ON A PROBLEM OF RANKIN ABOUT THE EPSTEIN ZETA-FUNCTION

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1. Introduction. Let

$$h(m, n) = \alpha m^2 + 2\chi mn + \beta n^2$$

be a positive definite quadratic form with determinant $\alpha\beta - \chi^2 = 1$. A special form of this kind is

$$Q(m, n) = 2 \,.\, 3^{-\frac{1}{2}}(m^2 + mn + n^2).$$

We consider the Epstein zeta-function

$$Z_h(s) = \sum_{\substack{m,\, n \text{ integers} \\ \text{not } m = n = 0}} \{h(m,n)\}^{-s},$$

the series converging for $s > 1$. For $s \geqslant 1{\cdot}035$ Rankin [1] proved the following

STATEMENT R.

$$Z_h(s) - Z_Q(s) \geqslant 0 \quad \text{...(1)}$$

The sign of equality is needed only when h is equivalent to Q.

When s is large, this statement suggests itself, since $Z_h(s)$ is dominated by those integer pairs (m, n) for which $h(m, n)$ is smallest, and the forms equivalent to $Q(m, n)$ are well known to be precisely the unimodular forms h for which

$$\min_{(m,n) \neq (0,0)} h(m, n)$$

is greatest. It is perhaps rather surprising that the statement R continues to hold so far as $s = 1{\cdot}035$, and Rankin asked if it continued to hold up to $s = 1$. In this note we shall show that this is the case and indeed rather more. The function $Z_h(s)$ may be analytically continued over the whole plane. Its only singularity is at $s = 1$, where it has a pole with residue π. We shall prove the following theorem:

THEOREM. *The statement* R *holds for all* $s \geqslant 0$.

We note that the statement R is meaningful even for $s = 1$, since $Z_h(s) - Z_Q(s)$ is regular there. This case has indeed a special interest since it is connected with the Kronecker Limit Theorem which plays a part in the old-fashioned treatment of modular functions (cf. Weber [3]; for an interesting application see Kronecker [6]). We shall, however, assume that $s \neq 1$ and leave to the reader the trivial modifications required to deal with $s = 1$.

For $s \leqslant 0$ it is easy to see how the statement R should be modified, since $Z_h(s)$ satisfies the functional equation.

$$\pi^{-s}\Gamma(s)Z_h(s) = \pi^{s-1}\Gamma(1-s)Z_h(1-s) \quad \text{..............................(2)}$$

(cf. Deuring [3]).

Our proof is a slight modification of Rankin's but we give incidentally a simplification in part of the range considered by him. When $s \geqslant 3$, Rankin gave an elementary proof on

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quite different lines from his proof for $1{\cdot}035 \leqslant s \leqslant 3$. As our proof here does not work, at least without modification, for large s, we shall consider only the case

$$0 \leqslant s \leqslant 3. \quad \text{(3)}$$

I am grateful to Professor Rankin for suggesting improvements and corrections to the first draft.

2. Preliminaries. Since $h(m, n)$ has unit determinant, it may be put in the shape

$$h(m, n) = y^{-1}\{(m+nx)^2 + n^2y^2\}$$

with $y > 0$. We write

$$Z_h(s) = G(x, y)(s),$$

and omit the (s) if it does not cause confusion. Put

$$z = x + iy.$$

Then, for fixed s, the function $G(x, y)$ is invariant under the substitutions of the modular group acting on z: it is not a modular function of z in the usual sense since it is not analytic. On developing $G(x, y)$ as a Fourier expansion for x, one obtains for $s > 1$ the expansion

$$G(x, y) = 2y^s\zeta(2s) + 2y^{1-s}\zeta(2s-1)\Gamma(\tfrac{1}{2})\Gamma(s-\tfrac{1}{2})/\Gamma(s)$$
$$+\frac{8\pi^s y^{\frac{1}{2}}}{\Gamma(s)} \sum_{r>0} r^{s-\frac{1}{2}}\, \sigma_{1-2s}(r)\, K_{s-\frac{1}{2}}(2\pi ry) \cos 2\pi rx, \quad \text{(4)}$$

