with the following property: there exists a positive integer n and a finite partition α such that the number of atom of $\bigvee_{i=0}^{n+1} T^i\alpha$ is one more than that of $\bigvee_{i=0}^{n} T^i\alpha$, and the probability of at least one of the atoms is irrational. Then there exists a unique (up to conjugacy) transformation S such that there is a partition β with S restricted to $\bigvee_{i=0}^{n+1} S^i\beta$ isomorphic to T restricted to $\bigvee_{i=0}^{n+1} T^i\alpha$, and the number of atoms in $V_{i=0}^{m+1} S^i\beta$ is one more than the number of atoms in $\bigvee_{i=0}^{m} S^i\beta$ for all $m \geq n$. Moreover this transformation has discrete spectrum with at most two generators. If there are two generators, one of them must be a root of unity.

1. Introduction. Hobby and Ylvisaker (1965) studied the problem of extending a stationary measure p_n on $\mathscr{A}^n$ to $\mathscr{A}^{n+1}$ where $\mathscr{A}$ is a finite alphabet (precise definition will be given in §2). One of their results states that for all p_1 on $\mathscr{A} = \{0, 1\}$ such that $p_1(0)$ is irrational, there exists a sequence $p_1, p_2, \ldots$ of stationary extensions such that N_n, the number of points in the support of p_n, is equal to $n+1$ for all n. We generalize this result by proving that for all ergodic p_n on $\mathscr{A}^n$ such that $N_n = N_{n-1}+1$ and such that some $p_n(x)$ is irrational, there exists a unique sequence of ergodic extentions, $p_{n+1}, p_{n+2}, \ldots$ such that $N_m = N_n + m - n$ for all $m \geqq n$. We further show that the class of measure preserving transformations that are obtainable in this way is identical with the class of transformations obtained by taking the cartesian product of a rotation of a finite number of points and an irrational rotation of the circle.

Hobby and Ylvisaker also show that for any stationary p_n on $\mathscr{A}^n$ there exists some m and a stationary extension p_m on $\mathscr{A}^m$ such that $N_m = N_{m-1}$. This result loses much of its interest because it generally produces nonergodic extensions. In fact we show that, in some sense, most p_n on $\mathscr{A}^n$ do not have ergodic extensions p_m to any $\mathscr{A}^m$ such that $N_m \leqq N_{m-1}+1$.

2. Preliminaries. In this section we shall state definitions and some known or easily derived results that will be used in the following sections.

Let $(X, \mathscr{B}, P)$ be a probability space and T a measure preserving transformation on X, i.e. for all $A \in \mathscr{B}$ $T^{-1}(A) \in \mathscr{B}$ and $P(T^{-1}(A)) = P(A)$. Let $\alpha = \{A(k) : k \in \mathscr{A}\}$ be a measurable partition of X, where $\mathscr{A}$ is finite set. For

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$n = 1, 2, \ldots$ and $x = (a_1, \ldots, a_n) \in \mathscr{A}^n$ we set

$$A_n(x) = \bigcap_{k=1}^{n} T^{1-k}(A(a_k)), \qquad p_n(x) = P(A_n(x)). \tag{2.1}$$

The functions p_n satisfy the following four conditions for all $n = 1, 2, \ldots$ and for all $x \in \mathscr{A}^n$:

$$p_n(x) \geqq 0 \qquad \sum_{a \in \mathscr{A}} p_1(a) = 1 \qquad \sum_{a \in \mathscr{A}} p_{n+1}(x, a) = p_n(x) \tag{2.2}$$

$$\sum_{a \in \mathscr{A}} p_{n+1}(a, x) = p_n(x).$$

Conversely, the well-known Kolmogorov extension theorem implies that if $\mathscr{A}$ is any finite set and $p_1, p_2, \ldots$ any sequence of functions defined respectively on $\mathscr{A}, \mathscr{A}^2, \ldots$ and satisfying (2.2), then there exists a probability space $(X, \mathscr{B}, P)$, a measure preserving transformation T, and a partition α such that (2.1) holds.

