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A FIXED POINT THEOREM IN H-SPACE AND RELATED RESULTS

E. Tarafdar

The equivalence of a fixed point theorem and the Fan-Knaster-Kuratowski-Mazurkiewicz theorem in H-space has been established. The fixed point theorem is then applied to obtain a theorem on sets with H-convex sections, and also results on minimax inequalities.

Introduction

Using the results of Horvath [6] and [7], Bardaro and Ceppitelli [2] have recently proved a version of the Fan-Knaster-Kuratowski-Mazurkiewicz theorem [4] in H-spaces and also given some generalisations of Fan's well-known minimax inequalities.

In this note we have proved that their version is equivalent to a fixed point theorem of a set valued mapping. Our result extends the result of the author [8] to the H-space situation. This necessitates the introduction of the H-convex hull of a subset in an H-space. Our definition of a H-KKM map is slightly different from theirs, but more in line with the usual one in a vector space. From our fixed point theorem we have also deduced a theorem on sets with H-convex sections which generalises a theorem of Fan (Theorem 16, [4]), Browder [3] and the author [9]. Finally, we have shown that Bardaro and Ceppitelli's generalisations of Fan's minimax inequalities can also be deduced from our fixed point theorem.

Let X be a topological space and $\mathcal{F}(X)$ the family of finite nonempty subsets of X. Let $\{F_A\}$ be a given family of nonempty contractible subsets of X, indexed by $A \in \mathcal{F}(X)$ such that $F_A \subset F_{A'}$, whenever $A \subset A'$. The pair $(X, \{F_A\})$ is called an H-space. Given an H-space $(X, \{F_A\})$, a nonempty subset D of X is called

(i) H-convex if $F_A \subset D$ for each finite subset A of D;
(ii) weakly H-convex if $F_A \cap D$ is nonempty and contractible for each finite subset A of D and
(iii) compactly open (closed) if $D \cap B$ is open (closed) in B for each compact subset B of X. Also a subset K of X is called H-compact if, for every finite subset A of X, there exists a compact, weakly H-convex subset D of X such that $K \cup A \subset D$.

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In this paper by a finite subset we will always mean nonempty finite subset.

Let $(X, \{F_A\})$ be an H-convex space. Then given a nonempty subset K of X, we define the H-convex hull of K, denoted by $H - \operatorname{co} K$ as

$$H - \operatorname{co} K = \cap\{D \subset X : D \text{ is } H\text{-convex and } D \supset K\}.$$

$H - \operatorname{co} K$ is H-convex. Indeed if A is a finite subset of $H - \operatorname{co} K$, then for every H-convex subset D of X with $D \supset K$, we have $H - \operatorname{co} K \subset D$ and thus $A \subset D$. Hence as D is H-convex, $F_A \subset D$ and hence $F_A \subset H - \operatorname{co} K$. It also follows that $H - \operatorname{co} K$ is the smallest H-convex subset containing K.

In what follows, we will need the following characterisation of the convex hull.

Lemma 1. *Let $(X, \{F_A\})$ be an H-space and K be a nonempty subset of X. Then $H - \operatorname{co} K = \cup\{H - \operatorname{co} A\colon A$ is a finite subset of $K\}$.*

Proof: Let A be a finite subset of K. Then $H - \operatorname{co} A$ is the smallest H-convex subset containing A and $H - \operatorname{co} K$ is the smallest H-convex subset containing K. Thus it follows that $H - \operatorname{co} A \subset H - \operatorname{co} K$. Hence $\cup\{H - \operatorname{co} A\colon A$ is a finite subset of $K\} \subset H - \operatorname{co} K$.

Next, let $\cup\{H - \operatorname{co} A\colon A$ is a finite subset of $K\} = L$. Then L contains K as a subset and we prove that L is H-convex.

Let $B = \{x_1, x_2, \dots, x_n\}$ be a finite subset of L. Then there are finite subsets $A_1, A_2, \dots, A_n$ of K such that $x_i \in H - \operatorname{co} A_i$, $i = 1, 2, \dots, n$. Obviously $A' = \bigcup_{i=1}^{n} A_i$ is a finite subset of K, and $x_i \in H - \operatorname{co} A'$ for $i = 1, 2, \dots, n$. Therefore, as $H - \operatorname{co} A'$ is H-convex, $F_B \subset F_{A'} \subset H - \operatorname{co} A' \subset L$. Thus L is an H-convex subset containing K. Hence $H - \operatorname{co} K \subset \cup\{H - \operatorname{co} A\colon A$ is a finite subset of $K\}$. ∎

Let $\{(X_\alpha, \{F^\alpha_{A_\alpha}\}) : \alpha \in I\}$ be a family of H-spaces where I is a finite or infinite index set. Let $X = \prod_{\alpha \in F} X_\alpha$ be the product space with product topology and for each $\alpha \in I$, let $P_\alpha\colon X \to X_\alpha$ be the projection of X onto X_α. For any finite subset A of X, we set $F_A = \prod_{\alpha \in I} F_{A_\alpha}$ where $A_\alpha = P_\alpha(A)$ for each $\alpha \in I$.

