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Whittaker Functions on Real Semisimple Lie Groups of Rank Two

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Abstract. We give explicit formulas for Whittaker functions on real semisimple Lie groups of real rank two belonging to the class one principal series representations. By using these formulas we compute certain archimedean zeta integrals.

Introduction

Whittaker functions on a real semisimple Lie group G play important roles in the study of automorhpic forms since they appear in Fourier expansions of automorphic forms. Motivated by number theoretic applications such as analysis of zeta integrals, explicit formulas for Whittaker functions have been studied by many authors, especially on the groups $\mathrm{SL}_n(\mathbb{R})$ and $\mathrm{Sp}_2(\mathbb{R})$.

In the recent progress on the study for higher rank cases, some relations between Whittaker functions for different Lie groups have been discovered. As for Whittaker functions for the class one principal series, recursive relations with respect to the real ranks are given in [12] ($G = \mathrm{SL}_n(\mathbb{R})$) and [10] ($G = \mathrm{SO}_{n+1,n}(\mathbb{R})$). In the paper [8], Whittaker functions for a generalized principal series of $\mathrm{Sp}_3(\mathbb{R})$ are described in terms of those for the class one principal series of $\mathrm{SO}_{3,2}(\mathbb{R})$. Some of these phenomena might be related to lifting of automorphic forms, and we think many relations still remain unknown.

The aim of this paper is to connect Whittaker functions for the class one principal series when the real rank of G is two. We express Whittaker functions on $\mathrm{Sp}_2(\mathbb{R})$ by using Whittaker functions on $\mathrm{SL}_3(\mathbb{R})$ and $\mathrm{SL}_2(\mathbb{R})$, or two Whittaker functions on $\mathrm{SO}_{2,2}(\mathbb{R})$. The latter result is essentially obtained by Niwa [14] (see also [11]) by considering the theta lifting from $\mathrm{SO}_{2,2}$ to Sp_2. We give explicit formulas for an exceptional group of type G_2, written in terms of two Whittaker functions on $\mathrm{SL}_3(\mathbb{R})$.

As in recent works [8, 10], our proof relies on the expansion formulas for Whittaker functions studied by Hashizume [7], that is, linear relations between the Jacquet integral representations of Whittaker functions (class one Whittaker functions) and the power series solutions (fundamental Whittaker functions) of the system of partial differential equations characterizing the Whittaker functions.

In the final section, we apply our formulas for $\mathrm{Sp}_2(\mathbb{R})$ to compute two kinds of archimedean zeta integrals. We discuss automophic L-functions on $\mathrm{GSp}_2 \times \mathrm{GL}_3$ with degree 12, and two fundamental L-functions on GSp_2. We show that the archimedean zeta integrals coincide with the appropriate Langlands L-factors.

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1 Preliminaries

We first recall the notion of Whittaker functions for class one principal series representations. Let G be a real semisimple Lie group with finite center and $\mathfrak{g}$ its Lie algebra. Fix a maximal compact subgroup K of G and put $\mathfrak{k} = \mathrm{Lie}(K)$. Let $\mathfrak{p}$ be the orthogonal complement of $\mathfrak{k}$ in $\mathfrak{g}$ and θ the corresponding Cartan involution. For a maximal abelian subalgebra $\mathfrak{a}$ of $\mathfrak{p}$ and a linear form α on $\mathfrak{a}$, put

$$\mathfrak{g}_\alpha = \{X \in \mathfrak{g} \mid [H, X] = \alpha(H)X \text{ for all } H \in \mathfrak{a}\}.$$

We denote by $\Delta = \Delta(\mathfrak{g}, \mathfrak{a}) = \{\alpha \in \mathfrak{a}^* \mid \mathfrak{g}_\alpha \neq 0\}$ the restricted root system. Let Δ^+ be a positive system in Δ and Π the set of the corresponding simple roots.

In the case of $G = \mathrm{SL}_3(\mathbb{R})$, $\mathrm{Sp}_2(\mathbb{R})$ and $G_2(\mathbb{R})$, we can take positive systems by

$$\Delta^+ = \begin{cases} \{\alpha, \beta, \alpha+\beta\} & \text{if } G = \mathrm{SL}_3(\mathbb{R}), \\ \{\alpha, \beta, \alpha+\beta, 2\alpha+\beta\} & \text{if } G = \mathrm{Sp}_2(\mathbb{R}), \\ \{\alpha, \beta, \alpha+\beta, 2\alpha+\beta, 3\alpha+\beta, 3\alpha+2\beta\} & \text{if } G = G_2(\mathbb{R}), \end{cases}$$

with the simple systems $\Pi = \{\alpha, \beta\}$ satisfying the following:

$$\Big(\frac{\langle\beta,\beta\rangle}{\langle\alpha,\alpha\rangle}, \frac{\langle\alpha,\beta\rangle}{\langle\alpha,\alpha\rangle}\Big) = \begin{cases} (1, -1/2) & \text{if } G = \mathrm{SL}_3(\mathbb{R}), \\ (2, -1) & \text{if } G = \mathrm{Sp}_2(\mathbb{R}), \\ (3, -3/2) & \text{if } G = G_2(\mathbb{R}). \end{cases}$$

Here $\langle\cdot\,,\cdot\rangle$ is the inner product on $\mathfrak{a}_\mathbb{C}^* = \mathfrak{a}^* \otimes_\mathbb{R} \mathbb{C}$ induced by the Killing form $B(\cdot\,,\cdot)$ on $\mathfrak{g}$.

If we put $\mathfrak{n} = \sum_{\alpha\in\Delta^+} \mathfrak{g}_\alpha$, then we have an Iwasawa decomposition $\mathfrak{g} = \mathfrak{n} \oplus \mathfrak{a} \oplus \mathfrak{k}$ and denote by $G = NAK$ the corresponding Iwasawa decomposition of G. We denote by $\mathcal{W}$ the Weyl group of the root system Δ.

Let $P_0 = MAN$ be a minimal parabolic subgroup of G with $M = Z_K(A)$ the centralizer of A in K. For a linear form $\nu \in \mathfrak{a}_\mathbb{C}^*$, we define a quasi-character e^ν on A by $e^\nu(a) = \exp(\nu(\log a))$ $(a \in A)$. When $G = \mathrm{SL}_3(\mathbb{R})$, $\mathrm{Sp}_2(\mathbb{R})$ and $G_2(\mathbb{R})$, we identify the linear form ν with two complex numbers (ν_1, ν_2) by

$$\nu_1 = \frac{\langle\nu,\alpha\rangle}{\langle\alpha,\alpha\rangle}, \quad \nu_2 = \frac{\langle\nu,\beta\rangle}{\langle\beta,\beta\rangle}.$$

We call the induced representation $\pi_\nu = \mathrm{Ind}_{P_0}^G(1_M \otimes e^{\nu+\rho} \otimes 1_N)$ the *class one principal series representation* of G. Here $\rho = \frac{1}{2}\sum_{\alpha\in\Delta^+}(\dim \mathfrak{g}_\alpha)\alpha$.

Let η be a unitary character of N. Since $\mathfrak{n} = [\mathfrak{n}, \mathfrak{n}] \oplus \sum_{\alpha\in\Pi} \mathfrak{g}_\alpha$, the character η is determined by its restriction to $\mathfrak{g}_\alpha$ $(\alpha \in \Pi)$. For $1 \le i \le \dim \mathfrak{g}_\alpha$, we define a real number $\eta_{\alpha,i}$ by $\eta(X_{\alpha,i}) = \sqrt{-1}\pi\eta_{\alpha,i}$, where the root vector $X_{\alpha,i}$ is chosen as $B(X_{\alpha,i}, \theta X_{\alpha,j}) = -\delta_{i,j}$ $(1 \le i, j \le \dim \mathfrak{g}_\alpha)$. The length $|\eta_\alpha|$ of η_α is defined such that $|\eta_\alpha| = (\sum_{1\le i\le \dim \mathfrak{g}_\alpha} \eta_{\alpha,i}^2)^{1/2}$.

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In this paper we assume that η is nondegenerate, that is, $|\eta_\alpha| \neq 0$ for all $\alpha \in \Pi$. Consider the space

$$C_\eta^\infty(N\backslash G) = \{f : G \to \mathbb{C},\ \text{smooth} \mid f(ng) = \eta(n)f(g)\ \forall(n,g) \in N \times G\},$$

and the intertwining space $\mathrm{Hom}_G(\pi_\nu, C_\eta^\infty(N\backslash G))$. Take a spherical vector v_0 in π_ν. The space of *Whittaker functions* for the class one principal series representation π_ν is

$$\mathrm{Wh}(\nu,\eta) := \{\Phi(v_0) \mid \Phi \in \mathrm{Hom}_G(\pi_\nu, C_\eta^\infty(N\backslash G))\}.$$

Because of the Iwasawa decomposition, a Whittaker function w in the space $\mathrm{Wh}(\nu,\eta)$ is determined by its restriction $w|_A$ to A. We call $w|_A$ the *radial part* of w.