where

$$\sigma_k(n) = \sum_{d|n} d^k$$

and

$$K_\nu(u) = \int_0^\infty e^{-u\cosh t} \cosh \nu t \, dt \quad (5_1)$$
$$= \frac{\Gamma(\nu+\frac{1}{2})2^\nu}{u^\nu \Gamma(\frac{1}{2})} \int_0^\infty \frac{\cos xu}{(x^2+1)^{\nu+\frac{1}{2}}}\, dx \quad (5_2)$$

is a Bessel function [cf. Rankin's paper, and Watson [4, § 6.3] for the equality of the two integrals for $K_\nu(u)$. The second, which is valid only when $\nu > \frac{1}{2}$, is the one which naturally arises in the development of $G(x, y)$ in a Fourier series. The first integral, which is valid for all ν provided that $\Re u > 0$, is the one which will be used in the sequel, as it was by Rankin.]

On applying the functional equation for the Riemann ζ-function to the second term, one obtains

$$\tfrac{1}{2}\Gamma(s)\pi^{-s} G(x, y) = \phi(s) + \phi(1-s) + 4y^{\frac{1}{2}} \sum_{r \geqslant 1} r^{s-\frac{1}{2}}\, \sigma_{1-2s}(r)\, K_{s-\frac{1}{2}}(2\pi ry) \cos 2\pi rx, \quad \text{(6)}$$

where

$$\phi(s) = (y/\pi)^s \Gamma(s)\zeta(2s). \quad \text{(6')}$$

This gives us the continuation of $G(x, y)(s)$ to the whole s-plane. Incidentally, since $K_\nu(u) = K_{-\nu}(u)$, it also gives the functional equation (2).

3. Outline of Proof. In §§ 4, 5 we shall prove the following two lemmas about the partial derivatives of $G(x, y)$ with respect to x and y.

LEMMA 1. $G_y(x, y) > 0$ *for* $y \geqslant \frac{3}{2}$.

LEMMA 2. $G_x(x, y) < 0$ *for* $y \geqslant \frac{3}{5}$ *and* $0 < x < \frac{1}{2}$.

Both of these lemmas play a part in Rankin's paper for one of the ranges $(1{\cdot}035 \leqslant s \leqslant 2)$ considered by him.

For the sake of completeness we reproduce Rankin's argument showing that Statement R follows from Lemmas 1 and 2.

When the form $h(m, n)$ is reduced, (x, y) lies in the modular region

$$D: \qquad 0 \leqslant x \leqslant \tfrac{1}{2}, \quad y > 0, \quad x^2 + y^2 \geqslant 1.$$

Since $G(x, y)$ is a continuous function, it must, by Lemma 1, attain its minimum at some point $(x', y') \in D$ with $y' < \frac{3}{2}$. By Lemma 2, we must have $x' = \frac{1}{2}$. But now

$$G(x', y') = G(x'', y''),$$

where

$$x'' + iy'' = \frac{1}{1 - (x' + iy')} = \frac{2}{4y'^2 + 1} + i\,\frac{4y'}{4y'^2 + 1},$$

so that

$$0 < x'' \leqslant \tfrac{1}{2}, \quad y'' \geqslant \tfrac{3}{5},$$

since $3^{1/2}/2 \leqslant y' \leqslant \frac{3}{2}$. By Lemma 2, we must have $x'' = \frac{1}{2}$. Hence $y' = 3^{1/2}/2$. That is, in the modular region D the function $G(x, y)$ attains its minimum at $x = \frac{1}{2}$, $y = 3^{1/2}/2$, and only there. This is just statement R.

In the rest of this note we shall prove Lemmas 1 and 2 by differentiating the identity of § 2, and estimating the resulting expressions.

