ON SUBSURFACES OF SOME RIEMANN SURFACES

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Introduction. In the theory of meromorphic functions, it is important to investigate the properties of covering surfaces generated by their inverse functions. For this purpose, the study of properties of a non-compact region of a Riemann surface is useful.

Recently Kuramochi has given in his paper [5] the following very interesting theorem. Let R be a Riemann surface and let R_0 be a compact domain on R with compact relative boundary ∂R_0. Then

Theorem. If R belongs to $O_{HB}-O_G$ ($O_{HD}-O_G$ resp.), then $R-R_0$ belongs to O_{AB} (O_{AD} resp.).

Here we use the following notations.

O_G: the class of Riemann surfaces which admit no Green function.

$O_{HB}(O_{AB})$: the class of Riemann surfaces on which there exists no non-constant single-valued bounded harmonic (analytic) function.

$O_{HD}(O_{AD})$: the class of Riemann surfaces on which there exists no non-constant single-valued harmonic (analytic) function with finite Dirichlet-integral.

Constantinescu-Cornea [1] have investigated this theorem in detail and obtained several results. Kuramochi [6] has extended this theorem again.

On the other hand, the method given by Heins [2] may be expected to contribute to the same purpose. He introduced the concept "locally of type-Bl" using the Green functions and gave many results concerning covering properties.

We shall give, in this article, simple proofs of extended Kuramochi's theorems in Constantinescu-Cornea's way and prove some properties of covering surfaces using them and Heins' method.

For simplicity, we shall call, in this article, a non-compact or compact domain G on a Riemann surface R a subregion on R when its relative boundary C with respect to R consists of at most an enumerable number of analytic non-compact or compact curves which cluster nowhere in R. We say that G belongs

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to the class SO_{HB} (SO_{HD}) if there exists no non-constant single-valued bounded (Dirichlet-bounded) harmonic function in G which vanishes continuously at every point on C.

1. Let R_1 and R_2 be two Riemann surfaces which admit Green functions and let f be a conformal mapping of R_1 into R_2. We denote by $\mathfrak{G}_{R_1}$ and $\mathfrak{G}_{R_2}$ Green functions of R_1 and R_2 respectively. Then holds the equality

$$\mathfrak{G}_{R_2}(f(p)\ ;\ q) = \sum_{f(r)=q} n(r)\mathfrak{G}_{R_1}(p\ ;\ r) + u_q(p),$$

where $n(r)$ is the multiplicity of f at $r \in R_1$, and $u_q(p)$ is the greatest harmonic minorant of $\mathfrak{G}_{R_2}(f(p)\ ;\ q)$ on R_1.

Generally, a positive harmonic function is representable uniquely by the sum of a non-negative quasi-bounded harmonic function which is defined as the limit of a monotone non-decreasing sequence of non-negative bounded harmonic functions, and a non-negative singular harmonic function which is defined as a non-negative harmonic function dominating no positive bounded harmonic function (Parreau [9]). Heins [2] proved that $u_q(p)$ is quasi-bounded except for a set of q of capacity zero and that the quasi-bounded component of $u_q(p)$ is either positive on $R_1 \times R_2$ or constantly zero.

According to Heins [2], we say that f is of type-Bl if the second alternative occurs for f.

Now, let R_1 and R_2 be arbitrary Riemann surfaces, and let f be a conformal mapping of R_1 into R_2. We shall say that f is of type-Bl at $q \in R_2$ provided that there exists a simply connected Jordan region Ω satisfying: (1) $q \in \Omega \subset R_2$, (2) $f^{-1}(\Omega) \neq \phi$ and (3) for each component Δ of $f^{-1}(\Omega)$, the restriction f_Δ of f to Δ is of type-Bl considering f_Δ as to be a conformal mapping of Δ into Ω. We shall say that f is locally of type-Bl if f is of type-Bl at each point of R_2. Then, we obtain the following:

THEOREM 1. *Let R_1 and R_2 be arbitrary Riemann surfaces, and let f be a conformal mapping of R_1 into R_2. Then, f is locally of type-*Bl *if and only if, for any compact subregion Ω on R_2 (we suppose that Ω has at least one exterior point when R_2 is compact), each component of $f^{-1}(\Omega)$ belongs to SO_{HB}.*

Proof. It is evident that f is locally of type-Bl if, for any compact subregion Ω on R_2, each component of $f^{-1}(\Omega)$ belongs to SO_{HB}.