Hashizume [7] constructed a basis of the space of Whittaker functions. Let us recall in our situation. From now on we sometimes use the subscript ε for the notation settled above:

$$\varepsilon = \begin{cases} 1 & \text{for } \mathrm{SL}_3(\mathbb{R}), \\ 2 & \text{for } \mathrm{Sp}_2(\mathbb{R}), \\ 3 & \text{for } G_2(\mathbb{R}). \end{cases}$$

We introduce a coordinate $y = (y_1, y_2)$ on A by

$$y_1 = \exp\alpha(H), \quad y_2 = \exp\beta(H)$$

for $H \in \mathfrak{a}$. Then $y^{\rho_1} = y_1y_2$, $y^{\rho_2} = y_1^2y_2^{3/2}$ and $y^{\rho_3} = y_1^5y_2^3$. Define a power series $M_{\nu,\eta}^\varepsilon(y)$ on A by

$$M_{\nu,\eta}^\varepsilon(y) = y^{\rho_\varepsilon}\sum_{m_1,m_2=0}^\infty C_{(m_1,m_2)}^\varepsilon(\nu)\cdot(\pi y_1)^{2m_1+l_1^\varepsilon}(\pi y_2)^{2m_2+l_2^\varepsilon}, \tag{1.1}$$

where the characteristic exponents $(l_1^\varepsilon, l_2^\varepsilon)$ are given by

$$(l_1^\varepsilon, l_2^\varepsilon) = \begin{cases} ((4\nu_1+2\nu_2)/3, (2\nu_1+4\nu_2)/3) & \text{if } \varepsilon = 1, \\ (2\nu_1+2\nu_2, \nu_1+2\nu_2) & \text{if } \varepsilon = 2, \\ (4\nu_1+6\nu_2, 2\nu_1+4\nu_2) & \text{if } \varepsilon = 3, \end{cases}$$

and the coefficients $C_{(m_1,m_2)}^\varepsilon(\nu)$ are determined by the recurrence relation

$$\begin{cases} C_{(0,0)}^\varepsilon(\nu) = 1, \\ Q_{(m_1,m_2)}^\varepsilon(\nu)\, C_{(m_1,m_2)}^\varepsilon(\nu) = \dfrac{|\eta_\alpha|^2}{2\langle\alpha,\alpha\rangle}C_{(m_1-1,m_2)}^\varepsilon(\nu) + \dfrac{\varepsilon|\eta_\beta|^2}{2\langle\beta,\beta\rangle}C_{(m_1,m_2-1)}^\varepsilon(\nu), \end{cases} \tag{1.2}$$

with

$$Q_{(m_1,m_2)}^\varepsilon(\nu) = m_1^2 + \varepsilon m_2^2 - \varepsilon m_1m_2 + \nu_1m_1 + \varepsilon\nu_2m_2.$$

If $Q_{(m_1,m_2)}^\varepsilon(\nu)$ does not vanish for all nonzero (m_1,m_2), we can inductively determine the coefficients $C_{(m_1,m_2)}^\varepsilon(\nu)$ via the relation (1.2). We extend the power series $M_{\nu,\eta}^\varepsilon(y)$ to the function on G by $M_{\nu,\eta}^\varepsilon(g) = \eta(n(g))M_{\nu,\eta}^\varepsilon(a(g))$ with $g = n(g)a(g)k(g)$ the Iwasawa decomposition of $g \in G$. In this paper we call the function $M_{\nu,\eta}^\varepsilon$ the *fundamental Whittaker function* on G.

Definition 1.1 The element $\nu = (\nu_1, \nu_2)$ in $\mathfrak{a}_{\mathbb{C}}^*$ is called *regular* if the following conditions are satisfied.

- $Q^{\varepsilon}_{(m_1,m_2)}(w\nu) \neq 0$ for all $(m_1, m_2) \in \mathbb{N}^2 \setminus \{(0,0)\}$ and $w \in \mathcal{W}_\varepsilon$,
- $w_1\nu - w_2\nu \notin \{m_1\alpha + m_2\beta \mid (m_1, m_2) \in \mathbb{Z}^2\}$ for all distinct pairs (w_1, w_2) in $\mathcal{W}_\varepsilon$.

Hashizume proved the following.

Proposition 1.2 [7, Theorem 5.4] *If ν is regular, then the set $\{M^{\varepsilon}_{w\nu,\eta} \mid w \in \mathcal{W}_\varepsilon\}$ forms a basis of* $\mathrm{Wh}(\nu, \eta)$.

Remark 1 The action of the Weyl group on $\mathfrak{a}_{\mathbb{C}}^*$ is described as follows. Let w_α and w_β be the simple reflections corresponding to the simple roots α and β, respectively. Then $w_\alpha(\nu_1, \nu_2) = (-\nu_1, \nu_1 + \nu_2)$ and

$$w_\beta(\nu_1, \nu_2) = \begin{cases} (\nu_1 + \nu_2, -\nu_2) & \text{if } G = \mathrm{SL}_3(\mathbb{R}), \\ (\nu_1 + 2\nu_2, -\nu_2) & \text{if } G = \mathrm{Sp}_2(\mathbb{R}), \\ (\nu_1 + 3\nu_2, -\nu_2) & \text{if } G = G_2(\mathbb{R}). \end{cases}$$

By replacing $C^{\varepsilon}_{(m_1,m_2)}(\nu) \to (|\eta_\alpha|^2/2\langle\alpha,\alpha\rangle)^{m_1}(\varepsilon|\eta_\beta|^2/2\langle\beta,\beta\rangle)^{m_2} C^{\varepsilon}_{(m_1,m_2)}(\nu)$, we may assume

$$|\eta_\alpha|^2 = 2\langle\alpha,\alpha\rangle \quad \text{and} \quad |\eta_\beta|^2 = 2\varepsilon^{-1}\langle\beta,\beta\rangle. \tag{1.3}$$

Hereafter we impose this assumption for simplicity and omit the symbol η from $M^{\varepsilon}_{\nu,\eta}$.

Jacquet [13] introduced an integral representation of the Whittaker function for π_ν of moderate growth [20]:

$$W_{\nu,\eta}(g) = \int_N \eta^{-1}(n) a(w_0^{-1} n g)^{\nu+\rho} dn.$$

Here w_0 is a longest element in the Weyl group. As a function on $\mathfrak{a}_{\mathbb{C}}^*$, the Jacquet integral converges absolutely and uniformly on the domain $\{\nu \in \mathfrak{a}_{\mathbb{C}}^* \mid \mathrm{Re}(\langle\nu,\alpha\rangle) > 0$ for all $\alpha \in \Delta^+\}$, and can be continued to a meromorphic function of ν. We call the Jacquet integral (and its constant multiple) the *class one Whittaker function*.

For applications to automorphic forms, Mellin–Barnes integral representations of class one Whittaker functions are useful. However, to find Mellin–Barnes integral representations by modifying Jacquet integrals is not easy in general. For the case of $G_2(\mathbb{R})$, we could not evaluate the Jacquet integral as in the cases of $\mathrm{SL}_3(\mathbb{R})$ ([19]) and $\mathrm{Sp}_2(\mathbb{R})$ ([9]). We will find a conjectural formula from the explicit formula for fundamental Whittaker function and prove it by using the expansion formulas of Hashizume. In our case, the expansion formulas can be written as follows.

Proposition 1.3 [7, Theorem 7.8] *Under assumption* (1.3), *we have*

$$W^{\varepsilon}_{\nu}(g) = c_\varepsilon \sum_{w \in \mathcal{W}_\varepsilon} w\Big\{ \prod_{\delta \in \Delta^+_\varepsilon} \Gamma\Big(\frac{\langle\delta,\nu\rangle}{\langle\alpha,\alpha\rangle} + 1\Big) \Gamma\Big(\frac{\langle\delta,-\nu\rangle}{\langle\alpha,\alpha\rangle}\Big) \cdot M^{\varepsilon}_{\nu}(g) \Big\},$$

with some constant c_ε.

We remark that Hashizume gives the constant c_ε explicitly.

2 Explicit Formulas for Fundamental Whittaker Functions

We solve the recurrence relation (1.2) under the assumption (1.3). The result is as follows.

Theorem 2.1 *When $\nu = (\nu_1, \nu_2)$ is regular, the radial parts of fundamental Whittaker functions are of the form* (1.1)*, where the coefficients $C^{\varepsilon}_{(m_1,m_2)}(\nu)$ are given by*

$$C^{1}_{(m_1,m_2)}(\nu) = \sum_{0\le n\le m_1} \frac{1}{(m_1-n)!\, m_2!\, n!} \times \frac{1}{(\nu_1+\nu_2+1)_{m_1}(\nu_2+1)_{m_2-n}(\nu_1+1)_n}, \tag{2.1}$$

$$C^{2}_{(m_1,m_2)}(\nu) = \sum_{\substack{0\le n_1+n_2\le m_1\\ 0\le n_2\le m_2}} \frac{1}{(m_1-n_1-n_2)!\,(m_2-n_2)!\, n_1!\, n_2!} \times \frac{1}{(\nu_1+\nu_2+1)_{m_1}(\nu_2+1)_{m_2-n_1}(\nu_1+1)_{n_1}(\nu_1+2\nu_2+1)_{n_2}}, \tag{2.2}$$

and

$$\begin{aligned} C^{3}_{(m_1,m_2)}(\nu) = \sum_{\substack{0\le n_1+n_2\le m_1\\ 0\le n_4\le n_3\le n_2\le m_2}} & \frac{1}{(m_1-n_1-n_2)!\,(m_2-n_2)!} \\ & \times \frac{1}{n_1!\,(n_2-n_3)!\,(n_3-n_4)!\,n_4!} \\ & \times \frac{1}{(\nu_1+\nu_2+1)_{m_1-n_3}(\nu_2+1)_{m_2-n_1}(\nu_1+1)_{n_1-n_4}} \\ & \times \frac{1}{(\nu_1+2\nu_2+1)_{n_2}(2\nu_1+3\nu_2+1)_{n_3}(\nu_1+3\nu_2+1)_{n_4}}. \end{aligned} \tag{2.3}$$

Here $(a)_n = \Gamma(a+n)/\Gamma(a)$ is the Pochhammer symbol.