For this reason we will refer to p_n as a *stationary* (*probability*) *measure* on $\mathscr{A}^n$ and call p_m, $m > n$, a *stationary extension* of p_n. The collection $\{A_n(x) : x \in \mathscr{A}^n\}$ is a partition denoted by $\bigvee_{k=1}^{n} T^{1-k}\alpha$. The sets $S_n = \{x \in \mathscr{A}^n : p_n(x) > 0\}$ and $\{A_n(x) : x \in S_n\}$ will be called the *support* of p_n, and N_n will denote the number of points in S_n.

A measure preserving transformation T is said to be ergodic if $T^{-1}(A) = A \in \mathscr{B}$ implies that $P(A) = 0$ or 1. We shall say p_n is *ergodic* if for all x and $y \in S_n$, there exist a number m and a chain $x = x_0, x_1, \ldots, x_m = y$ of vectors $x_i \in S_n$ such that for each $i = 1, \ldots, m$ there exist $a, b \in \mathscr{A}$ and $z \in \mathscr{A}^{n-1}$ with $x_{i-1} = (a, z)$ and $x_i = (z, b)$. We then have

2.3. Lemma. *A stationary measure p_n on $\mathscr{A}^n$ is ergodic if and only if there exists an ergodic transformation T such that* (2.1) *holds.*

Sketch of proof. If p_n is not ergodic, then some union of sets $A_n(x)$ will produce a proper invariant set for every induced transformation T. Conversely if p_n is ergodic there is an n-dependent ergodic Markov extension. This is the extension obtained by taking for each $m > n$ the stationary extension p_m of p_{m-1} with maximal entropy, $-\sum_{x \in \mathscr{A}} p_m(x) \ln p_m(x)$.

3. Unique minimal extensions. In this section we prove the following.

3.1. Theorem. *Let p_n be an ergodic stationary measure on $\mathscr{A}^n$ such that $N_n \leqq N_{n-1} + 1$. Then there exists a unique ergodic stationary extension of p_n to $\mathscr{A}^{n+1}$ such that $N_{n+1} \leqq N_n + 1$.*

Of course, $N_n \geqq N_{n-1}$, and if $N_n = N_{n-1}$, then any ergodic measure preserving transformation T determining p_n must be a rotation of N_n points: $X = \{1, \ldots, N_n\}$, $T(k) = k + 1 \bmod(N_n)$. For the case $N_n = N_{n-1} + 1$, we shall use the following.

3.2. LEMMA. *Let p_n be an ergodic stationary extension of p_{n-1} such that $N_n = N_{n-1}+1$, and let T be an ergodic measure preserving transformation such that* (2.1) *holds. Then there exist non-negative integers r, s, and t with $s>0$ and a partition $\alpha = (A_1, \ldots, A_{r+s}, B_1, \ldots, B_s, C_1, \ldots, C_t)$ such that*

$$\begin{aligned} T^{-1}(A_i) &= A_{i+1}, \qquad 1 \leqq i < r+s \\ T^{-1}(B_i) &= B_{i+1}, \qquad 1 \leqq i < s \\ T^{-1}(A_{r+s} \cup B_s) &= C_1 \\ T^{-1}(C_i) &= C_{i+1}, \qquad 1 \leqq i < t \\ T^{-1}(C_t) &= A_1 \cup B_1. \end{aligned}$$

Proof of lemma. By (2.2) for every $x \in S_{n-1}$ there exist a_x and $b_x \in \mathscr{A}$ such that (a_x, x) and $(x, b_x) \in S_n$. Conversely if (a_x, x) or $(x, b_x) \in S_n$, then $x \in S_{n-1}$. Since $N_n = N_{n-1}+1$, there exist unique x_0 and $y_0 \in S_{n-1}$ and distinct $b_{x_0}, b'_{x_0} \in \mathscr{A}$ (resp. distinct $a_{y_0}, a'_{y_0} \in \mathscr{A}$) such that (x_0, b_{x_0}) and (x_0, b'_{x_0}) (resp. (a_{y_0}, y_0) and (a'_y, y_0)) are in S_n. If $x \notin \{x_0, y_0\}$, then a_x and b_x are uniquely determined. Suppose now that $x = (a_1, \ldots, a_{n-1}) \in S_{n-1}$ and $x' = (x, b) \in S_n$. Then

$$\begin{aligned} T^{-1}(A_n(x')) &= \bigcap_{k=1}^{n-1} T^{-k}(A(a_k)) \cap T^{-n}(A(b)) \\ &\subseteq \bigcap_{k=2}^{n} T^{1-k}(A(a_{k-1})) = \bigcup_{a \in \mathscr{A}} A(a, x). \end{aligned}$$