Suppose that f is locally of type-Bl. Let Ω be an arbitrary compact subregion on R_2, and let $\{R_2^i\}$ be an exhaustion of R_2 with compact relative boundaries ∂R_2^i. As Ω is compact in R_2, there exists an integer i_0 such that $R_2^{i_0} \supset \Omega$. (When R_2 is compact, we take as $R_2^{i_0}$ a subregion on R_2 containing Ω and having at least one exterior point.) Let Δ be any component of $f^{-1}(\Omega)$ and let Δ^* be the component of $f^{-1}(R_2^{i_0})$ containing Δ. And we put $A = \min_{s \in \bar{\Omega}} \mathfrak{G}_{R^{i_0}_2}(s\ ;\ q)$, where q is an arbitrary point of R_2. Consider a bounded positive harmonic function u on Δ vanishing continuously on $\partial\Delta$, and denote by u^* the subharmonic function which is equal to u on Δ and to zero on $\Delta^* - \Delta$. Without loss of generality, we can suppose that $\sup u^* \leqq 1$. Then, we have

$$Au^* \leqq \mathfrak{G}_{R^{i_0}_2}(f_{\Delta^*}\ ;\ q)$$

on Δ^*. The least harmonic majorant of Au^* on Δ^* is dominated by the quasi-bounded component of the greatest harmonic minorant of $\mathfrak{G}_{R^{i_0}_2}(f_{\Delta^*}\ ;\ q)$. By Theorem 16. 1 in [2], f_{Δ^*} is of type-Bl considering f_{Δ^*} as to be a conformal mapping of Δ^* into $R_2^{i_0}$, and hence the quasi-bounded component of the greatest harmonic minorant of $\mathfrak{G}_{R^{i_0}_2}(f_{\Delta^*}\ ;\ q)$ is identically zero in Δ^*. Consequently, we can conclude that $u \equiv 0$ and therefore we have $\Delta \in SO_{HB}$. Thus our proof is complete.

2. Let R be a Riemann surface which admits a Green function, let $\mathfrak{G}_R(p\ ;\ q)$ be the Green function on R with a pole at $q \in R$ and let $p = \varphi(t)$ be the mapping which maps the universal covering surface R^∞ of R onto $|t| < 1$ one-to-one conformally. Then $\mathfrak{G}_R(\varphi(t)\ ;\ q)$ has angular limit zero a.e. on $|t| = 1$. We denote by $\mathfrak{F}$ the set of all points on $|t| = 1$ of such kind and classify $\mathfrak{F}$ into classes by the following equivalence relation. Let t_1 and t_2 be points of $\mathfrak{F}$. We say that t_1 and t_2 belong to the same class provided that there exists a covering transformation T of R^∞ such that $t_2 = T'(t_1)$, where T' is the linear transformation of $|t| < 1$ onto itself corresponding to T. We call each class an ideal boundary point and call all points of $\mathfrak{F}$ belonging to an ideal boundary point its image. We denote by F all ideal boundary points.

If the image $\mathfrak{M}$ of a subset M of F is measurable on $|t| = 1$, we say that M is measurable and call $\omega(p\ ;\ M, R) = \omega^*(\varphi^{-1}(p)\ ;\ \mathfrak{M})$ the harmonic measure of M with respect to R, where $\omega^*(t\ ;\ \mathfrak{M})$ is the harmonic measure of $\mathfrak{M}$ with respect to $|t| < 1$. Let M be a set of positive measure. According to Constantinescu-

Cornea [1], we say that M is $HB(HD)$-indivisible if, for any bounded (Dirichlet-bounded) harmonic function $u(p)$ on R, $u(\varphi(t))$ has the same angular limit a.e. on the image $\mathfrak{M}$ of M. For instance, F is $HB(HD)$-indivisible if R belongs to $O_{HB}-O_G(O_{HD}-O_G)$. It is known that if M is HB-indivisible, then M is HD-indivisible.