Proof The formula for $C^{1}_{(m_1,m_2)}(\nu)$ is given in [12, Proposition 5] by arranging with the formula of Bump [2, Chapter II]:

$$C^{1}_{(m_1,m_2)}(\nu) = \frac{(\nu_1+\nu_2+1)_{m_1+m_2}}{m_1!\, m_2!\, (\nu_1+\nu_2+1)_{m_1}(\nu_1+1)_{m_1}(\nu_1+\nu_2+1)_{m_2}(\nu_2+1)_{m_2}}. \tag{2.4}$$

We can show (2.2) as in [12, Theorem 15]. If we denote the summand in (2.2) by $1/X_{m_1,m_2;n_1,n_2}$, then the identity

$$\frac{X_{m_1,m_2;n_1,n_2}}{X_{m_1-1,m_2;n_1,n_2}} + \frac{X_{m_1,m_2;n_1,n_2}}{X_{m_1,m_2-1;n_1,n_2}} - \frac{X_{m_1,m_2;n_1,n_2}}{X_{m_1,m_2;n_1+1,n_2}} - \frac{X_{m_1,m_2;n_1,n_2}}{X_{m_1,m_2;n_1,n_2+1}} = Q^{2}_{(m_1,m_2)}(\nu)$$

implies that the right-hand side of (2.2) satisfies the recurrence relation (1.2). The proof of (2.3) can be similarly performed. ■

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Remark 2 The following formula for $C^2_{(m_1,m_2)}(\nu)$ is given by the author [9, Theorem 2.1]:

$$C^2_{(m_1,m_2)}(\nu) = \frac{1}{m_1!\, m_2!\, (\nu_1+1)_{m_1}(\nu_1+2\nu_2+1)_{m_2}} \times {}_3F_2\left(\begin{matrix} -m_2,\ m_1+\nu_1+\nu_2+1,\ -m_1-\nu_1-\nu_2 \\ \nu_1+\nu_2+1,\ \nu_2+1 \end{matrix} \,\middle|\, 1 \right).$$

Here

$${}_3F_2\left(\begin{matrix} a,\ b,\ c \\ d,\ e \end{matrix} \,\middle|\, z \right) = \sum_{m=0}^{\infty} \frac{(a)_m (b)_m (c)_m}{(d)_m (e)_m} \cdot \frac{z^m}{m!}$$

is the generalized hypergeometric function.

We can find the following relations between $M^2_\nu(y)$, $M^3_\nu(y)$, and $M^1_\nu(y)$.

Corollary 2.2 *If we set*

$$C^\varepsilon_{(m_1,m_2)}(\nu) = \Gamma^\varepsilon(\nu)\, \widetilde{C}^\varepsilon_{(m_1,m_2)}(\nu) \quad \textit{and} \quad M^\varepsilon_\nu(y) = y^{\rho_\varepsilon}\Gamma^\varepsilon(\nu)\widetilde{M}^\varepsilon_\nu(y)$$

with

$$\Gamma^\varepsilon(\nu) = \prod_{\delta\in\Delta^+_\varepsilon} \Gamma\left(\frac{\langle\delta,\nu\rangle}{\langle\alpha,\alpha\rangle} + 1 \right),$$

then we have the following:

$$\widetilde{M}^2_{(\nu_1,\nu_2)}(y) = \sum_{k=0}^{\infty} (\pi^3 y_1^2 y_2)^{(2k+\nu_1)/3} \widetilde{C}^0_k(\nu_1) \widetilde{M}^1_{(k+\nu_1+\nu_2,-k+\nu_2)}(y), \tag{2.5}$$

$$\widetilde{M}^3_{(\nu_1,\nu_2)}(y) = \sum_{k_1,k_2=0}^{\infty} (\pi^3 y_1^2 y_2)^{2(k_1+k_2+2\nu_1+3\nu_2)/3} \widetilde{C}^1_{(k_1,k_2)}(\nu_1, \nu_1+3\nu_2) \times \widetilde{M}^1_{(k_1+\nu_1+\nu_2,-k_1+k_2+\nu_2)}(y). \tag{2.6}$$

Here $\widetilde{C}^0_k(\nu_1) = 1/(k!\,\Gamma(k+\nu_1+1))$ *is the coefficient of the fundamental Whittaker function on* $\mathrm{SL}_2(\mathbb{R})$.

Proof Let us prove formula (2.6). By [12, Proposition 5], the coefficient $\widetilde{C}^1_{(m_1,m_2)}(\nu)$ has the following expressions in addition to (2.1):

$$\sum_{n=0}^{m_2} \frac{1}{m_1!\,(m_2-n)!\,n!\Gamma(m_1-n+\nu_1+1)} \times \frac{1}{\Gamma(m_2+\nu_1+\nu_2+1)\Gamma(n+\nu_2+1)}, \tag{2.7}$$

$$\sum_{n=0}^{\min\{m_1,m_2\}} \frac{1}{(m_1-n)!\,(m_2-n)!\,n!\Gamma(m_1+\nu_1+1)} \times \frac{1}{\Gamma(m_2+\nu_2+1)\Gamma(n+\nu_1+\nu_2+1)}. \tag{2.8}$$

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We apply (2.7) for $\widetilde{C}^1_{(k_1,k_2)}(\nu_1,\nu_1+3\nu_2)$, and (2.8) for $\widetilde{M}^1_{(k+\nu_1+\nu_2,-k_1+k_2+\nu_2)}(y)$ to rewrite the right-hand side of (2.6) as

$$
\begin{aligned}
&\sum_{k_1,k_2=0}^{\infty} \frac{(\pi^3 y_1^2 y_2)^{2(k_1+k_2+2\nu_1+3\nu_2)/3}}{k_1!\,\Gamma(k_2+2\nu_1+3\nu_2+1)} \\
&\quad \times \sum_{i=0}^{k_2} \frac{1}{(k_2-i)!\,i!\,\Gamma(k_1-i+\nu_1+1)\Gamma(i+\nu_1+3\nu_2+1)} \\
&\quad \times \sum_{l_1,l_2=0}^{\infty} \frac{(\pi y_1)^{2l_1+2(k_1+k_2+2\nu_1+3\nu_2)/3}(\pi y_2)^{2l_2+2(-k_1+2k_2+\nu_1+3\nu_2)/3}}{\Gamma(l_1+k_1+\nu_1+\nu_2+1)\Gamma(l_2-k_1+k_2+\nu_2+1)} \\
&\quad \times \sum_{j=0}^{\min\{l_1,l_2\}} \frac{1}{(l_1-j)!\,(l_2-j)!\,j!\,\Gamma(j+k_2+\nu_1+2\nu_2+1)}.
\end{aligned} \tag{2.9}
$$

After the substitutions $k_2 \to k_2+i$, $l_1 \to l_1+j$ and $l_2 \to l_2+j$, (2.9) becomes

$$
\begin{aligned}
&\sum_{\substack{k_1,k_2,i\\ l_1,l_2,j=0}}^{\infty} \frac{(\pi y_1)^{2(k_1+k_2+i+l_1+j)+4\nu_1+6\nu_2}(\pi y_2)^{2(k_2+i+l_2+j)+2\nu_1+4\nu_2}}{k_1!\,k_2\,l_1!\,l_2!\,i!\,j!\,\Gamma(k_1-i+\nu_1+1)\Gamma(k_2+i+2\nu_1+3\nu_2+1)} \\
&\quad \times \frac{1}{\Gamma(i+\nu_1+3\nu_2+1)\Gamma(l_1+j+k_1+\nu_1+\nu_2+1)} \\
&\quad \times \frac{1}{\Gamma(l_2+j-k_1+k_2+i+\nu_2+1)\Gamma(j+k_2+i+\nu_1+2\nu_2+1)}.
\end{aligned}
$$

It is equivalent to formula (2.3) by the replacement

$$
i=n_4,\ k_2=n_3-n_4,\ j=n_2-n_3,\ k_1=n_1,\ l_1=m_1-n_1-n_2,\ l_2=m_2-n_2,
$$

and we get the assertion. ∎

3 Explicit Formulas for Class One Whittaker Functions

In this section we give Mellin–Barnes type integral representations of the radial parts $W^{\varepsilon}_{\nu}(y)$ of the class one Whittaker functions. The case of $\mathrm{SL}_3(\mathbb{R})$ was studied by Bump [2].