The fact that both $\{A(x) : x \in S_n\}$ and $\{T^{-1}(A(x)) : x \in S_n\}$ are partitions yields the following relations for $x \in S_{n-1}$.

$$\begin{aligned} & T^{-1}(A(x, b_x)) = A(a_x, x) \quad \text{if} \quad x \notin \{x_0, y_0\} \\ & T^{-1}(A(x, b_x)) = A(a_{y_0}, x) \qquad A(a'_{y_0}, x) \quad \text{if} \quad x = y_0 \neq x_0 \\ (3.3) \quad & T^{-1}(A(x, b_{x_0}) \cup A(x, b'_{x_0})) = A(a, x) \quad \text{if} \quad x = x_0 \neq y_0 \\ & T^{-1}(A(x, b_{x_0}) \cup A(x, b'_{x_0})) = A(a_{y_0}, x) \cup A(a'_{y_0}, x) \quad \text{if} \quad x = x_0 = y_0. \end{aligned}$$

If there is a chain $D_1, \ldots, D_k$ of sets $A(x)$ such that $T^{-1}(D_i) = D_{i+1}$ for $1 \leqq i < k$ and $T^{-1}(D_k) = D_1$, then by ergodicity $k = N_n$ and the theorem is proved with $A_1 = D_1$, $r = k$, and $s = t = 0$. Otherwise there exists a finite number of chains of maximal length of the form $D_1, \ldots, D_k$ where $T^{-1}(D_i) = D_{i+1}$ for $1 \leqq i < k$. By (3.3) D_k must be one of the at most three sets $A(x_0, b_{x_0})$, $A(x_0, b'_{x_0})$, and $A(y_0, b_{y_0})$. Thus there are at most three such chains. If $x_0 \neq y_0$, then set $C_1 = D_1$ where $D_k = A(y_0, b_{y_0})$, A_1 and B_1 are defined to be the first elements of the other two chains. If $x_0 = y_0$, then there are only two possible chains, and the lemma is completed by setting $t = 0$. (In this case $T^{-1}(A_{r+s} \cup B_s) + A_1 \cup B_1$.)

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Proof of Theorem 3.1. Suppose $N_n = N_{n-1}+1$. Then the partition $\bigvee_{i=1}^{n} T^{1-i}\alpha$ is of the form $(A_1, \ldots, A_{r+s}, B_1, \ldots, B_s, C_1, \ldots, C_t)$ given by Lemma 3.2. The partition $\bigvee_{k=1}^{n+1} T^{1-k}\alpha$ then consists of the following sets:

$A_1, \ldots, A_{r+s}, T^{-1}(A_{r+s}), B_1, \ldots, B_s, T^{-1}(B_s), C_2, \ldots, C_t$ if $t \neq 0$. In this case $N_{n+1} = N_n + 1$. If $t = 0$, then $\bigvee_{k=1}^{n+1} T^{1-k}\alpha$ consist of the following sets:

$$A_2, \ldots, A_{r+s}, B_2, \ldots, B_s, \quad \text{and}$$
$$A_1 \cap T^{-1}(A_{r+s}), A_1 \cap T^{-1}(B_s), B_1 \cap T^{-1}(A_{r+s}), B_1 \cap T^{-1}(B_s).$$

If $N_{n+1} \leqq N_n + 1$, then at least one of these last four sets must have probability 0. Let $p = P(A_1)$. Then $P(B_1) = [1-(r+s)p]/s$. Set $P(A_1 \cap T^{-1}(A_{r+s})) = \alpha$. Then

$$P(A_1 \cap T^{-1}(A_{r+s})) = \alpha$$
$$P(A_1 \cap T^{-1}(B_s)) = p - \alpha$$
$$P(B_1 \cap T^{-1}(A_{r+s})) = p - \alpha$$
$$P(B_1 \cap T^{-1}(B_s)) = \alpha + (1-(r+2s)p)/s.$$

Ergodicity requires that $\alpha \neq p$. Thus either $\alpha = 0$ or $\alpha = ((r+2s)p-1)/s$. If $(r+2s)p = 1$, then $N_{n+1} = N_n$. Otherwise $\alpha = 0$ or $((r+2s)p-1)/s$ according as $((r+2s)p-1)/s$ is <0 or >0. Thus there is a unique choice of α and the theorem is proved.