Since for each $\alpha \in I$, F_{A_α} is contractible, it is easy to see that F_A is contractible. [To see this, let for each $\alpha \in I$, F_{A_α} be contractible to $x^0_\alpha \in X_\alpha$ through the homotopy $h_\alpha : A_\alpha \times [0, 1] \to A_\alpha$, that is h_α is continuous, $h_\alpha(x_\alpha, 1) = x_\alpha$ for all $x_\alpha \in A_\alpha$ and $h_\alpha(x_\alpha, 0) = x^0_\alpha$ for all $x_\alpha \in A_\alpha$. Then the mapping $h : A \times [0, 1] \to A$ defined by $h(x, t) = \prod_{\alpha \in I} h_\alpha(x_\alpha, t)$ is clearly a homotopy map and A is contractible to $\prod_\alpha x^0_\alpha \in X$ where $P_\alpha(x) = x_\alpha$]. Moreover if A and B are two finite subsets of X with $A \subset B$, then for each $\alpha \in I$, $P_\alpha(A) \subset P_\alpha(B)$, that is, $A_\alpha \subset B_\alpha$ and consequently $F_{A_\alpha} \subset F_{B_\alpha}$. Hence $F_A = \prod_{\alpha \in I} F_{A_\alpha} \subset \prod_{\alpha \in I} F_{B_\alpha} = F_B$. Thus $(X, \{F_A\})$ is an H-space.

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Now let D_α be an H-convex subset of X_α for each $\alpha \in I$; then $D = \prod_{\alpha\in I} D_\alpha$ is an H-convex subset of X. To see this let A be a finite subset of D. Then for each $\alpha \in I$, $A_\alpha = P_\alpha(A)$ is a finite subset of D_α and $F_{A_\alpha} \subset D_\alpha$ as D_α is H-convex. Hence $F_A = \prod_{\alpha\in I} F_{A_\alpha} \subset \prod_{\alpha\in I} D_\alpha = D$.

Then we have proved the following:

Lemma 2. *The product of any number of H-spaces is an H-space and the product of H-convex subsets is H-convex.* □

A set valued mapping $T\colon X \to 2^X$ is said to be H-KKM if for each finite subset A of X, $H - \operatorname{co} A \subset \bigcup_{x\in A} T(x)$.

We should point out that in [2] T is called H-KKM if for each finite subset A of X, $F_A \subset \bigcup_{x\in A} T(x)$. Thus if T is H-KKM in our sense, then T is H-KKM in the sense of [2].

The following theorem is proved by Bardaro and Ceppitelli [2].

Theorem 1. *Let $(X, \{F_A\})$ be an H-space and $T\colon X \to 2^X$ an H-KKM set valued mapping such that*

(a) *for $x \in X$, $T(x)$ is compactly closed;*
(b) *there is a compact subset L of X and an H-compact subset K of X such that for every weakly H-convex subset D with $K \subset D \subset X$, we have*
$$\bigcap_{x\in D} (T(x) \cap D) \subset L.$$
Then
$$\bigcap_{x\in X} T(x) \neq \emptyset.$$

In what follows we prove that this theorem is equivalent to the following fixed point theorem:

Theorem 2. *Let $(X, \{F_A\})$ be an H-space and $f\colon X \to 2^X$ be a set-valued mapping such that*

(i) *for each $x \in X$, $f(x)$ is non-empty and H-convex;*
(ii) *for each $y \in X$, $f^{-1}(y) = \{x \in X : y \in f(x)\}$ contains a compactly open subset O_y of X (O_y could be empty for some y);*
(iii) $\bigcup_{x\in X} O_x = X$; *and*
(iv) *there exists a compact subset L of X and an H-compact subset K of X such that for every weakly H-convex subset D with $K \subset D \subset X$, we*

have

$$\bigcap_{x\in D}(O_x^c\cap D)\subset L,$$

where O_x^c denotes the complement of O_x in X.

Then there is a point $x_0\in X$ such that $x_0\in f(x_0)$.