4. Proof of Lemma 2. On differentiating the identity (4) for $G(x, y)$ term by term we obtain

$$G_x(x, y) = -\frac{16\pi^{s+1} y^{\frac{1}{2}}}{\Gamma(s)} \Lambda,$$

where we have written

$$\Lambda = \sum_{r \geqslant 1} r^{s+\frac{1}{2}} \sigma_{1-2s}(r) K_{s-\frac{1}{2}}(2\pi r y) \sin 2\pi r x.$$

On substituting the integral (5_1) for $K_{s-\frac{1}{2}}(2\pi r y)$ and interchanging summation and integration we obtain

$$\Lambda = \int_0^\infty \psi(\delta_t) \cosh\,(s - \tfrac{1}{2})t\, dt,$$

where

$$\delta_t = e^{-2\pi y \cosh^2 t}$$

and

$$\psi = \psi(\delta) = \sum_{r \geqslant 1} r^{s+\frac{1}{2}} \sigma_{1-2s}(r) \delta^r \sin 2\pi r x. \quad \text{(7)}$$

We note that

$$\delta_t \leqslant e^{-2\pi y} \leqslant e^{-6\pi/5} < 40^{-1}, \quad \text{(8)}$$

since $y \geqslant 3/5$. Hence it will be enough to show that

$$\psi(\delta) > 0$$

whenever

$$\left.\begin{array}{l} 0 < x < \frac{1}{2}, \\ 0 < \delta < 40^{-1}, \\ 0 \leqslant s \leqslant 3. \end{array}\right\} \quad \text{(9)}$$

In (7) we have

$$\sigma_{1-2s}(r) = \sum_{d|r} d^{1-2s}.$$

Put $r = df$ and change the order of summation in (7). Then we have

$$\psi = \sum_{d \geqslant 1} d^{\frac{3}{2}-s} \omega_d,$$

where

$$\omega_d = \sum_{f \geqslant 1} f^{s+\frac{1}{2}} \delta^{df} \sin 2\pi d f x. \quad \text{(10)}$$

We now obtain various estimates for ω_d. In the first place, quite trivially,

$$|\omega_d| \leqslant \sum_{f \geqslant 1} f^{s+\frac{1}{2}} \delta^{df} \leqslant \sum_{f \geqslant 1} f^4 \delta^{df}$$

$$\leqslant \frac{\delta^d}{(1-\delta^d)^{16}}, \quad \text{(10')}$$

the last inequality holding because the expansion of the last line majorizes the previous line.

On applying partial summation following Rankin, one also obtains

$$4 \sin^2(\pi d x) \omega_d = \sum_{f \geqslant 1} g_f \{(f+1) \sin 2\pi d x - \sin 2\pi (f+1) d x\}, \quad \text{(11)}$$

where

$$\begin{aligned} g_f &= f^{s+\frac{1}{2}} \delta^{df} - 2(f+1)^{s+\frac{1}{2}} \delta^{d(f+1)} + (f+2)^{s+\frac{1}{2}} \delta^{d(f+2)} \quad \text{(12)} \\ &\geqslant f^{s+\frac{1}{2}} \delta^{df} \{1 - 2[(f+1)/f]^{s+\frac{1}{2}} \delta^d\} \\ &\geqslant f^{s+\frac{1}{2}} \delta^{df} (1 - 2^5 \delta^d) \\ &> 0, \end{aligned}$$

by (9). We deduce from (11) that

$$\omega_d > 0$$

for all d such that

$$0 < dx < \tfrac{1}{2}. \quad \text{(13)}$$

By hypothesis, (13) is true with $d = 1$. Since $x > 0$, there is a greatest d, say d_0, such that (13) holds, so that

$$\tfrac{1}{4} \leqslant d_0 x < \tfrac{1}{2}.$$

Then, by (11),

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$$\begin{aligned} 4\omega_{d_0} &\geqslant 4\sin^2(\pi d_0 x)\omega_{d_0} \\ &= \sum_{f\geqslant 1} g_f\{(f+1)\sin 2\pi d_0 x - \sin 2\pi(f+1)d_0 x\} \\ &\geqslant \sum_{f\geqslant 1} g_f\{(f+1)2^{-\frac{1}{2}} - 1\} \\ &= (2^{\frac{1}{2}}-1)\delta^{d_0} + (1-2^{-\frac{1}{2}})2^{s+\frac{1}{2}}\delta^{2d_0}, \end{aligned}$$