3.4. Corollary. *Let p_n be as in Theorem* 3.1 *and suppose in addition that some $p_n(x)$ is irrational. Then there exists a unique sequence of ergodic stationary extensions $p_{n+1}, p_{n+2}, \ldots$ such that $N_m = N_n + m - n$.*

Proof. If in some extension p_m, given inductively by Theorem 3.1, $N_m = N_{m-1}$, then all probabilities must be rational which is a contradiction. Thus $N_m = N_{m-1}+1$ for all $m > n$.

4. Stacking Methods. In this section we describe the transformations of Section 3 in terms of stacking methods. For an excellent introduction to these methods the reader is refered to Friedman (1970).

We shall define a measure preserving transformation T on $[0, 1]$ by inductively extending the domain of T. At step n we suppose that $\alpha = (A_1, \ldots, A_r, B_1, \ldots, B_s)$ is a measurable partition of $[0, 1]$ such that $\lambda(A_i) = \lambda(A_j)$ and $\lambda(B_i) = \lambda(B_j)$ for all i and j where λ is Lebesgue measure. T is defined on $\bigcup_{k=1}^{r-1} A_k \cup \bigcup_{k=1}^{s-1} B_k$ by mapping A_k (resp. B_k) onto A_{k+1} (resp. B_{k+1}) in a measure preserving manner. Figure 4.1 describes this situation.

Suppose $A_1, \ldots, A_r$ is the stack with $\lambda(A_1) \leqq \lambda(B_1)$. Then T is extended to A_r by mapping it into B_1 in a measure preserving manner. Thus the B-stack is divided into two stacks of "widths" $\lambda(A_1)$ and $\lambda(B_1) - \lambda(A_1)$, and the stack of width $\lambda(A_1)$ is placed on top of the A-stack (cf. Fig. 4.1).

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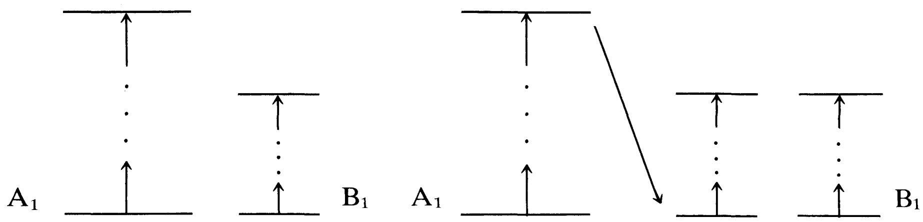

Figure 4.1

If $\lambda(A_1)$ is rational, this procedure will eventually terminate in a single stack after $\lambda(A_1)=\lambda(B_1)$. If $\lambda(A_1)$ is irrational, then the procedure will continue indefinitely, defining T on all $[0, 1]$ by induction.

If we start initially with two stacks, $\alpha=(A_1, \ldots, A_r, B_1, \ldots, B_s)$, then it is easily seen that the number of sets in $\bigvee_{k=1}^{n} T^{1-k}\alpha$ is $r+s+n$. Thus these transformations are the same as those determined by Theorem 3.1. Moreover, Lemma 3.2 proves that any transformation determined by Theorem 3.1 is the same as one obtained by the above stacking method.