PROOF: We first prove that Theorem 1 implies Theorem 2. Let the conditions of Theorem 2 hold. For each $x\in X$, we set $T(x)=O_x^c$. If for each finite subset A of X, $H-\mathrm{co}\,A\subset\bigcup_{x\in A}T(x)$, then for each finite subset A of X, $F_A\subset\bigcup_{x\in A}T(x)$ as $H-\mathrm{co}\,A$ is an H-convex subset. Thus the set-valued mapping $T\colon X\to 2^X$ would satisfy all the conditions of Theorem 1 and hence $\bigcap_{x\in X}T(x)\neq\emptyset$ which would contradict the condition (iii). Hence there must exist at least one finite subset A of X such that $H-\mathrm{co}\,A\not\subset\bigcup_{x\in A}T(x)$, that is, there exists a point $y\in H-\mathrm{co}\,A$ such that $y\notin\bigcup_{x\in A}T(x)$, that is, $y\in[T(x)]^c$ for each $x\in A$, that is, $y\in O_x\subset f^{-1}(x)$ for each $x\in A$. Hence $x\in f(y)$ for each $x\in A$, that is, $A\subset f(y)$. But as $f(y)$ is H-convex, $H-\mathrm{co}\,A\subset f(y)$ which implies that $y\in f(y)$.

Next we prove that Theorem 2 implies Theorem 1. Assume that the conditions of Theorem 1 hold. If possible, suppose that $\bigcap_{x\in X}T(x)=\emptyset$. Then we can define a set-valued mapping $g\colon X\to 2^X$ by $g(y)=\{x\in X: y\notin T(x)\}$. Clearly $g(y)$ is a nonempty subset of X for each $y\in Y$. Also for each $x\in X$, $g^{-1}(x)=(T(x))^c=O_x$,say which is open subset of X. Let $f\colon X\to 2^X$ be the set-valued mapping defined by $f(y)=H-\mathrm{co}\,g(y)$ for each $y\in X$. Thus for each $y\in X$, $f(y)$ is an H-convex subset of X with $g(y)\subset f(y)$, and for each $x\in X$, $f^{-1}(x)\supset g^{-1}(x)=O_x$. Moreover, $\bigcap_{x\in X}T(x)=\emptyset$ implies $\bigcup_{x\in X}O_x=X$. Finally, $\bigcap_{x\in D}(O_x^c\cap D)=\bigcap_{x\in D}(T(x)\cap D)\subset L$. Hence the mapping f satisfies the conditions of the Theorem 2. Thus there exists a point $x_0\in X$ such that $x_0\in f(x_0)=H-\mathrm{co}\,g(x_0)$, that is, there is by Lemma 1 a finite subset $A=\{x_1,x_2,\dots,x_n\}$ of $g(x_0)$ such that $x_0\in H-\mathrm{co}\,A\subset f(x_0)$. But $x_i\in g(x_0)$, $i=1,2,\dots,n\Rightarrow x_0\notin T(x_i)$, $i=1,2,\dots,n$, that is, $x_0\notin\bigcup_{i=1}^{n}T(x_i)$, that is, $H-\mathrm{c_0}A\not\subset\bigcup_{x\in A}T(x)$ which contradicts that T is H-K.K.M. This proves our assertion. ∎

Our next theorem generalises a theorem of Fan (Theorem 16, [**4**]), Browder [**3**] and the author [**9**].

THEOREM 3. *Let $X_1,X_2,\dots,X_n$ be $n\geqslant 2$ H-spaces and let $X=\prod_{j=1}^{n}X_j$.*

Let $\{A_j\}_{j=1}^{n}$ and $\{B_j\}_{j=1}^{n}$ be two families of subsets of X having the following

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properties:

(a) *Let* $\widehat{X}_j = \prod_{i \neq j} X_i$ *and let* $\widehat{x}_j$ *denote a generic element of* $\widehat{X}_j$. *For each* $j = 1, 2, \ldots, n$ *and for each point* $\widehat{x}_j \in \widehat{X}_j$, *the set* $B_j(\widehat{x}_j) = \{x_j \in X_j : [x_j, \widehat{x}_j] \in B_j\}$ *is nonempty and the set* $A_j(\widehat{x}_j) = \{x_j \in X_j : [x_j, \widehat{x}_j] \in A_j\}$ *contains the* H*-convex hull of* $B_j(\widehat{x}_j)$.