on substituting the values (12) for g_f and arranging in powers of δ. Hence

$$\omega_{d_0} \geqslant \tfrac{1}{4}(2^{\frac{1}{2}}-1)\delta^{d_0}.$$

Since $\omega_d > 0$ for $d < d_0$, we deduce that

$$\psi \geqslant \sum_{d\geqslant d_0} d^{\frac{3}{2}-s}\omega_d.$$

Hence

$$d_0^{s-\frac{3}{2}}\delta^{-d_0}\psi \geqslant \tfrac{1}{4}(2^{\frac{1}{2}}-1) - \sum_{d>d_0}\left(\frac{d}{d_0}\right)^{\frac{3}{2}-s}\frac{\delta^{d-d_0}}{(1-\delta^d)^{16}}. \quad \text{......................(14)}$$

Here

$$\left(\frac{d}{d_0}\right)^{\frac{3}{2}-s} \leqslant \left(\frac{d}{d_0}\right)^{\frac{3}{2}} \leqslant (d-d_0+1)^{\frac{3}{2}} \leqslant 2^{\frac{1}{2}}(d-d_0)(d-d_0+1)$$

and

$$(1-\delta^d)^{16} \geqslant (1-\delta^2)^{16}.$$

On substituting these estimates in (14) we obtain

$$\begin{aligned} d_0^{s-\frac{3}{2}}\delta^{-d_0}\psi &\geqslant \tfrac{1}{4}(2^{\frac{1}{2}}-1) - 2^{\frac{3}{2}}(1-\delta^2)^{-16}\sum_{k\geqslant 1}\tfrac{1}{2}k(k+1)\delta^k \\ &= \tfrac{1}{4}(2^{\frac{1}{2}}-1) - 2^{\frac{3}{2}}(1-\delta^2)^{-16}(1-\delta)^{-3}\delta \\ &> 0, \end{aligned}$$

since $\delta < 40^{-1}$. This concludes the proof that $\psi > 0$ and so of Lemma 2.

5. Proof of Lemma 1. This lemma was already proved simply by Rankin for all $s > 1$ (his Lemma 7). His proof does not naturally extend to $s \leqslant 1$. We may thus confine ourselves to the range

$$0 \leqslant s \leqslant 1. \quad \text{..(15)}$$

However, it would probably not be difficult to extend our proof to all $s \geqslant 0$.

On differentiating the identity (6) of § 2 term by term with respect to $\log y$ we obtain

$$\tfrac{1}{2}y\Gamma(s)\pi^{-s}G_y(x,y) = \theta(s) + \theta(1-s) + 2M, \quad \text{..........................(16)}$$

where

$$\theta(s) = s(y/\pi)^s\Gamma(s)\xi(2s) \quad \text{.......................................(17)}$$

and

$$\begin{aligned} M = {} & y^{\frac{1}{2}}\sum_{r\geqslant 1} r^{s-\frac{1}{2}}\sigma_{1-2s}(r)K_{s-\frac{1}{2}}(2\pi r y)\cos 2\pi r x \\ & + 4\pi y^{\frac{3}{2}}\sum_{r\geqslant 1} r^{s+\frac{1}{2}}\sigma_{1-2s}(r)K_{s-\frac{1}{2}}(2\pi r y)\cos 2\pi r x. \quad \text{.....................(18)} \end{aligned}$$

We shall show that $G_y(x, y) > 0$ by showing that $\theta(s) + \theta(1-s)$ is fairly large and M is fairly small in the range

$$0 \leqslant s \leqslant 1, \quad y \geqslant \tfrac{3}{2} \quad \text{...(19)}$$

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under consideration. Most of the time we can estimate quite crudely.

We consider first $\theta(s)$ and write

$$\eta = \frac{y}{\pi} \geqslant \frac{3}{2\pi}. \qquad \text{(20)}$$

Since $\theta(s)$ has a pole at $s = \frac{1}{2}$ with residue $\frac{1}{2}\eta^{\frac{1}{2}}\Gamma(\frac{1}{2}) = \frac{1}{2}(\eta\pi)^{\frac{1}{2}}$, it is convenient to treat

$$\theta^*(s) = \theta(s) - \frac{(\eta\pi)^{\frac{1}{2}}}{2(2s-1)}$$

$$= \eta^s\Gamma(s+1)\zeta(2s) - [(\eta\pi)^{\frac{1}{2}}/2(2s-1)]. \qquad \text{(21)}$$