Proposition 3.1 [2] *Up to a constant multiple, the radial part of the class one Whittaker function on* $\mathrm{SL}_3(\mathbb{R})$ *can be written by* $W^1_\nu(y) = y^{\rho_1}\widetilde{W}^1_\nu(y)$, *where*

$$
\widetilde{W}^1_{(\nu_1,\nu_2)}(y) = \frac{1}{(2\pi\sqrt{-1})^2}\int_{L(\sigma_1)}\int_{L(\sigma_2)} (\pi y_1)^{-2s_1}(\pi y_2)^{-2s_2}T_{(\nu_1,\nu_2)}(s_1,s_2)\,ds_2 ds_1,
$$

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with

$$T_{(\nu_1,\nu_2)}(s_1,s_2)=\frac{\Gamma(s_1+\frac{2\nu_1+\nu_2}{3})\Gamma(s_1+\frac{-\nu_1+\nu_2}{3})\Gamma(s_1+\frac{-\nu_1-2\nu_2}{3})}{\Gamma(s_1+s_2)}$$
$$\times\Gamma\Big(s_2-\frac{2\nu_1+\nu_2}{3}\Big)\Gamma\Big(s_2-\frac{-\nu_1+\nu_2}{3}\Big)\Gamma\Big(s_2-\frac{-\nu_1-2\nu_2}{3}\Big).$$

Here the path $L(\sigma)$ $(\sigma\in\mathbb{R})$ means the vertical line from $\sigma-\sqrt{-1}\infty$ to $\sigma+\sqrt{-1}\infty$ and σ_1,σ_2 are taken such that

$$\sigma_1>\max\Big\{-\operatorname{Re}\Big(\frac{2\nu_1+\nu_2}{3}\Big),\operatorname{Re}\Big(\frac{\nu_1-\nu_2}{3}\Big),\operatorname{Re}\Big(\frac{\nu_1+2\nu_2}{3}\Big)\Big\},$$
$$\sigma_2>\max\Big\{\operatorname{Re}\Big(\frac{2\nu_1+\nu_2}{3}\Big),\operatorname{Re}\Big(\frac{-\nu_1+\nu_2}{3}\Big),-\operatorname{Re}\Big(\frac{\nu_1+2\nu_2}{3}\Big)\Big\}.$$

Now we give two integral representations in the case of $\mathrm{Sp}_2(\mathbb{R})$. The first formula (3.1) can be seen to be the analogue of the formula (2.5), and the second one (3.2) is given by Moriyama and the author [11, Proposition 2.2], which is essentially the same as the formula obtained by Niwa ([14], *cf.* [9, Theorem 3.2]).

Theorem 3.2 *Up to a constant multiple, the radial part of the class one Whittaker function on $\mathrm{Sp}_2(\mathbb{R})$ can be written by $W^2_\nu(y)=y^{\rho_2}\widetilde{W}^2_\nu(y)$ where*

$$(3.1)\quad \widetilde{W}^2_{(\nu_1,\nu_2)}(y)=\frac{1}{2\pi\sqrt{-1}}\int_{L(\sigma)}(\pi^3y_1^2y_2)^{2s/3}\widetilde{W}^1_{(s+\frac{\nu_1}{2}+\nu_2,-s+\frac{\nu_1}{2}+\nu_2)}(y)\,T^0_{\nu_1}(-s)\,ds,$$

with $\sigma<|\operatorname{Re}(\nu_1/2)|$. We also have the expression

$$(3.2)\quad \widetilde{W}^2_{(\nu_1,\nu_2)}(y)=\frac{1}{(2\pi\sqrt{-1})^2}\int_{L(\sigma_1)}\int_{L(\sigma_2)}(\pi^2y_1y_2)^{s_1+s_2}\widetilde{W}^0_{s_1+s_2}(y_1)\widetilde{W}^0_{s_1-s_2}(y_2)$$
$$\times\,T^0_{\nu_1}(-s_1)T^0_{\nu_1+2\nu_2}(-s_2)\,ds_2ds_1,$$

with $\sigma_1<|\operatorname{Re}(\nu_1/2)|$ and $\sigma_2<|\operatorname{Re}(\nu_1/2+\nu_2)|$. Here $W^0_\nu(y)=y^{1/2}\widetilde{W}^0_\nu(y)$ is the radial part of the class one Whittaker function on $\mathrm{SL}_2(\mathbb{R})$:

$$\widetilde{W}^0_\nu(y)=2K_\nu(2\pi y)=\frac{1}{2\pi\sqrt{-1}}\int_{L(\sigma)}(\pi y)^{-2s}T^0_\nu(s)\,ds,$$

with $T^0_\nu(s)=\Gamma(s+\nu/2)\Gamma(s-\nu/2)$.

Proof In the formula (3.2), we integrate with respect to s_2 by means of the formula

$$\widetilde{W}^1_{(\nu_1,\nu_2)}(y)=\frac{1}{2\pi\sqrt{-1}}\int_{L(\sigma)}(\pi y_1)^{s-(\nu_1-\nu_2)/6}(\pi y_2)^{s+(\nu_1-\nu_2)/6}$$
$$\times\,\widetilde{W}^0_{s+(\nu_1-\nu_2)/2}(y_1)\widetilde{W}^0_{-s+(\nu_1-\nu_2)/2}(y_2)\,T^0_{\nu_1+\nu_2}(-s)\,ds$$

([12, Corollary 3]) to reach expression (3.1). ∎

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Finally we discuss the case of $G_2(\mathbb{R})$.

Theorem 3.3 *Up to a constant multiple, the radial part of the class one Whittaker function on $G_2(\mathbb{R})$ can be written by $W^3_\nu(y) = y^{\rho_3}\widetilde{W}^3_\nu(y)$, where*

$$(3.3)\quad \widetilde{W}^3_{(\nu_1,\nu_2)}(y) = \frac{1}{(2\pi\sqrt{-1})^2}\int_{L(\sigma_1)}\int_{L(\sigma_2)} (\pi^3 y_1^2 y_2)^{2(s_1+s_2)/3}\widetilde{W}^1_{(s_1,-s_1+s_2)}(y) \times T_{(\nu_1,\nu_1+3\nu_2)}(-s_1,-s_2)\,ds_2 ds_1,$$

with

$$\sigma_1 < \min\{\mathrm{Re}(\nu_1+\nu_2), \mathrm{Re}(\nu_2), \mathrm{Re}(\nu_1+2\nu_2)\},$$
$$\sigma_2 < \min\{\mathrm{Re}(-\nu_1-\nu_2), \mathrm{Re}(-\nu_2), \mathrm{Re}(-\nu_1-2\nu_2)\}.$$

Proof As in the proof in [8, 10], we use the expansion formula in Proposition 1.3. Our task is to show

$$(3.4)\quad \widetilde{W}^3_{(\nu_1,\nu_2)}(y) = \sum_{w\in\mathcal{W}_3} w\Big\{\frac{\pi^3}{\sin\pi(-\nu_1)\sin\pi(-\nu_2)\sin\pi(-\nu_1-\nu_2)} \times \frac{\pi^3}{\sin\pi(-\nu_1-2\nu_2)\sin\pi(-\nu_1-3\nu_2)\sin\pi(-2\nu_1-3\nu_2)}\widetilde{M}^3_{(\nu_1,\nu_2)}(y)\Big\}.$$

Here we used the formula $\Gamma(x)\Gamma(1-x) = \pi/\sin\pi x$.

Let $\mathcal{W}_3'$ be the subset of $\mathcal{W}_3$ defined by

$$\mathcal{W}_3' = \{1,\ w_\alpha,\ w_\beta w_\alpha w_\beta,\ w_\alpha w_\beta w_\alpha w_\beta,\ w_\beta w_\alpha w_\beta w_\alpha,\ w_\alpha w_\beta w_\alpha w_\beta w_\alpha\}.$$

Then $\mathcal{W}_3'$ is isomorphic to the Weyl group $\mathcal{W}_1$ and we have $\mathcal{W}_3 = \mathcal{W}_3' \amalg w_\beta\mathcal{W}_3' = \mathcal{W}_3' \amalg \mathcal{W}_3' w_\beta$.