(b) *For each* $j = 1, 2, \ldots, n$ *and for each point* $x_j \in X_j$, *the set* $B_j(x_j) = \{\widehat{x}_j \in \widehat{X}_j : [x_j, \widehat{x}_j] \in B_j\}$ *is compactly open in* $\widehat{X}_j$.

(c) *There exists an* H*-compact subset* X_0 *of* X *such that* $\bigcap_{x \in X_0} O_x^c$ *is compact where* $O_x = \bigcap_{j=1}^{n} \{B_j(x_j) \times X_j\}$ *and* x_j *is the projection of* x *into* X_j *for each* $j = 1, 2, \ldots, n$.

Then $\bigcap_{j=1}^{n} A_j \neq \emptyset$.

PROOF: We define two set-valued mappings $f : X \to 2^X$ and $g : X \to 2^X$ by $f(x) = \prod_{j=1}^{n} H\text{-co}\, B_j(\widehat{x}_j)$ and $g(x) = \prod_{j=1}^{n} B(\widehat{x}_j)$ for each $x = [x_j, \widehat{x}_j] \in X$ where x_j and $\widehat{x}_j$ are respectively the projections of x into X_j and $\widehat{X}_j$. Clearly for $x \in X$, by Lemma 2 $f(x)$ is H-convex, and by (a) $g(x) \neq \emptyset$ and $f(x) \supset g(x)$. For each $y \in X$, we consider the set $g^{-1}(y) = \{x \in X : y \in g(x)\}$. Now $x \in g^{-1}(y) \Leftrightarrow y = (y_1, y_2, \ldots, y_n) \in g(x) = \prod_{j=1}^{n} B_j(\widehat{x}_j) \Leftrightarrow y_j \in B_j(\widehat{x}_j)$ for each $j = 1, 2, \ldots, n \Leftrightarrow \widehat{x}_j \in B_j(y_j)$ for each $j = 1, 2, \ldots, n$. Thus for each $y \in X$, $g^{-1}(y) = \bigcap_{j=1}^{n} \{B_j(y_j) \times X_j\} = O_y$, which is compactly open. To show this it would suffice that $B_j(y_j) \times X_j$ is compactly open. Let K be a compact subset of X. Let $\widehat{P}_j(K) = \widehat{K}_j$ and $P_j(K) = K_j$ where $\widehat{P}_j$ and P_j are respectively the projections of X onto $\widehat{X}_j$ and X_j. Then $\widehat{K}_j$ and K_j are compact subsets of $\widehat{X}_j$ and X_j respectively and $(B_j(y_j) \times X_j) \cap \left(\widehat{K}_j \times K_j\right) = \left(B_j(y_j) \cap \widehat{K}_j\right) \times K_j$. This shows that $(B_j(y_j) \times X_j)$ is open in $\widehat{K}_j \times K_j$ by virtue of (b). Now since $\widehat{K}_j \times K_j \subset K$, it follows that $B_j(y_j) \times X_j$ is open in K. Now since $g(x) \subset f(x)$ for each $x \in x$, it follows that for each $y \in X$, $f^{-1}(y)$ contains a compactly open subset $g^{-1}(y) = O_y$. Furthermore $\bigcup_{y \in X} O_y = X$. [For let $x \in X$. Since $g(x) \neq \emptyset$, $g(x)$ contains a point $y \in X$. Thus $x \in g^{-1}(y) = O_y$]. Finally by (e) there exists an H-compact subset X_0 of X such that $\bigcap_{x \in X_0} O_x^c = L$ is compact. Clearly with this pair (X_0, L) the condition (iv) of Theorem 2 is satisfied. Thus by Theorem 2 there exists a point $x \in X$ such that

$$x \in f(x) = \prod_{j=1}^{n} H-\operatorname{co} B(\widehat{x}_j) \subset \prod_{j=1}^{n} A_j(\widehat{x}_j)$$

by (a), that is, $x_j \in A_j(\widehat{x}_j)$ for $j = 1, 2, \ldots, n$, that is $[x_j, \widehat{x}_j] \in A_j$ for $j = 1, 2, \ldots, n$. Thus $x \in \bigcap_{j=1}^{n} A_j$. ∎

REMARK. The theorem dual, in the sense of [**11**], to the above theorem can similarly be stated and proved.

Bardaro and Ceppitelli [**2**] proved some generalisations of Fan's minimax inequalities in Riesz space. We prove a variant of one of these (Theorem 3, [**2**]) by means of our Theorem 2.

Let (E, C) be a Riesz space, where C is the positive cone, provided with a linear, order compatible topology (for example, see [**5**]) and C, the interior of C is assumed to be nonempty.