Clearly

$$\theta^*(s) + \theta^*(1-s) = \theta(s) + \theta(1-s).$$

It is probably well-known that

$$\zeta(t) \geqslant \frac{1}{t-1} + \frac{1}{2} = \frac{t+1}{2(t-1)} \qquad \text{(22)}$$

for $t \geqslant 0$. [For the identity

$$\zeta(t) = \frac{1}{t-1} + \tfrac{1}{2} + t\sum_{n>0}\int_0^{\frac{1}{2}}\left\{\frac{1}{(n+\frac{1}{2}-u)^{t+1}} - \frac{1}{(n+\frac{1}{2}+u)^{t+1}}\right\}u\,du,$$

which is an immediate consequence of Euler's summation formula when $\Re\, t > 1$, continues to hold by analytic continuation when $\Re\, t \geqslant 0$]. By (21) and (22),

$$\theta^*(s) \geqslant \frac{(2s+1)\eta^s\Gamma(s+1) - (\eta\pi)^{\frac{1}{2}}}{2(2s-1)}.$$

We may now apply the mean-value theorem to

$$f(s) = (2s+1)\eta^s\Gamma(s+1), \qquad \text{(23)}$$

since

$$f(\tfrac{1}{2}) = (\eta\pi)^{\frac{1}{2}}.$$

Hence

$$\inf_{0\leqslant s\leqslant 1} \theta^*(s) \geqslant \tfrac{1}{2} \inf_{0\leqslant t\leqslant 1} f'(t). \qquad \text{(24)}$$

Now

$$\frac{f'(t)}{f(t)} = \frac{2}{2t+1} + \frac{\Gamma'}{\Gamma}(t+1) + \log\eta. \qquad \text{(25)}$$

From the tables of Γ'/Γ (e.g. in Jahnke and Emde [5]) one readily sees that

$$\frac{2}{2[(r+1)/10]+1} + \frac{\Gamma'}{\Gamma}\left(\frac{r}{10}+1\right) \geqslant 0{\cdot}9$$

for $r = 0, 1, 2, \ldots, 9$ and so, by the monotonicity of $2/(2t+1)$ and $\Gamma'(t+1)/\Gamma(t+1)$, we have

$$\frac{2}{2t+1} + \frac{\Gamma'}{\Gamma}(t+1) \geqslant 0{\cdot}9 \quad (0 \leqslant t \leqslant 1). \qquad \text{(26)}$$

Further,

$$\log\eta \geqslant \log\frac{3}{2\pi} \geqslant -0{\cdot}75.$$

Hence

$$f'(t) \geqslant 0{\cdot}15 f(t) \quad (0 \leqslant t \leqslant 1). \qquad \text{(27)}$$

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Further, $\Gamma(t+1) \geqslant 0{\cdot}8$ for $0 \leqslant t \leqslant 1$, and so

$$f(t) \geqslant (0{\cdot}8)(2t+1)\eta^t$$
$$\geqslant (0{\cdot}8)(2t+1)\eta_0^t, \quad \text{......(28)}$$

where

$$\eta_0 = \frac{3}{2\pi}.$$

Now $\log\{(2t+1)\eta_0^t\}$ is convex in $0 \leqslant t \leqslant 1$ and takes the values 0 and $\log 3\eta_0 > 0$ at the two ends of the range. Hence, by (28),

$$f(t) \geqslant 0{\cdot}8. \quad \text{......(29)}$$

To sum up, from (24), (27), (29) we have

$$\theta^*(s) \geqslant \tfrac{1}{4}(0{\cdot}15)(0{\cdot}8) = 0{\cdot}03 \quad (0 \leqslant s \leqslant 1). \quad \text{......(30)}$$

[From the signs of the coefficients of η^s and $\eta^{\frac{1}{2}}$ in (21), it is clear that for fixed s in $0 \leqslant s \leqslant 1$ the function $\theta^*(s)$ increases when y increases, provided that it is positive, so it would have been enough to consider $y = \frac{3}{2}$. The numerical evidence suggests that then $\theta^*(s)$ increases in $0 \leqslant s \leqslant 1$. If so, the 0·03 in (30) could be replaced by the value of $\theta^*(0)$ when $y = \frac{3}{2}$, namely $\frac{1}{2}(\frac{3}{2})^{\frac{1}{2}} - \frac{1}{2} \risingdotseq 0{\cdot}1124$. But the inequality (30) is much more than we in fact need.]