We shall consider the class $U_{HB}(U_{HD})$ of Riemann surfaces which contain at least one $HB(HD)$-indivisible set on their ideal boundaries. Heins [3] introduced a class O_L of Riemann surfaces, on which there exists no non-constant single-valued Lindelöfian meromorphic function. Here we say a conformal mapping of a Riemann surface R_1 into another Riemann surface R_2 is Lindelöfian if

$$\sum_{f(r)=q} n(r)\,\mathfrak{G}_{R_1}(p\,;\,r) < +\infty$$

for p and q satisfying $f(p) \neq q$. It was proved by Heins that the relation

$$O_{HB} \subset O_L \subset O_{AB}$$

holds and that, for the class of Riemann surfaces with finite genus,

$$O_G = O_{HB} = O_L$$

holds.

Let R be a Riemann surface belonging to U_{HB}, let M be an HB-indivisible set on its ideal boundary and let f be a single-valued Lindelöfian meromorphic function. Then we have for $w=f(\varphi(t))$

$$\begin{aligned}\sum_{f(\varphi(s))=w} n(w)\,\mathfrak{G}(t\,;\,s) &= \sum_{f(r)=w} n(r\,;\,f)\{\sum_{\varphi(s)=r}\mathfrak{G}(t\,;\,s)\}\\ &= \sum_{f(r)=w} n(r\,;\,f)\,\mathfrak{G}_R(\varphi(t)\,;\,r) < +\infty,\end{aligned}$$

and $f(\varphi(t))$ is Lindelöfian on $|t|<1$. Hence, we see that $f(\varphi(t))$ is meromorphic of bounded type in Nevanlinna's sense in $|t|<1$ from Heins' result: A Lindelöfian meromorphic function of the unit disc is of bounded type. So $f(\varphi(t))$ has the same angular limit a.e. on the image $\mathfrak{M}$ of M and we can conclude that f is constant by the theorem of Lusin and Priwaloff [8].

Similary we can see that there exists no non-constant single-valued meromorphic function with finite Dirichlet-integral on any Riemann surface belonging to U_{HD}. Thus, we have the following relations;

$$(*)\qquad \begin{aligned}&O_{HB}-O_G \subset U_{HB} \subset O_L - O_G \subset O_{AB}-O_G\\ &O_{HD}-O_G \subset U_{HD} \subset O_{AD}-O_G.\end{aligned}$$

3. We shall deal with some operations introduced by Kuramochi [4] and Heins [2] for the sequel. Let G be a subregion on a Riemann surface R, let u be a positive harmonic function on R and let U be a positive harmonic function on G vanishing continuously on ∂G such that there exists at least one positive superharmonic function on R dominating U on G (we shall call such a function U admissible). We denote by $I_G(u)$ and $E_G(U)$ the upper envelope of the non-negative subharmonic functions on ∂G dominated by u and vanishing continuously on ∂G and the lower envelope of the positive superharmonic functions on R dominating U on G, respectively. It is easily verified that $I_G(u)$ and $E_G(U)$ are harmonic in G and in R respectively, and that $I_G(u)$ vanishes continuously on ∂G.

We shall state some properties of these operations as lemmas.

LEMMA 1. *Operations I_G and E_G have the property of linearity.*

Proof. We shall give a proof only for I_G.

For any positive number a, obviously the equality

$$I_G(au) = aI_G(u)$$

holds. Let v be the same one as u. Then

$$I_G(u) + I_G(v) \leqq u + v \qquad \text{on } G.$$

Hence

$$I_G(u) + I_G(v) \leqq I_G(u + v) \leqq u + v$$

on G. Consider $\max(I_G(u+v) - u, 0)$ on G. It is subharmonic in G, vanishes continuously on ∂G and is dominated by v on G. Hence

$$I_G(u+v) - u \leqq \max(I_G(u+v) - u, 0) \leqq I_G(v)$$

and

$$I_G(u+v) - I_G(v) \leqq u.$$

Hence we have

$$I_G(u+v) - I_G(v) \leqq I_G(u), \quad \text{i.e.} \quad I_G(u+v) \leqq I_G(u) + I_G(v),$$

and therefore we can conclude that

$$I_G(u+v) = I_G(u) + I_G(v).$$

We can prove the linearity of E_G in the similar way.