THEOREM 4. *Let $(X, \{F_A\})$ be an H-space and $f, g\colon X \times X \to (E, C)$ two functions such that with a given $\lambda \in E$ the following conditions hold:*

(a) *$g(x, y) \leqslant f(x, y)$ for all $x, y \in X$;*

(b) *$f(x, x) \notin \lambda + \overset{\circ}{C}$ for all $x \in X$;*

(c) *for every $y \in X$, the set $\{x \in X : f(x, y) \in \lambda + \overset{\circ}{C}\}$ is H-convex;*

(d) *for every $x \in X$, the set $\{y \in X : g(x, y) \in \lambda + \overset{\circ}{C}\}$ is compactly open;*

(e) *there exists an H-compact subset X_0 of X such that $\{y \in X : g(x, y) \notin \lambda + \overset{\circ}{C}$, for each $x \in X_0\}$ is a compact subset of X.*

Then the set $S = \{y\colon g(x, y) \notin \lambda + \overset{\circ}{C}$ for all $x \in X\}$ is a nonempty compactly closed subset of X.

PROOF: For each $x \in X$, let $F(x) = \{y \in X : f(x, y) \notin \lambda + \overset{\circ}{C}\}$ and $G(x) = \{y \in X\colon g(x, y) \notin \lambda + \overset{\circ}{C}\}$. Then by (d), for each $x \in X$, $G(x)$ is compactly closed. It is clear that $S = \bigcap_{x \in X} G(x)$ and S is compactly closed. So we need to show that $S \neq \emptyset$. If possible, let $S = \emptyset$. Then for each $y \in X$, the set $h(y) = \{x \in X : y \notin G(x)\} = \{x \in X : g(x, y) \in \lambda + \overset{\circ}{C}\}$ is non-empty. Hence for each $y \in X$, the set

$$k(y) = \{x \in X : f(x, y) \in \lambda + \overset{\circ}{C}\} \supset h(y) = \{x \in X : g(x, y) \in \lambda + \overset{\circ}{C}\}.$$

The last inclusion follows from the inclusion $G(x)^c \subset F(x)^c$ which in turn follows from (b). [To see this let $y \notin G(x)$, that is, $g(x, y) \in \lambda + \overset{\circ}{C}$. Then there is a neighbourhood

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V of O in E such that $g(x,y)+V \subset \lambda+\overset{\circ}{C}$. Now $g(x,y) \leqslant f(x,y) \Rightarrow \lambda < g(x,y)+v \leqslant f(x,y)+v$ for each $v \in V$. Thus $f(x,y)+V \subset \lambda+\overset{\circ}{C}$, that is $y \notin F(x)$]. Now for each $x \in X$,

$$h^{-1}(x) = \{y \in X : x \in h(y)\} = \{y \in X : g(x,y) \in \lambda + \overset{\circ}{C}\} = O_x,$$

say, is compactly open by (d). Thus for the set-valued mapping $k\colon X \to 2^X$, $k(y)$ is nonempty and H-convex (by (c)) and for each $x \in X$, $k^{-1}(x)$ contains a compactly open subset $O_x = h^{-1}(x)$. [That $h^{-1}(x) \subset k^{-1}(x)$ follows from the fact that $h(x) \subset k(x)$]. Also $\bigcup_{x\in X} h^{-1}(x) = \bigcup_{x\in X} O_x = X$. [To see this let $y \in X$. Since $h(y) \neq \emptyset$, we can assume $x \in h(y)$. Then $y \in h^{-1}(x) = O_x$]. Finally

$$\text{(e)} \Rightarrow \bigcap_{x\in X_0} O_x^c = \bigcap_{x\in X_0} \left(h^{-1}(x)\right)^c = \bigcap_{x\in X_0} \{y \in X : g(x,y) \notin \lambda + \overset{\circ}{C}\} = L,$$

say, is compact. Thus the pair (L, X_0) satisfies the condition (iv) of Theorem 2 for the mapping k. Hence this mapping $k\colon X \to 2^X$ fulfils all the conditions of Theorem 2 and, therefore, there is a point $x_0 \in X$ such that $x_0 \in k(x_0)$, that is, $f(x_0, x_0) \in \lambda+\overset{\circ}{C}$ which contradicts (b). Thus we have proved the theorem. ∎

REMARKS. In the same way we can deduce the Theorem 4 and Corollary 1 of [2] from our Theorem 2. The Theorem 4 here includes a theorem of Allen [1] and also of Tarafdar [10].

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Department of Mathematics
The University of Queensland
Queensland 4072 Australia