If we shift the path of integration to the left in the right-hand side of (3.3), it becomes sum of the residues at

$$(s_1,s_2) = (m_1 + w(\nu_1+\nu_2), m_2 + w(\nu_1+2\nu_2)),\quad m_1,m_2\in\mathbb{N},\ w\in\mathcal{W}_3'$$

(see [16, p. 707]). Then in view of $\mathrm{Res}_{s=-n}\Gamma(s) = (-1)^n/n!$, the right-hand side of (3.3) becomes

$$(3.5)\quad \sum_{w\in\mathcal{W}_3'} w\Big\{\sum_{m_1,m_2=0}^{\infty}(\pi^3 y_1^2 y_2)^{2(m_1+m_2+2\nu_1+3\nu_2)/3} \times \widetilde{W}^1_{(m_1+\nu_1+\nu_2,-m_1+m_2+\nu_2)}(y)\cdot S_{m_1,m_2}(\nu)\Big\},$$

with

$$S_{m_1,m_2}(\nu) = \frac{(-1)^{m_1+m_2}\Gamma(-m_1-\nu_1)\Gamma(-m_1-2\nu_1-3\nu_2)}{m_1!\,m_2!\,\Gamma(-m_1-m_2-2\nu_1-3\nu_2)} \times \Gamma(-m_2-\nu_1-3\nu_2)\Gamma(-m_2-2\nu_1-3\nu_2).$$

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Now we apply the expansion formula

$$\widetilde{W}^1_{(\nu_1,\nu_2)}(y) = \sum_{w\in\mathcal{W}_1} w\Big\{ \frac{\pi^3}{\sin\pi(-\nu_1)\sin\pi(-\nu_2)\sin\pi(-\nu_1-\nu_2)} \widetilde{M}^1_{(\nu_1,\nu_2)}(y) \Big\},$$

for $\mathrm{SL}_3(\mathbb{R})$ to express $\widetilde{W}^1_{(m_1+\nu_1+\nu_2,-m_1+m_2+\nu_2)}(y)$ as a linear combination of six fundamental Whittaker functions. Since the set $\{w(\mu_1,\mu_2) \mid w\in\mathcal{W}_1\}$ is

$$\{(\mu_1,\mu_2),(\mu_1+\mu_2,-\mu_2),(\mu_2,-\mu_1-\mu_2),\\ (-\mu_2,-\mu_1),(-\mu_1,\mu_1+\mu_2),(-\mu_1-\mu_2,\mu_1)\},$$

(3.5) can be written as

$$\sum_{w\in\mathcal{W}_3'} \sum_{\mathrm{I}\le \mathrm{J}\le \mathrm{VI}} \sum_{m_1,m_2,n_1,n_2=0}^{\infty} P^{\mathrm{J}}_{m_1,m_2,n_1,n_2}(w\nu), \tag{3.6}$$

where

$$P^{\mathrm{I}}_{m_1,m_2,n_1,n_2}(\nu) = \frac{\pi^3\cdot(\pi y_1)^{2(n_1+m_1+m_2)+4\nu_1+6\nu_2}(\pi y_2)^{2(n_2+m_2)+2\nu_1+4\nu_2}}{\sin\pi(-m_1-\nu_1-\nu_2)\sin\pi(m_1-m_2-\nu_2)\times\sin\pi(-m_2-\nu_1-2\nu_2)}\\ \times S_{m_1,m_2}(\nu)\,\widetilde{C}^1_{(n_1,n_2)}(m_1+\nu_1+\nu_2,-m_1+m_2+\nu_2),$$

$$P^{\mathrm{II}}_{m_1,m_2,n_1,n_2}(\nu) = \frac{\pi^3\cdot(\pi y_1)^{2(n_1+m_1+m_2)+4\nu_1+6\nu_2}(\pi y_2)^{2(n_2+m_1)+2\nu_1+2\nu_2}}{\sin\pi(-m_2-\nu_1-2\nu_2)\sin\pi(-m_1+m_2+\nu_2)\times\sin\pi(-m_1-\nu_1-\nu_2)}\\ \times S_{m_1,m_2}(\nu)\,\widetilde{C}^1_{(n_1,n_2)}(m_2+\nu_1+2\nu_2,m_1-m_2-\nu_2),$$

$$P^{\mathrm{III}}_{m_1,m_2,n_1,n_2}(\nu) = \frac{\pi^3\cdot(\pi y_1)^{2(n_1+m_2)+2\nu_1+4\nu_2}(\pi y_2)^{2n_2}}{\sin\pi(m_1-m_2-\nu_2)\sin\pi(m_2+\nu_1+2\nu_2)\sin\pi(m_1+\nu_1+\nu_2)}\\ \times S_{m_1,m_2}(\nu)\,\widetilde{C}^1_{(n_1,n_2)}(-m_1+m_2+\nu_2,-m_2-\nu_1-2\nu_2),$$

$$P^{\mathrm{IV}}_{m_1,m_2,n_1,n_2}(\nu) = \frac{\pi^3\cdot(\pi y_1)^{2(n_1+m_1)+2\nu_1+2\nu_2}(\pi y_2)^{2n_2}}{\sin\pi(-m_1+m_2+\nu_2)\sin\pi(m_1+\nu_1+\nu_2)\sin\pi(m_2+\nu_1+2\nu_2)}\\ \times S_{m_1,m_2}(\nu)\,\widetilde{C}^1_{(n_1,n_2)}(m_1-m_2-\nu_2,-m_1-\nu_1-\nu_2),$$

$$P^{\mathrm{V}}_{m_1,m_2,n_1,n_2}(\nu) = \frac{\pi^3\cdot(\pi y_1)^{2(n_1+m_2)+2\nu_1+4\nu_2}(\pi y_2)^{2(n_2+m_2)+2\nu_1+4\nu_2}}{\sin\pi(m_1+\nu_1+\nu_2)\sin\pi(-m_2-\nu_1-2\nu_2)\sin\pi(m_1-m_2-\nu_2)}\\ \times S_{m_1,m_2}(\nu)\,\widetilde{C}^1_{(n_1,n_2)}(-m_1-\nu_1-\nu_2,m_2+\nu_1+2\nu_2),$$

$$P^{\mathrm{VI}}_{m_1,m_2,n_1,n_2}(\nu) = \frac{\pi^3\cdot(\pi y_1)^{2(n_1+m_1)+2\nu_1+2\nu_2}(\pi y_2)^{2(n_2+m_1)+2\nu_1+2\nu_2}}{\sin\pi(m_2+\nu_1+2\nu_2)\sin\pi(-m_1-\nu_1-\nu_2)\sin\pi(-m_1+m_2+\nu_2)}\\ \times S_{m_1,m_2}(\nu)\,\widetilde{C}^1_{(n_1,n_2)}(-m_2-\nu_1-2\nu_2,m_1+\nu_1+\nu_2).$$

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Because $S_{m_1,m_2}(\nu) = S_{m_2,m_1}(w_\beta \nu)$, we can see that

$$P^{\mathrm{II}}_{m_1,m_2,n_1,n_2}(\nu) = P^{\mathrm{I}}_{m_2,m_1,n_1,n_2}(w_\beta \nu),$$
$$P^{\mathrm{IV}}_{m_1,m_2,n_1,n_2}(\nu) = P^{\mathrm{III}}_{m_2,m_1,n_1,n_2}(w_\beta \nu),$$
$$P^{\mathrm{VI}}_{m_1,m_2,n_1,n_2}(\nu) = P^{\mathrm{V}}_{m_2,m_1,n_1,n_2}(w_\beta \nu).$$

Then (3.5) equals (3.6) equals

$$\sum_{w\in\mathcal{W}_3'} \sum_{\mathrm{J}=\mathrm{I},\mathrm{III},\mathrm{V}} \sum_{m_1,m_2,n_1,n_2=0}^{\infty} \big(P^{\mathrm{J}}_{m_1,m_2,n_1,n_2}(w\nu) + P^{\mathrm{J}}_{m_1,m_2,n_1,n_2}(w_\beta w\nu)\big)$$
$$= \sum_{w\in\mathcal{W}_3} \sum_{\mathrm{J}=\mathrm{I},\mathrm{III},\mathrm{V}} \sum_{m_1,m_2,n_1,n_2=0}^{\infty} P^{\mathrm{J}}_{m_1,m_2,n_1,n_2}(w\nu).$$

Hence the following claim implies our theorem for $G_2(\mathbb{R})$.

Claim (i) $\displaystyle\sum_{w\in\mathcal{W}_3} \sum_{m_1,m_2,n_1,n_2=0}^{\infty} P^{\mathrm{I}}_{m_1,m_2,n_1,n_2}(w\nu) = \widetilde{W}^3_\nu(y),$

(ii) $\displaystyle\sum_{w\in\mathcal{W}_3} \sum_{m_1,m_2,n_1,n_2=0}^{\infty} P^{\mathrm{III}}_{m_1,m_2,n_1,n_2}(w\nu) = 0,$

(iii) $\displaystyle\sum_{w\in\mathcal{W}_3} \sum_{m_1,m_2,n_1,n_2=0}^{\infty} P^{\mathrm{V}}_{m_1,m_2,n_1,n_2}(w\nu) = 0.$

We begin with (i). In view of

$$\Gamma(-a-n) = \frac{(-1)^n}{\Gamma(a+n+1)} \cdot \frac{\pi}{\sin\pi(-a)} \quad (a \notin \mathbb{Z}, n \in \mathbb{Z}),$$

$S_{m_1,m_2}(\nu)$ can be written as

$$\frac{\pi^3}{\sin\pi(-\nu_1)\sin\pi(-2\nu_1-3\nu_2)\sin\pi(-\nu_1-3\nu_2)} \cdot \frac{1}{m_1!\,m_2!}$$
$$\times \frac{\Gamma(m_1+m_2+2\nu_1+3\nu_2+1)}{\Gamma(m_1+\nu_1+1)\Gamma(m_1+2\nu_1+3\nu_2+1)\Gamma(m_2+\nu_1+3\nu_2+1)\Gamma(m_2+2\nu_1+3\nu_2+1)}$$
$$= \frac{\pi^3}{\sin\pi(-\nu_1)\sin\pi(-2\nu_1-3\nu_2)\sin\pi(-\nu_1-3\nu_2)} \widetilde{C}^1_{(m_1,m_2)}(\nu_1, \nu_1+3\nu_2)$$

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(see (2.4)). Then

$$P^{\mathrm{I}}_{m_1,m_2,n_1,n_2}(\nu) = \frac{\pi^3}{\sin\pi(-\nu_1-\nu_2)\sin\pi(-\nu_2)\sin\pi(-\nu_1-2\nu_2)} \times \frac{\pi^3}{\sin\pi(-\nu_1)\sin\pi(-2\nu_1-3\nu_2)\sin\pi(-\nu_1-3\nu_2)} \times (\pi y_1)^{2(n_1+m_1+m_2)+4\nu_1+6\nu_2}(\pi y_2)^{2(n_2+m_2)+2\nu_1+4\nu_2} \times \widetilde{C}^1_{(m_1,m_2)}(\nu_1,\nu_1+3\nu_2)\,\widetilde{C}^1_{(n_1,n_2)}(m_1+\nu_1+\nu_2,-m_1+m_2+\nu_2),$$

and hence (i) follows from Corollary 2.2 and (3.4).