LEMMA 2. *$I_G \cdot E_G$ is an identity, that is, for any admissible positive harmonic function U on G,*

$$I_G[E_G(U)] = U.$$

Proof. It is evident that $E_G(U) \geqq U$ on G and we have on G

$$E_G(U) \geqq I_G[E_G(U)] \geqq U.$$

Hence we have

$$E_G(U) \geqq E_G[I_G(E_G(U))] \geqq E_G(U),$$

and, by Lemma 1,

$$\begin{aligned} E_G[I_G(E_G(U))] &= E_G[I_G(E_G(U)) - U + U] \\ &= E_G[I_G(E_G(U)) - U] + E_G(U). \end{aligned}$$

Therefore

$$E_G[I_G(E_G(U)) - U] = 0,$$

and we can infer that

$$I_G[E_G(U)] = U.$$

LEMMA 3. *Let v be a positive harmonic function on R. If there exists an admissible positive harmonic function U on G such that v is dominated by $E_G(U)$, then we can find an admissible function V on G such that*

$$v = E_G(V).$$

Proof. From $v \leqq E_G(U)$, we have

$$U = I_G[E_G(U)] = I_G[(E_G(U) - v) + v] = I_G[E_G(U) - v] + I_G(v).$$

Hence we have

$$E_G[I_G(v)] + E_G[I_G(E_G(U) - v)] = E_G(U).$$

On the other hand, obviously

$$E_G[I_G(v)] \leqq v \quad \text{and} \quad E_G[I_G(E_G(U) - v)] \leqq E_G(U) - v,$$

and we can conclude that

$$v = E_G[I_G(v)].$$

Putting $V = I_G(v)$, we see that V satisfies the conditions of the lemma.

LEMMA 4. *Let U and U_i $(i = 1, 2, \ldots)$ be admissible positive harmonic*

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functions on G *and let* u *and* u_i $(i=1, 2, \dots)$ *be positive harmonic functions on* R. *If* $U=\sum_{i=1}^{\infty} U_i$ *exists, then*

$$E_G(U) = \sum_{i=1}^{\infty} E_G(U_i).$$

If $u=\sum_{i=1}^{\infty} u_i$ *exists, then*

$$I_G(u) = \sum_{i=1}^{\infty} I_G(u_i).$$

Proof. For any integer n, $U \geqq \sum_{i=1}^{n} U_i$ and $u \geqq \sum_{i=1}^{n} u_i$. Hence we have

$$E_G(U) \geqq E_G(\sum_{i=1}^{n} U_i) = \sum_{i=1}^{n} E_G(U_i)$$

and

$$I_G(u) \geqq I_G(\sum_{i=1}^{n} u_i) = \sum_{i=1}^{n} I_G(u_i).$$

Therefore

$$E_G(U) \geqq \sum_{i=1}^{\infty} E_G(U_i) \quad \text{and} \quad I_G(u) \geqq \sum_{i=1}^{\infty} I_G(u_i).$$

By Lemma 3, we can find a positive harmonic function V on G vanishing continuously on ∂G such that $E_G(U) \geqq E_G(V) = \sum_{i=1}^{\infty} E_G(U_i)$. Hence, for any integer n,

$$U = I_G[E_G(U)] \geqq V = I_G[E_G(V)] \geqq I_G[\sum_{i=1}^{n} E_G(U_i)] = \sum_{i=1}^{n} U_i.$$

Hence we can see that $U=V$ and therefore

$$E_G(U) = E_G(V) = \sum_{i=1}^{\infty} E_G(U_i).$$

Next we shall prove the latter equality. If we take an arbitrary point p on R, then we can find an integer n for given positive number ε such that $\sum_{i=n+1}^{\infty} u_i(p) < \varepsilon$. From $I_G(\sum_{i=n+1}^{\infty} u)(p) \leqq \sum_{i=n+1}^{\infty} u_i(p) < \varepsilon$, we have

$$I_G(u)(p) - \varepsilon \leqq (\sum_{i=1}^{\infty} I_G(u_i))(p) \leqq (\sum_{i=1}^{\infty} I_G(u_i))(p).$$

Since we can take ε as small as we please and p is an arbitrary point on R, we have

$$I_G(u) \leqq \sum_{i=1}^{\infty} I_G(u_i),$$

and hence

$$I_G(u) = \sum_{i=1}^{\infty} I_G(u_i).$$

We shall say that a positive harmonic function u is minimal if, for any positive harmonic function v dominated by u, there exists a constant c $(0<c\leqq 1)$ such that $v = cu$. Then we obtain the following lemma.