This stacking method can sometimes be reversed. Let $\alpha = (A_1, \ldots, A_r, B_1, \ldots, B_s)$ as above, and suppose that $r>s$. Then set $\beta = (A_1, \ldots, A_{r-s}, A_{r-s+1}\cup B_1, \ldots, A_r \cup B_s)$. Then if we apply the above stacking method once to β, we obtain α, and moreover we have the relation $\alpha = \bigvee_{k=0}^{s} T^{-k}\beta$. This reverse procedure can continue until the two stacks have the same height $(r=s)$. Thus we have the following

4.2. Theorem. *Let T be a measure preserving transformation determined by Theorem 3.1. Let α be a partition and m an integer such that $N_{k+1}(\alpha)=N_k(\alpha)+1$ for all $k \geqq m$. Then there exist integers r and n and a partition $\beta=(A_1, \ldots, A_r, B_1, \ldots, B_r)$ such that: $T^{-1}(A_i)=A_{i+1}$ and $T^{-1}(B_i)=B_{i+1}$ for $1 \leqq i<r$, $N_k(\beta)=2r+k-1$ for all $k \geqq 1$, and $\bigvee_{i=0}^{n} T^{-i}\beta=\alpha$.*

5. Rotations. In this section we shall identify the transformation determined by Theorem 3.1.

Let r be a positive integer and denote the ergodic rotation on r points by T_r, i.e., $X_r=\{1, \ldots, r\}$, $\mathcal{B}_r=$ all subsets of X_r, $P_r(A)=r^{-1}$ times the number of points in A, and $T_r(x)=x+1 \bmod (r)$.

Next let b be an irrational number in $[0, 1)$ and denote the rotation through an angle b by T_b, i.e., $X_b=[0, 1)$, $\mathcal{B}_b=$ the Borel sets of X_b, $P_b=$ Lebesgue measure, and $T_b(x)=x+b \bmod (1)$.

Finally set $T=T_r \times T_b$, i.e., $T(x, y)=(T_r(x), T_b(y))$ for $x \square X_r$ and $y \in X_b$. Consider the partition $\beta=\{A_1, \ldots, A_r, B_1, \ldots, B_r\}$ where $A_i=\{i\}\times T_b[0, b)$ and $B_i=\{i\}\times T_b[b, 1)$. The transformation T is then an ergodic transformation with discrete spectrum (cf. Halmos (1956)).

By definition $T(A_i)=A_{i+1}$ and $T(B_i)=B_{i+1}$ for $1 \leqq i<r$. It is also easily seen that the partition $\bigvee_{k=1}^{n} T^{n-k}\beta$ consist of intervals separated by the points

$\{T^i(1,0): 0 \leqq i \leqq 2r+n-1\}$. Thus β is as in Theorem 4.2, and we have

5.1. THEOREM. *Let T be a measure preserving transformation, and α a finite partition such that $\bigvee_{k=0}^{\infty} T^{-k}\alpha$ generates the σ algebra $\mathcal{B}$. Let N be an integer such that for all $n > N$, $N_n \leqq N_{n-1}+1$ and p_n is ergodic. Then T is an ergodic transformation with discrete spectrum. The group of eigenvalues has at most two generators, $e^{2\pi 0\lambda}$ and $e^{2\pi i\mu}$, where λ is rational and μ is irrational. Conversely for any ergodic measure preserving transformation with such a discrete spectrum, there exists a partition α with the above properties.*

REMARKS. Another problem considered by Hobby and Ylvisaker was, starting from an arbitrary stationary measure p_n on $\mathcal{A}^n$ one could extend it in such a way as to eventually have a small or no increase in N_n. Without insisting on the ergodicity of the extention, they show that one can always do this. However, because of the nonergodicity, this solution is not too interesting.

Consider, for example, $\mathcal{A} = \{0, 1\}$ with $p(1) = p$. An arbitrary stationary extension of p to $\mathcal{A}^2$ has the form: $p_2(1,1) = p - r$, $p_2(1,0) = p_2(0,1) = r$, and $p_2(0,0) = 1 - p - r$ where r is any number between 0 and $\min\{p, 1-p\}$. If p_2 has an *ergodic* stationary extension to some $\mathcal{A}^m$ such that $N_m \leqq N_{m-1}+1$, then Lemma 3.2 would imply that every nonzero probability $p_m(z)$ is a rational affine function of any other, i.e. $p_m(z) = a + bp_m(z')$ where a and b are rational numbers. It follows that for a given number p there exists at most a countable number of r such that p_2, given above, has such an extension. Thus "most" stationary measures on $\mathcal{A}^2$ do not have such extensions.

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