We can now estimate $|M|$ using the techniques of § 3 but more crudely. For $|\nu| \leqslant 1$ we have

$$0 \leqslant K_\nu(u) = \int_0^\infty e^{-u\cosh t}\cosh \nu t\, dt$$
$$\leqslant \int_0^\infty e^{-u\cosh t}\cosh t \cosh \nu t\, dt$$
$$= -K_\nu'(u)$$
$$\leqslant \int_0^\infty e^{-u\cosh t}\cosh^2 t\, dt. \quad \text{......(31)}$$

On applying these inequalities to M and observing that

$$y^{\frac{1}{2}} \leqslant y^{\frac{3}{2}}, \quad r^{s-\frac{1}{2}} \leqslant r^{s+\frac{1}{2}}, \quad |\cos 2\pi\nu x| \leqslant 1,$$

we obtain

$$|M| \leqslant (4\pi+1)y^{\frac{3}{2}} \int_0^\infty \Psi(\delta_t)\cosh^2 t\, dt, \quad \text{......(32)}$$

where

$$\delta_t = e^{-2\pi y\cosh t} \leqslant e^{-2\pi y} \leqslant e^{-3\pi} \leqslant 10^{-4}, \quad \text{......(33)}$$

and $\Psi(\delta)$ is defined by replacing $\sin 2\pi rx$ by 1 on the right-hand side of (7) in § 4. But now as in § 4, we have

$$|\Psi(\delta)| = \Big|\sum_{d\geqslant 1} d^{\frac{3}{2}-s}\Omega_d\Big| \leqslant \sum_{d\geqslant 1} d^{\frac{3}{2}}|\Omega_d|, \quad \text{......(33')}$$

where Ω_d is defined by replacing $\sin 2\pi rx$ by 1 on the right-hand side of (10). The estimate (10′) holds with Ω_d instead of ω_d. Hence by (33) and (33′),

$$|\Psi(\delta)| \leqslant \sum_{d\geqslant 1} d^{\frac{3}{2}}\delta^d(1-\delta^d)^{-16} \leqslant (1-\delta)^{-20}\,\delta$$
$$\leqslant (1{\cdot}1)\delta. \quad \text{......(34)}$$

From (32) and (34), we have

$$| M | \leqslant (4\pi + 1)(1{\cdot}1) y^{\frac{3}{2}} e^{-2\pi y} I, \qquad \text{.................................(35)}$$

where

$$I = \int_0^\infty e^{-2\pi y (\cosh t - 1)} \cosh^2 t \, dt. \qquad \text{.................................(36)}$$

On making the substitution $v = \cosh t$ and observing that

$$e^{-2\pi y(v-1)} \leqslant v^{-2\pi y} \leqslant v^{-9},$$

one readily sees that

$$I \leqslant \int_1^\infty \frac{v^{-7}}{(v^2-1)^{\frac{1}{2}}} \, dv \leqslant 1. \qquad \text{.................................(37)}$$

From (35) and (37) we have

$$\begin{aligned} | M | &\leqslant (4\pi + 1)(1{\cdot}1) y^{\frac{3}{2}} e^{-2\pi y} \\ &< 0{\cdot}005, \qquad \text{.................................(38)} \end{aligned}$$

since $y \geqslant \frac{3}{2}$. Thus finally, by (30) and (38),

$$\begin{aligned} \tfrac{1}{2} y \Gamma(s) \pi^{-s} G_y(x, y) &= \theta^*(s) + \theta^*(1-s) + 2M \\ &\geqslant 0{\cdot}03 + 0{\cdot}03 - 2(0{\cdot}005) \\ &> 0. \end{aligned}$$

This concludes the proof of Lemma 1 and so of the theorem.

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