LEMMA 5. *Let u be a positive minimal harmonic function on R. If $I_G(u)$ is positive, then $I_G(u)$ is also minimal on G.*

Proof. Let U be a positive harmonic function on G dominated by $I_G(u)$. Then U vanishes continuously on ∂G. We have

$$E_G(U) \leqq E_G[I_G(u)] \leqq u,$$

and on account of the minimality of u we can find a constant c $(0 < c \leqq 1)$ such that

$$E_G(U) = cu.$$

Hence

$$U = I_G[E_G(U)] = cI_G(u).$$

Let $\underline{HD}$ be the class of non-negative harmonic functions, each of which is the limiting function of a monotone non-increasing sequence of positive harmonic functions with finite Dirichlet-integrals. We shall say that a positive harmonic function u belonging to $\underline{HD}$ is minimal in $\underline{HD}$ if, for any positive member v of $\underline{HD}$ dominated by u, there exists a constant c $(0 < c \leqq 1)$ such that $v = cu$.

Constantinescu and Cornea [1] proved that if u and v belong to $\underline{HD}$, the greatest harmonic minorant $u \wedge v$ of the superharmonic function min (u, v) and the least harmonic majorant $u \vee v$ of the subharmonic function max (u, v) also belong to $\underline{HD}$.

LEMMA 6. *Let u be a positive $\underline{HD}$-minimal harmonic function on R, and let G be a subregion not belonging to SO_{HD}. If there exists an admissible positive harmonic function U on G having a finite Dirichlet-integral such that $E_G(U)$ dominates u on R, then $I_G(u)$ is also minimal in $\underline{HD}$ on G.*

Proof. By Lemma 3 we can see that there exists an admissible function V on G such that $E_G(V) = u$, because $E_G(U) \geqq u$. Hence $U \geqq V$ and $u \geqq u \wedge U \geqq V$ on G. Obviously $u \wedge U$ vanishes continuously on ∂G. We see that $u \wedge U = V$ because V is the upper envelope of positive subharmonic functions dominated

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by u and vanishing continuously on ∂G. Therefore V belongs to $\underline{HD}$.

If W is a positive harmonic function on G belonging to $\underline{HD}$ and dominated by V, then $E_G(W)$ also belongs to $\underline{HD}$ on R and $E_G(W) = cu$ for some constant c $(0 < c \leqq 1)$. In fact, let $\{W_i\}$ be a monotone non-increasing sequence of harmonic functions with finite Dirichlet-integrals having W as their limiting function. Then the sequence $\{U \wedge W_i\}$ also has W as their limiting function. It is seen that $E_G(U \wedge W_i) \in HD$ and $\lim_{i\to\infty} E_G(U \wedge W_i) = E_G(W) \leqq E_G(V) = u$. Since u is minimal in $\underline{HD}$ on R, there exists a constant c such that $E_G(W) = cu$.

Hence we have $W = I_G[E_G(W)] = cI_G(u) = cV$. Thus we can conclude that $I_G(u)$ is minimal in $\underline{HD}$ on G.

If M is a HD-indivisible set such that, for any HD-indivisible set M' containing M, the harmonic measure of $M' - M$ with respect to R is zero, then we call M a maximal HD-indivisible set. Constantinescu-Cornea [1] proved that M is HB (maximal HD)-indivisible if and only if the harmonic measure $\omega(p\,;M)$ of M with respect to R is minimal (minimal in $\underline{HD}$). For the problem when subregions on a Riemann surface belonging to U_{HB} or U_{HD} belong to U_{HB} or U_{HD}, Lemmas 5 and 6 with this result give some answers.

The condition of the last lemma is equivalent to the condition "frei" given by Constantinescu-Cornea [1].

4. According to Constantinescu and Cornea [1], we denote by $O_{HB_n}(O_{HD_n})$ $(1 \leqq n \leqq \infty)$ the class of Riemann surfaces, the ideal boundary of which is null or consists of at most n HB (maximal HD)-indivisible sets. These classes are the same ones considered by Kuramochi [6]. In fact, as Constantinescu and Cornea proved, $O_{HB_n}(O_{HD_n})$ $(1 \leqq n < \infty)$ coincides with the class of Riemann surfaces on which there exist at most n number of linearly independent bounded (Dirichlet-bounded) harmonic functions. We note that $O_{HB_1} = O_{HB}$ and $O_{HD_1} = O_{HD}$.