Let us consider (ii). To prove the vanishing for $P^{\mathrm{III}}_{m_1,m_2,n_1,n_2}$, it is enough to show

$$\sum_{m_1,m_2,n_1,n_2=0}^{\infty} \bigl(P^{\mathrm{III}}_{m_1,m_2,n_1,n_2}(\nu) + P^{\mathrm{III}}_{m_1,m_2,n_1,n_2}(w_\alpha\nu)\bigr) = 0.$$

We substitute $n_1 \to n_1 - m_2$ to get

$$\sum_{m_1,m_2,n_1,n_2=0}^{\infty} P^{\mathrm{III}}_{m_1,m_2,n_1,n_2}(\nu) = \frac{\pi^3}{\sin\pi(-\nu_2)\ \sin\pi(\nu_1+2\nu_2)\ \sin\pi(\nu_1+\nu_2)} \times \sum_{n_1,n_2=0}^{\infty} p_{n_1,n_2}(\nu)(\pi y_1)^{2n_1+2\nu_1+4\nu_2}(\pi y_2)^{2n_2},$$

with

$$p_{n_1,n_2}(\nu) = \sum_{\substack{0\le m_1\\ 0\le m_2\le n_1}} S_{m_1,m_2}(\nu)\widetilde{C}^1_{(n_1-m_2,n_2)}(-m_1+m_2+\nu_2,-m_2-\nu_1-2\nu_2).$$

For our purpose, it suffices to show $p_{n_1,n_2}(\nu) + p_{n_1,n_2}(w_\alpha\nu) = 0$. By using (2.1), we have

$$\begin{aligned}
&\widetilde{C}^1_{(n_1-m_2,n_2)}(-m_1+m_2+\nu_2,-m_2-\nu_1-2\nu_2)\\
&\quad= \sum_{j=0}^{n_1-m_2} \frac{1}{(n_1-m_2-j)!n_2!j!}\cdot\frac{1}{\Gamma(n_1-m_1-m_2-\nu_1-\nu_2+1)}\\
&\qquad\times\frac{1}{\Gamma(n_2-j-m_2-\nu_1-2\nu_2+1)\Gamma(j-m_1+m_2+\nu_2+1)}\\
&\quad= \sum_{j=m_2}^{n_1} \frac{1}{(n_1-j)!n_2!(j-m_2)!}\cdot\frac{1}{\Gamma(n_1-m_1-m_2-\nu_1-\nu_2+1)}\\
&\qquad\times\frac{1}{\Gamma(n_2-j-\nu_1-2\nu_2+1)\Gamma(j-m_1+\nu_2+1)}.
\end{aligned}$$

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Then we have

$$
\begin{aligned}
p_{n_1,n_2}(\nu) = \sum_{\substack{0\le m_2\le j\le n_1\\0\le m_1}} & \frac{(-1)^{m_1+m_2}\Gamma(-m_1-\nu_1)\Gamma(-m_1-2\nu_1-3\nu_2)}{m_1!\,m_2!\,\Gamma(-m_1-m_2-2\nu_1-3\nu_2)}\\
&\times \Gamma(-m_2-\nu_1-3\nu_2)\Gamma(-m_2-2\nu_1-3\nu_2)\\
&\times \frac{1}{(n_1-j)!n_2!(j-m_2)!}\cdot\frac{1}{\Gamma(n_1-m_1-m_2-\nu_1-\nu_2+1)}\\
&\times \frac{1}{\Gamma(n_2-j-\nu_1-2\nu_2+1)\Gamma(j-m_1+\nu_2+1)}.
\end{aligned}
$$

We apply the relation $\Gamma(a-m_1)=(-1)^{m_1}\Gamma(a)/(1-a)_{m_1}$ to find that

$$
\begin{aligned}
p_{n_1,n_2}(\nu) = \sum_{0\le m_2\le j\le n_1} & \frac{(-1)^{m_2}\Gamma(-m_2-\nu_1-3\nu_2)\Gamma(-m_2-2\nu_1-3\nu_2)}{m_2!\,n_2!\,(j-m_2)!\,(n_1-j)!\,\Gamma(n_2-j-\nu_1-2\nu_2+1)}\\
&\times \frac{\Gamma(-\nu_1)\Gamma(-2\nu_1-3\nu_2)}{\Gamma(-m_2-2\nu_1-3\nu_2)\Gamma(-m_2+n_1-\nu_1-\nu_2+1)\Gamma(j+\nu_2+1)}\\
&\times \sum_{m_1=0}^{\infty}\frac{(m_2+2\nu_1+3\nu_2+1)_{m_1}(m_2-n_1+\nu_1+\nu_2)_{m_1}(-j-\nu_2)_{m_1}}{m_1!\,(\nu_1+1)_{m_1}(2\nu_1+3\nu_2+1)_{m_1}}.
\end{aligned}
$$

The summation over m_1 becomes the generalized hypergeometric series ${}_3F_2(1)$. We use a transformation formula [1, p. 98, 7]

$$
{}_3F_2\left(\begin{matrix}a,\ b,\ c\\ d,\ e\end{matrix}\,\middle|\,1\right)=\frac{\Gamma(d)\Gamma(d+e-a-b-c)}{\Gamma(d+e-a-b)\Gamma(d-c)}\,{}_3F_2\left(\begin{matrix}e-a,\ e-b,\ c\\ d+e-a-b,\ e\end{matrix}\,\middle|\,1\right)
$$

with $a=m_2+2\nu_1+3\nu_2+1$, $b=-j-\nu_2$, $c=m_2-n_1+\nu_1+\nu_2$, $d=\nu_1+1$, and $e=2\nu_1+3\nu_2+1$, to get

$$
\begin{aligned}
\sum_{m_1} = & \frac{\Gamma(\nu_1+1)\Gamma(n_1+j-2m_2+1)}{\Gamma(-m_2+j+\nu_1+\nu_2+1)\Gamma(n_1-m_2-\nu_2+1)}\\
&\times {}_3F_2\left(\begin{matrix}-m_2,\ j+2\nu_1+4\nu_2+1,\ m_2-n_1+\nu_1+\nu_2\\ -m_2+j+\nu_1+\nu_2+1,\ 2\nu_1+3\nu_2+1\end{matrix}\,\middle|\,1\right).
\end{aligned}
$$

Further we use a relation for terminating hypergeometric series ([15, 7.4.4.86])

$$
{}_3F_2\left(\begin{matrix}-m_2,\ a,\ b\\ c,\ d\end{matrix}\,\middle|\,1\right)=\frac{(c-a)_{m_2}}{(c)_{m_2}}\,{}_3F_2\left(\begin{matrix}-m_2,\ a,\ d-b\\ d,\ a-c-m_2+1\end{matrix}\,\middle|\,1\right)
$$

with $a=j+2\nu_1+4\nu_2+1$, $b=m_2-n_1+\nu_1+\nu_2$, $c=-m_2+j+\nu_1+\nu_2+1$, and