Now, we give proofs of Kuramochi's Theorems [5], [6].

THEOREM 2. (Kuramochi) *If a Riemann surface R belongs to $O_{HB_n} - O_G$ $(1 \leqq n \leqq \infty)$ and a subregion G on R does not belong to SO_{HB}, then G belongs to O_L.*

Proof. Suppose that the ideal boundary of R consists of just m $(\leqq n)$ number of HB-indivisible sets M_i $(i = 1, 2, \ldots, m)$. Let ω_i $(i = 1, 2, \ldots, m)$

be the harmonic measure of M_i in R. Then each ω_i is minimal and $\sum_{i=1}^{m}\omega_i \equiv 1$. Since G does not belong to SO_{HB}, $I_G 1 = \sum_{i=1}^{m} I_G(\omega_i)$ is positive. Consequently for some i_0, $I_G(\omega_{i_0})$ is positive and minimal on G by Lemma 5.

We map the universal covering surface G^∞ of G onto $|t| < 1$, and denote the mapping function by $p = \varphi(t)$. Let M be the set on $|t| = 1$ such that $I_G(\omega_{i_0}) \circ \varphi$ has angular limit 1 a.e. on it and 0 a.e. on $(|t| = 1) - M$. Then M is of measure positive and on account of the minimulity of $I_G(\omega_{i_0})$, M is an HB-indivisible set. Hence the region G belongs to U_{HB} and by the relation (*) we can see that $G \in O_L$. Thus the proof is complete.

Kuroda [7] introduced a class O^0_{AB} of Riemann surfaces, on every subregion of which there exists no non-constant single-valued bounded analytic function with a real part vanishing continuously on its relative boundary. He proved that each Riemann surface belonging to O^0_{AB} has Iversen property and gave the relation

$$O_{HB} \subset O^0_{AB} \subset O_{AB}$$

and for the class of Riemann surfaces with finite genus,

$$O_G = O_{HB} \subset O^0_{AB} \subseteqq O_{AB}.$$

The subregion G of Theorem 2 obviously does not belong to O^0_{AB}, because there exist non-constant single-valued meromorphic functions on G not having Iversen property. Hence we have

$$O_L \not\subseteqq O^0_{AB}.$$

Further, O_{HD} is not a subclass of O_L in virtue of Tôki's example [10] and we obtain

$$O_L \not\subseteqq Q_{HD}.$$

THEOREM 3. (Kuramochi) *If a Riemann surface R belongs to $O_{HD_n} - O_G$ $(1 \leqq n \leqq \infty)$ and a subregion G on R does not belong to SO_{HD}, then G belongs to O_{AD}.*

Proof. Suppose that the ideal boundary of R consists of just m $(\leqq n)$ number of maximal HD-indivisible sets M_i $(i = 1, 2, \ldots, m)$. Let ω_i $(i = 1, 2, \ldots, m)$ be the harmonic measure of M_i with respect to R. Then ω_i belongs to HD and is minimal in HD (cf. [1]). Since G does not belong to SO_{HD} and since $SO_{HD} = SO_{HBD}$, there exists a positive bounded harmonic function U having a finite Dirichlet-integral and vanishing continuously on ∂G. By Dirichlet principle we see that $E_G(U)$ has also a finite Dirichlet-integral and $E_G(U) = \sum_{i=1}^{m} \alpha_i \omega_i$. Since $E_G(U)$ is positive, for some i_0, α_{i_0} is positive and $\frac{1}{\alpha_{i_0}} E_G(U) = E_G\left(\frac{1}{\alpha_{i_0}} U\right) \geqq \omega_{i_0}$. Hence by Lemma 6, we can conclude that $I_G(\omega_{i_0})$ is minimal in HD on G.

We map the universal covering surface G^∞ of G onto $|t| < 1$ by φ and denote by M the set on $|t| = 1$ such that $I_G(\omega_{i_0}) \circ \varphi$ has angular limit 1 a.e. on M and 0 a.e. on $(|t| = 1) - M$. It is seen that M is of positive measure and is maximal HD-indivisible because of the HD-minimality of $I_G(\omega_{i_0})$ (cf. [1]). Hence $G \in U_{HD}$ and by the relation (*) we can see that $G \in O_{AD}$. Thus our theorem is proved.

5. In this section we shall state some results which are deduced from Theorems 1 and 2.

THEOREM 4. *If a Riemann surface R belongs to O_{HB_n} $(1 \leqq n \leqq \infty)$, then any non-constant single-valued meromorphic function f on R is locally of type-*Bl.

Proof. Let Ω be an arbitrary subregion on the w-plane having at least one exterior point. Then all components of $f^{-1}(\Omega)$ belong to SO_{HB} by Theorem 2. Thus we can see that f is locally of type-Bl by Theorem 1.

COROLLARY. *Let R be a Riemann surfuce belonging to O_{HB_n} $(1 \leqq n \leqq \infty)$, and let Φ be the covering surface of the w-plane generated by a non-constant single-valued meromorphic function f on R. Then every connected piece Φ_Δ of Φ on any disc Δ in the w-plane covers each point of Δ the same number of times except for at most an F_σ-set of capacity zero.*

Proof. This corollary is immediate from Theorem 4 and Theorem 21.2 in [2].

THEOREM 5. *Let R be a Riemann surface belonging to O_{HB_n} $(1 \leqq n \leqq \infty)$ and let G be a subregion on R not belonging to SO_{HB}. Then the cluster set of any non-constant single-valued meromorphic function f on G at the ideal boundary of G is the whole w-plane, and the range of values of f contains all values of the w-plane except for at most an F_σ-set of capacity zero.*

Proof. Without loss of generality, we may suppose that f is analytic on

∂G. By Theorem 2, G belongs to O_L and f is not Lindelöfian. Heins proved in [3] that if, for some $p_0 \in G$, $\sum_{f(r)=w} n(r) \mathfrak{G}_G(p_0, r) < +\infty$ for a set of w of positive capacity, then f is Lindelöfian on G. Hence f takes each value infinitely often except for an F_σ-set of capacity zero.

6. Here we shall be concerned with the subsurfaces on Riemann surfaces of the class O_{HD_n}.

THEOREM 6. *Let f be a non-constant single-valued meromorphic function on a Riemann surface R. If there exist a point w_0, $n-1$ $(n < \infty)$ number of subregions c_i and a sequence of Jordan regions Ω_i of the w-plane such that $c_i \cap c_j = \phi$ for $i \neq j$, $w_0 \notin \bigcup_{i=1}^{n-1} \bar{c}_i$, $\Omega_i \supset \bar{\Omega}_{i+1}$ and $\bigcap_{i=1}^{\infty} \Omega_i = w_0$, and that, for each i, at least one component δ_i of $f^{-1}(c_i)$ and one component Δ_i of $f^{-1}(\Omega_i)$ do not belong to SO_{HD}, then R does not belong to O_{HD_n}.*[1)]

To prove this theorem, we give the following:

THEOREM 7. *Let R be a Riemann surface. Then R does not belong to O_{HB_n} (O_{HD_n} resp.) $(n < \infty)$ if there exist $n+1$ subregions G_i $(i = 0, 1, 2, \ldots, n)$ disjoint from each other on R such that $G_i \notin SO_{HB}$ for all i ($G_0 \notin SO_{HB}$ and $G_i \notin SO_{HD}$ for $i = 1, 2, \ldots, n$ resp.).*

Proof. Suppose that R belongs to $O_{HB_n}(O_{HD_n})$. Then the boundary of R consists of just m $(\leqq n)$ number of HB (maximal HD)-indivisible sets M_k $(k = 1, 2, \ldots, m)$. Since $G_i \notin SO_{HB}(SO_{HD})$ $(i = 1, 2, \ldots, n)$, we can find for each $i \neq 0$ in the same way as in the proofs of Theorems 2 and 3 a harmonic measure $\omega_k(p) = \omega(p\,;\, M_k)$ of M_k such that $I_{G_i}(\omega_k) > 0$. Furthermore we can see that $I_{G_j}(\omega_k) = 0$ for $j = 0, \ldots, i-1, i+1, \ldots, n$. In fact, for $i \neq j$,

$$E_{G_i} I_{G_i}(\omega_k) + E_{G_j} I_{G_j}(\omega_k) \leqq \omega_k,$$

and from the minimality of ω_k and the fact that $\sup_{G_i} I_{G_i}(\omega_k) = 1$

$$E_{G_i} I_{G_i}(\omega_k) = \omega_k.$$

Hence we have $E_{G_j} I_{G_j}(\omega_k) = 0$ and $I_{G_j}(\omega_k) = I_{G_j} E_{G_j} I_{G_j}(\omega_k) = 0$. Thus we can see that, for any ω_k, $I_{G_0}(\omega_k) = 0$ and $I_{G_0}(1) = I_{G_0}(\sum_{k=1}^{m} \omega_k) = \sum_{k=1}^{m} I_{G_0}(\omega_k) = 0$. This contradicts the condition: $G_0 \notin SO_{HB}$, which proves the theorem.