$d = 2\nu_1 + 3\nu_2 + 1$. Then we get

$$p_{n_1,n_2}(\nu) = \sum_{0\le m_2\le j\le n_1} \frac{(-1)^{m_2}\Gamma(-m_2-\nu_1-3\nu_2)\Gamma(-m_2-2\nu_1-3\nu_2)}{m_2!\,n_2!\,(j-m_2)!\,(n_1-j)!\,\Gamma(n_2-j-\nu_1-2\nu_2+1)}$$
$$\times \frac{\Gamma(-\nu_1)\Gamma(-2\nu_1-3\nu_2)}{\Gamma(-m_2-2\nu_1-3\nu_2)\Gamma(-m_2+n_1-\nu_1-\nu_2+1)\Gamma(j+\nu_2+1)}$$
$$\times \frac{\Gamma(\nu_1+1)\Gamma(n_1+j-2m_2+1)}{\Gamma(-m_2+j+\nu_1+\nu_2+1)\Gamma(-m_2+n_1-\nu_2+1)}$$
$$\times \frac{\Gamma(-m_2+j+\nu_1+\nu_2+1)\Gamma(-\nu_1-3\nu_2)}{\Gamma(j+\nu_1+\nu_2+1)\Gamma(-m_2-\nu_1-3\nu_2)}$$
$$\times {}_3F_2\left(\begin{matrix}-m_2,\ j+2\nu_1+4\nu_2+1,\ -m_2+n_1+\nu_1+2\nu_2+1\\ 2\nu_1+3\nu_2+1,\ \nu_1+3\nu_2+1\end{matrix}\ \middle|\ 1\right)$$
$$= \Gamma(-\nu_1)\Gamma(\nu_1+1)\Gamma(-2\nu_1-3\nu_2)\Gamma(-\nu_1-3\nu_2)$$
$$\times \sum_{0\le m_2\le j\le n_1} \frac{(-1)^{m_2}(n_1+j-2m_2)!}{m_2!\,n_2!\,(j-m_2)!\,(n_1-j)!}$$
$$\times \frac{1}{\Gamma(j+\nu_1+\nu_2+1)\Gamma(j+\nu_2+1)}$$
$$\times \frac{1}{\Gamma(-m_2+n_1-\nu_1-\nu_2+1)\Gamma(-m_2+n_1-\nu_2+1)}$$
$$\times \frac{1}{\Gamma(n_2-j-\nu_1-2\nu_2+1)}$$
$$\times {}_3F_2\left(\begin{matrix}-m_2,\ j+2\nu_1+4\nu_2+1,\ -m_2+n_1+\nu_1+2\nu_2+1\\ 2\nu_1+3\nu_2+1,\ \nu_1+3\nu_2+1\end{matrix}\ \middle|\ 1\right).$$

Since $w_\alpha(\nu_1,\nu_2) = (-\nu_1, \nu_1+\nu_2)$, the expression above implies

$$p_{n_1,n_2}(\nu) + p_{n_1,n_2}(w_\alpha\nu) = 0.$$

By using (2.8) to rewrite $\widetilde{C}^1_*(*)$ we can similarly show the cancellation (iii). Thus, the claim is verified, and this completes the proof of Theorem 3.3. ■

By using Theorems 3.2 and 3.3, we can get the following integral representations.

Corollary 3.4 *Under the same notation as in Theorems* 3.2 *and* 3.3, *we have*

(3.7) $$\widetilde{W}^2_{(\nu_1,\nu_2)}(y) = \int_0^\infty\int_0^\infty\int_0^\infty \exp\Big\{-(\pi y_1)^2 t_1 - (\pi y_2)^2 t_2 - (\pi y_2)^2 t_3 - \frac{1}{t_1} - \frac{1}{t_2} - \frac{1}{t_3}\frac{t_2}{t_1}\Big\}$$
$$\times \widetilde{W}^0_{\nu_1}\big(y_1y_2\sqrt{t_1t_3}\big) \times \left((\pi y_1)\sqrt{\frac{t_3}{t_1}}t_2\right)^{\nu_1+2\nu_2} \frac{dt_1}{t_1}\frac{dt_2}{t_2}\frac{dt_3}{t_3},$$

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and

(3.8) $\widetilde{W}^{3}_{(\nu_1,\nu_2)}(y)$

$$= \int_0^\infty \int_0^\infty \int_0^\infty \exp\Big\{ -(\pi y_1)^2 t_1 - (\pi y_2)^2 t_2 - (\pi y_2)^2 t_3 - \frac{1}{t_1} - \frac{1}{t_2} - \frac{1}{t_3}\frac{t_2}{t_1} \Big\} \times \widetilde{W}^{1}_{(\nu_1,\nu_1+3\nu_2)}\Big(y_1 y_2 \sqrt{t_1 t_3}, y_1 \sqrt{\frac{t_2}{t_3}} \Big) \frac{dt_1}{t_1}\frac{dt_2}{t_2}\frac{dt_3}{t_3}.$$

Proof We substitute the formula [12, Corollary 4]

$$\widetilde{W}^{1}_{(\nu_1,\nu_2)}(y) = \int_0^\infty \int_0^\infty \int_0^\infty \exp\Big\{ -(\pi y_1)^2 t_1 - (\pi y_2)^2 t_2 - (\pi y_2)^2 t_3 - \frac{1}{t_1} - \frac{1}{t_2} - \frac{1}{t_3}\frac{t_2}{t_1} \Big\} \times (\pi y_1)^{(4\nu_1+2\nu_2)/3} (\pi y_2)^{(2\nu_1-2\nu_2)/3} t_1^{\nu_1} t_2^{\nu_1+\nu_2} t_3^{-\nu_2} \frac{dt_1}{t_1}\frac{dt_2}{t_2}\frac{dt_3}{t_3}$$

into (3.1) and (3.3). Then we can obtain (3.7) and (3.8) by Mellin inversion. ■

4 Applications to Archimedean Zeta Integrals

Here we give two examples of computations of archimedean zeta integrals as an application of the integral representations of $W^2_\nu(y)$.

The first one is the degree 12 automorphic L-function on $\mathrm{GSp}_2 \times \mathrm{GL}_3$. The Rankin–Selberg integral for this L-function is constructed in Bump's survey [3, §3.5] (*cf.* [5]). When the infinite types of automorphic representations of $\mathrm{GL}_3(\mathbb{A}_\mathbb{Q})$ and $\mathrm{GSp}_2(\mathbb{A}_\mathbb{Q})$ are isomorphic to the class one principal series representations of $\mathrm{GL}_3(\mathbb{R})$ and $\mathrm{GSp}_2(\mathbb{R})$, respectively, the archimedean zeta integral we should compute becomes

$$Z_\infty(s) = \int_0^\infty \int_0^\infty \widetilde{W}^{1}_{(\mu_1,\mu_2)}((y_1,y_2)) \widetilde{W}^{2}_{(\nu_1,\nu_2)}((y_2,y_1)) \cdot (y_1 y_2^2)^s \frac{dy_1}{y_1}\frac{dy_2}{y_2}.$$

In view of the formula (3.1), our computation can be reduced to the archimedean zeta integral on $\mathrm{GL}_3 \times \mathrm{GL}_3$, which is calculated by Stade [17, 18].

Theorem 4.1 *Let* $\Gamma_\mathbb{R}(s) = \pi^{-s/2}\Gamma(s/2)$ *and*

$$(a_1,a_2,a_3) = \Big(\frac{4\mu_1 + 2\mu_2}{3}, \frac{-2\mu_1 + 2\mu_2}{3}, \frac{-2\mu_1 - 4\mu_2}{3} \Big),$$
$$(b_1,b_2) = (\nu_1, \nu_1 + 2\nu_2).$$

Then we have

$$Z_\infty(s) = \frac{\prod_{1\le i\le 3}\prod_{1\le j\le 2}\{\Gamma_\mathbb{R}(s+a_i+b_j)\Gamma_\mathbb{R}(s+a_i-b_j)\}}{\prod_{1\le i\le 3}\Gamma_\mathbb{R}(2s-a_i)}.$$

Proof By means of the formula (3.1), we have

$$
\begin{aligned}
Z_\infty(s) = & \int_0^\infty \int_0^\infty \widetilde{W}^1_{(\mu_1,\mu_2)}((y_1,y_2)) \times \frac{1}{2\pi\sqrt{-1}} \int_{L(\tau)} (\pi^3 y_1 y_2^2)^{2t/3} \\
& \times \widetilde{W}^1_{(t+\frac{\nu_1}{2}+\nu_2, -t+\frac{\nu_1}{2}+\nu_2)}((y_2,y_1)) \\
& \times \Gamma\Big(-t+\frac{\nu_1}{2}\Big)\Gamma\Big(-t-\frac{\nu_1}{2}\Big)(y_1y_2^2)^s dt \frac{dy_1}{y_1}\frac{dy_2}{y_2} \\
= & \frac{1}{2\pi\sqrt{-1}} \int_{L(\tau)} \pi^{2t}\Gamma\Big(-t+\frac{\nu_1}{2}\Big)\Gamma\Big(-t-\frac{\nu_1}{2}\Big) \\
& \times \int_0^\infty \int_0^\infty \widetilde{W}^1_{(\mu_1,\mu_2)}((y_1,y_2))\widetilde{W}^1_{(-t-\frac{\nu_1}{2}-\nu_2, t-\frac{\nu_1}{2}-\nu_2)}((y_1,y_2)) \\
& \times (y_1y_2^2)^{s+2t/3}\frac{dy_1}{y_1}\frac{dy_2}{y_2} dt.
\end{aligned}
$$

Here we uesd $\widetilde{W}^1_{(\mu_1,\mu_2)}((y_2,y_1)) = \widetilde{W}^1_{(-\mu_1,-\mu_2)}((y_1,y_2))$, which is obvious from the Mellin–Barnes integral representation in Proposition 3.1.

Here is the result of Stade [17]:

$$
\begin{aligned}
\int_0^\infty \int_0^\infty & \widetilde{W}^1_{(\mu_1,\mu_2)}((y_1,y_2))\widetilde{W}^1_{(\mu_1',\mu_2')}((y_1,y_2)) \cdot (y_1y_2^2)^s \frac{dy_1}{y_1}\frac{dy_2}{y_2} \\
& = \frac{\pi^{-3s}}{\Gamma(\frac{3s}{2})} \prod_{1\le i,j\le 3} \Gamma\Big(\frac{s}{2}+\lambda_i+\lambda_j'\Big),
\end{aligned}
$$

where

$$
\begin{aligned}
(\lambda_1,\lambda_2,\lambda_3) &= \Big(\frac{2\mu_1+\mu_2}{3}, \frac{-\mu_1+\mu_2}{3}, \frac{-\mu_1-2\mu_2}{3}\Big), \\
(\lambda_1',\lambda_2',\lambda_3') &= \Big(\frac{2\mu_1'+\mu_2'}{3}, \frac{-\mu_1'+\mu_2'}{3}, \frac{-\mu_1'-2\mu_2'}{3}\Big).
\end{aligned}
$$

Then we get

$$
\begin{aligned}
Z_\infty(s) = \pi^{-3s} & \prod_{1\le i\le 3} \Big\{\Gamma\Big(\frac{s}{2}+\frac{\nu_1}{2}+\nu_2+a_i\Big)\Gamma\Big(\frac{s}{2}-\frac{\nu_1}{2}-\nu_2+a_i\Big)\Big\} \\
& \times \frac{1}{2\pi\sqrt{-1}} \int_{L(\tau)} \frac{\Gamma(-t+\frac{\nu_1}{2})\Gamma(-t-\frac{\nu_1}{2})}{\Gamma(\frac{3s}{2}+t)} \prod_{1\le i\le 3} \Gamma\Big(\frac{s}{2}+t+a_i\Big)\, dt.
\end{aligned}
$$

The Barnes integral with respect to t is carried out by Barnes' second lemma [1, 6.2]:

$$
\begin{aligned}
(4.1) \quad \frac{1}{2\pi\sqrt{-1}} \int_{L(\tau)} & \frac{\Gamma(t-a)\Gamma(t-b)\Gamma(t-c)\Gamma(p-t)\Gamma(q-t)}{\Gamma(t+p+q-a-b-c)} dt \\
& = \frac{\Gamma(p-a)\Gamma(p-b)\Gamma(p-c)\Gamma(q-a)\Gamma(q-b)\Gamma(q-c)}{\Gamma(p+q-b-c)\Gamma(p+q-c-a)\Gamma(p+q-a-b)},
\end{aligned}
$$

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and thus we complete the proof. ■

Next we discuss the Rankin–Selberg integral on GSp_2 considered by Bump, Friedberg and Ginzburg [4], which contains two complex variables s, w and, interestingly, it interpolates the product of the standard L-function and the spinor L-function.

As in the computation above, we assume that the infinite type is the class one principal series representation. Then the archimedean zeta integral is

$$Z_\infty(s,w) = \int_0^\infty \int_0^\infty \int_{\mathbb{R}^2} W^{(2)}_{(\nu_1,\nu_2)}((y_1,y_2)) \cdot (1+x_1^2)^{-(s+1)/2}(1+x_2^2)^{-(w+1/2)} \times \exp\{2\pi\sqrt{-1}(x_1y_2+x_2y_1)\} \cdot y_1^{s-2}y_2^{w-3/2}\, dx_1dx_2 \frac{dy_1}{y_1}\frac{dy_2}{y_2}.$$

Theorem 4.2 *We have*

$$Z_\infty(s,w) = 2^{-2}\frac{L_\infty(s,\mathrm{st})\cdot L_\infty(w,\mathrm{spin})}{\Gamma_\mathbb{R}(s+1)\Gamma_\mathbb{R}(2s)\Gamma_\mathbb{R}(2w+1)},$$

where

$$L_\infty(s,\mathrm{st}) = \Gamma_\mathbb{R}(s)\Gamma_\mathbb{R}(s+2\nu_1+2\nu_2)\Gamma_\mathbb{R}(s-2\nu_1-2\nu_2)\Gamma_\mathbb{R}(s+2\nu_2)\Gamma_\mathbb{R}(s-2\nu_2),$$
$$L_\infty(w,\mathrm{spin}) = \Gamma_\mathbb{R}(w+\nu_1)\Gamma_\mathbb{R}(w-\nu_1)\Gamma_\mathbb{R}(w+\nu_1+2\nu_2)\Gamma_\mathbb{R}(w-\nu_1-2\nu_2).$$

Proof By the integral representation (3.2), we have

$$\begin{aligned} Z_\infty(s,w) = \frac{2^2}{(2\pi\sqrt{-1})^2}&\int_{L(\sigma_1)}\int_{L(\sigma_2)} T^0_{\nu_1}(-s_1)\,T^0_{\nu_1+2\nu_2}(-s_2)\,\pi^{2(s_1+s_2)} \\ &\times \int_0^\infty\int_\mathbb{R} K_{s_1+s_2}(2\pi y_1)(1+x_2^2)^{-(w+1/2)}\exp(2\pi\sqrt{-1}x_2y_1)\,y_1^{s_1+s_2+s}\,dx_2\frac{dy_1}{y_1} \\ &\times \int_0^\infty\int_\mathbb{R} K_{s_1-s_2}(2\pi y_2)(1+x_1^2)^{-(s+1)/2}\exp(2\pi\sqrt{-1}x_1y_2)\,y_2^{s_1+s_2+w}\,dx_1\frac{dy_2}{y_2} \\ &\times ds_2ds_1. \end{aligned}$$

By using the formulas

$$\int_\mathbb{R}(1+x^2)^{-\rho}\exp(2\pi\sqrt{-1}xy)\,dx = \frac{2\pi^\rho}{\Gamma(\rho)}\cdot y^{\rho-1/2}K_{\rho-1/2}(2\pi y)$$

for $\mathrm{Re}(\rho) > 0$ and $y > 0$ [6, 3.771.2], and

$$\int_0^\infty K_\mu(2\pi y)K_\nu(2\pi y)y^s\frac{dy}{y} = 2^{-3}\pi^{-s}\frac{\Gamma(s+\mu+\nu)\Gamma(s+\mu-\nu)\Gamma(s-\mu+\nu)\Gamma(s-\mu-\nu)}{\Gamma(s)}$$

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for $\mathrm{Re}(s) > |\mathrm{Re}(\mu)| + |\mathrm{Re}(\nu)|$ [6, 6.576.4], we get

$$\begin{aligned}Z_\infty(s,w) = \frac{2^2}{(2\pi\sqrt{-1})^2}\int_{L(\sigma_1)}\int_{L(\sigma_2)} \Gamma\Big(-s_1+\frac{\nu_1}{2}\Big)\Gamma\Big(-s_1-\frac{\nu_1}{2}\Big)\\ \times\Gamma\Big(-s_2+\frac{\nu_1}{2}+\nu_2\Big)\Gamma\Big(-s_2-\frac{\nu_1}{2}-\nu_2\Big)\pi^{2(s_1+s_2)}\\ \times 2^{-2}\pi^{-(s_1+s_2+s)+1/2}\frac{\Gamma(s_1+s_2+\frac{s}{2}+w)\Gamma(\frac{s}{2}+w)\Gamma(s_1+s_2+\frac{s}{2})\Gamma(\frac{s}{2})}{\Gamma(w+\frac{1}{2})\Gamma(s_1+s_2+s+w)}\\ \times 2^{-2}\pi^{-(s_1+\frac{s}{2}+w)+1/2}\frac{\Gamma(s_1+\frac{s+w}{2})\Gamma(s_1+\frac{w}{2})\Gamma(s_2+\frac{s+w}{2})\Gamma(s_2+\frac{w}{2})}{\Gamma(\frac{s+1}{2})\Gamma(s_1+s_2+\frac{s}{2}+w)}\,ds_2ds_1.\end{aligned}$$

Now the integration with respect to s_1 can be performed by Barnes' second lemma (4.1). Then we have

$$\begin{aligned}Z_\infty(s,w) = 2^{-2}\pi^{-s-w+1}\frac{\Gamma(\frac{s}{2})\Gamma(\frac{w+\nu_1}{2})\Gamma(\frac{w-\nu_1}{2})\Gamma(\frac{s+w+\nu_1}{2})\Gamma(\frac{s+w-\nu_1}{2})}{\Gamma(\frac{s+1}{2})\Gamma(w+\frac{1}{2})}\\ \times\frac{1}{2\pi\sqrt{-1}}\int_{L(\sigma_2)}\frac{\Gamma(-s_2+\frac{\nu_1}{2}+\nu_2)\Gamma(-s_2-\frac{\nu_1}{2}-\nu_2)}{\Gamma(s_2+s+\frac{w}{2})}\\ \times\Gamma(s_2+\frac{w}{2})\Gamma(s_2+\frac{s+\nu_1}{2})\Gamma(s_2+\frac{s-\nu_1}{2})\,ds_2.\end{aligned}$$

We use the formula (4.1) again to reach the assertion. ■

Remark 3 The denominator in $Z_\infty(s,w)$ is the product of the normalizing factors of two Eisenstein series used in the construction of the Rankin–Selberg integral. Since the global functional equation for the spinor L-function for generic cusp forms is established in [11, Theorem 1.2], we get the global functional equation for the standard L-function.

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