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1) The auther proved only the case $n=1$ and the extension of the present form is due to Kuroda.

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Proof of Theorem 6. By Theorem 1, f is not locally of type-Bl, so by Theorem 17.1 in [2] the set of points w in any closed neighbourhood of w_0, at which f is not of type-Bl, is of positive capacity. Let $w_1 \neq w_0$ be such a point, satisfying $w_1 \notin \bigcup_{i=1}^{n-1} \bar{c}_i$, then for some i, Ω_i does not contain w_1 and $\Omega_i \cap (\bigcup_{i=1}^{n-1} c_i) = \phi$. Choosing a positive number ρ satisfying that $(\Omega_i \cup (\bigcup_{i=1}^{n-1} c_i)) \cap (|w - w_1| < \rho) = \phi$, we can find among components of $f^{-1}(|w - w_1| < \rho)$, a component Δ_0 not belonging to SO_{HB} and satisfying $\Delta_0 \cap \Delta_i = \phi$ and $\Delta_0 \cap \delta_i = \phi$. By Theorem 7, R does not belong to O_{HD_n}.

THEOREM 8. *Let R be a Riemann surface belonging to O_{HD_n} $(1 \leqq n \leqq \infty)$, let Φ be the covering surface of the w-plane generated by a non-constant single-valued meromorphic function f on R, and let Φ_ρ be a connected piece of Φ on $|w - w_0| < \rho$. If the area of Φ_ρ is finite, then the restriction f_ρ of f to the component Δ_ρ of $f^{-1}(|w - w_0| < \rho)$ corresponding to Φ_ρ is of type*-Bl *of Δ_ρ. Hence Φ_ρ covers each point of $|w - w_0| < \rho$ the same number of times except for at most a closed set of capacity zero, and Φ_ρ is finitely sheeted.*

Proof. Suppose that f_ρ is not of type-Bl. Then, by Theorem 1, there exists a positive number $\rho_0 < \rho$ such that a component Δ_{ρ_0} of $f^{-1}(|w - w_0| < \rho_0)$ exists and does not belong to SO_{HB}. Let ω be the harmonic measure of $|w - w_0| = \rho_0$ with respect to the ring domain $(\rho_0 < |w - w_0| < \rho)$, and let ω^* be the superharmonic function such that ω^* is equal to ω on $\rho_0 < |w - w_0| < \rho$ and to 1 on $|w - w_0| \leqq \rho_0$. Put $A = \max|\mathrm{grad}\, \omega^*|$. Then A is finite and $D(\omega^* \circ f) \leqq A^2 D(f_\rho) = A^2 \times$ (the area of Φ_ρ) $< +\infty$. Hence, by Dirichlet principle, the greatest harmonic minorant u of $\omega^* \circ f$ of Δ_ρ has a finite Dirichlet-integral. Since Δ_{ρ_0} does not belong to SO_{HB}, there exists a positive bounded harmonic function u_0 such that $u_0 = 0$ on $\partial \Delta_{\rho_0}$ and $\sup u_0 = 1$. Denote by u_0^* the subharmonic function such that $u_0^* = u_0$ on Δ_{ρ_0} and $u_0^* = 0$ on $\Delta_\rho - \Delta_{\rho_0}$, then $u_0^* \leqq \omega^* \circ f_\rho$, $0 < Eu_0^* \leqq \omega^* \circ f$ because of superharmonicity of $\omega^* \circ f$, and we can conclude that $0 < Eu_0^* \leqq u$ and Δ_ρ does not belong to SO_{HD}. This contradicts Theorem 3. Thus our theorem is established.

It is evident that this theorem implies Kuramochi's result (Theorem 12 in [6]).

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